COHOMOLOGY OF ARTIN GROUPS

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ABSTRACT. We survey the $K(\pi, 1)$ conjecture and cohomology of Artin groups. We also present a formula for the second mod 2 homology of all Artin groups without assuming the $K(\pi, 1)$ conjecture.

1. Introduction

The study of *braid groups* is an active topic in diverse areas of mathematics and theoretical physics. In 1925, E. Artin [Art25] introduced the notion of braids in a geometric picture. Fox and Neuwirth [FN62] showed that the configuration space of unordered n-tuples of distinct points in $\mathbb{C}$ is a classifying space of the braid group $Br(n)$. This led to extensive investigations of the cohomology of braid groups by Arnol'd [Arn69], Fuks [Fuk70], Vainstein [Vaĭ78] and many others. It is remarkable to note that the braid group is closely related to the hyperplane arrangement associated to the symmetric group, known as the braid arrangement. The above mentioned configuration space is nothing but the orbit space of the complement to the complexified braid arrangement with respect to the action of the symmetric group by permutation of coordinates.

A generalization of the relation between braid groups and symmetric groups is that of *Artin groups* and *Coxeter groups*. For a Coxeter graph Γ and the associated Coxeter system $(W(\Gamma), S)$, we associate an Artin group $A(\Gamma)$ obtained by, informally speaking, dropping the relations that each generator has order 2 from the standard presentation of $W(\Gamma)$. The braid group $Br(n)$ is the Artin group of type A_{n-1} and the symmetric group $\mathfrak{S}_n$ is the Coxeter group of type A_{n-1}. When $W(\Gamma)$ is a finite Coxeter group, we say that $A(\Gamma)$ is of finite type (or spherical type). Recall that a finite Coxeter group $W(\Gamma)$ can be geometrically realized as an orthogonal reflection group acting on $\mathbb{R}^n$ where $n = \#S$ is the rank of W. Let $\mathcal{A}$ be the collection of the reflection hyperplanes (in $\mathbb{R}^n$) determined by W, known as the Coxeter arrangement associated to W. Topologically, it is more interesting to consider the complexified Coxeter arrangement $\mathcal{A}_{\mathbb{C}} = \{H \otimes \mathbb{C} \mid H \in \mathcal{A}\}$ and its complement $M(\Gamma) = \mathbb{C}^n \setminus \cup_{H \in \mathcal{A}} H \otimes \mathbb{C}$. The Coxeter group $W(\Gamma)$ acts freely on $M(\Gamma)$. Set $N(\Gamma) = M(\Gamma)/W(\Gamma)$ to be the quotient space. Brieskorn [Bri71] proved that the fundamental group of $N(\Gamma)$ is isomorphic to the Artin group $A(\Gamma)$. Furthermore, as a consequence of a theorem of Deligne [Del72], $N(\Gamma)$ is a $K(A(\Gamma), 1)$ space. Hence the (co)homology of $N(\Gamma)$ is the (co)homology of the Artin group $A(\Gamma)$ of spherical type. There are many computations in the literature. Besides the above mentioned references for braid groups (type A_n), see [Gor78] for types C_n and D_n, and [Sal94] for exceptional types. Cohomology ring structure is computed in [Lan00].

When $W(\Gamma)$ is an infinite Coxeter group, we say that the associated Artin group $A(\Gamma)$ is of infinite type (or non-spherical type). In this case, the Coxeter group $W(\Gamma)$ can be realized as a (non-orthogonal) reflection group acting on a convex cone U (called Tits cone, see Subsection 2.1) in $\mathbb{R}^n$ with $n = \#S$ the rank of W. Let $\mathcal{A}$ be the collection of reflection hyperplanes. Consider the complement $M(\Gamma) = (\mathrm{int}(U) + \sqrt{-1}\mathbb{R}) \setminus \cup_{H \in \mathcal{A}} H \otimes \mathbb{C}$ and the $W(\Gamma)$-action on $M(\Gamma)$, the resulting quotient space $N(\Gamma) = M(\Gamma)/W(\Gamma)$ has fundamental group isomorphic to $A(\Gamma)$ ([vdL83]). However, it is only conjectured that $N(\Gamma)$ is a $K(A(\Gamma), 1)$ space in general. See Subsection 2.3 for a summary for the progress of this conjecture.

The most effective tool in the computation of cohomology of Artin group is the so-called *Salvetti complex* introduced by Salvetti in [Sal87]. In that paper, Salvetti associated a CW-complex (known as Salvetti complex) to each real hyperplane arrangement which has the homotopy type of the complement to the complexified arrangement. Later, Salvetti [Sal94] and joint with De Concini [DCS96] applied the construction of Salvetti complex to (possibly infinite) arrangements associated to Coxeter groups and obtained a very useful algebraic complex that computes the (co)homology of the quotient space $N(\Gamma)$ of the complement $M(\Gamma)$ with respect to the Coxeter group $W(\Gamma)$. Whenever $N(\Gamma)$ is known to be a $K(\pi, 1)$ space, this provides a standard

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method to compute the (co)homology of the Artin group $A(\Gamma) \cong \pi_1(N(\Gamma))$ over both trivial and twisted coefficients. See Subsection 3.3 for a list of results using this method.

Existing results about (co)homology of Artin groups all rely on the affirmative solution of the $K(\pi, 1)$ conjecture, since the computations are actually the (co)homology of the quotient space $N(\Gamma)$. However in Section 4, we present an explicit formula for the second mod 2 homology of an arbitrary Artin group,

$$H_2(A(\Gamma); \mathbb{Z}_2) \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)},$$

which does not rely on the $K(\pi, 1)$ conjecture. See Theorem 3.5 for definitions of the numbers $p(\Gamma)$ and $q(\Gamma)$.

As a corollary, we obtain a sufficient condition of the triviality of the Hurewicz homomorphism

$$h_2 : \pi_2(N(\Gamma)) \to H_2(N(\Gamma); \mathbb{Z}).$$

Furthermore, we conclude that the induced Hurewicz homomorphism

$$h_2 \otimes \mathrm{id}_{\mathbb{Z}_2} : \pi_2(N(\Gamma)) \otimes \mathbb{Z}_2 \to H_2(N(\Gamma); \mathbb{Z}) \otimes \mathbb{Z}_2$$

is always trivial. This provides affirmative evidence for the $K(\pi, 1)$ conjecture.

2. Basic definitions

We collect relevant definitions and properties of Coxeter groups and Artin groups. We refer to [Bou68], [Hum90] and [Par12] for details.

2.1. **Coxeter groups.** Let S be a finite set. A *Coxeter matrix* over S is a symmetric matrix $M = (m(s,t))_{s,t\in S}$ such that $m(s,s) = 1$ for all $s \in S$ and $m(s,t) \in \{2, 3, \cdots\} \cup \{\infty\}$ for distinct $s, t \in S$. It is convenient to represent M by a labeled graph Γ, called the *Coxeter graph* of M defined as follows:

- The vertex set $V(\Gamma) = S$;
- The edge set $E(\Gamma) = \{\{s, t\} \subset S \mid m(s,t) \geq 3\}$;
- The edge $\{s, t\}$ is labeled by $m(s,t)$ if $m(s,t) \geq 4$.

Let Γ_{odd} be the subgraph of Γ with $V(\Gamma_{odd}) = V(\Gamma)$ and $E(\Gamma_{odd}) = \{\{s,t\} \in E(\Gamma) \mid m(s,t) \text{ is odd}\}$ inheriting labels from Γ. By abuse of notations, we frequently regard Γ (hence also Γ_{odd}) as its underlying 1-dimensional CW-complex with the set of 0-cells S and the set of 1-cells $\{\langle s,t\rangle \mid \{s,t\} \in E(\Gamma)\}$.

For two letters s, t and an integer $m \geq 2$, we shall use the following notation of the word of length m consisting of s and t in an alternating order.

$$(st)_m := \overbrace{sts\cdots}^{m}.$$

Definition 2.1. Let Γ be a Coxeter graph and S its vertex set. The *Coxeter system* associated to Γ is by definition the pair $(W(\Gamma), S)$, where $W(\Gamma)$ is the group presented by

$$W(\Gamma) = \langle S \mid \overline{R_W} \cup Q_W \rangle.$$

The sets of relations are $\overline{R_W} = \{R(s,t) \mid m(s,t) < \infty\}$ and $Q_W = \{Q(s) \mid s \in S\}$, where $R(s,t) := (st)_{m(s,t)}(ts)^{-1}_{m(s,t)}$ and $Q(s) := s^2$.

Note that since $R(s,t) = R(t,s)^{-1}$, we may reduce the relation set $\overline{R_W}$ by introducing a total order on S and put $R_W := \{R(s,t) \mid m(s,t) < \infty, s < t\}$. We have the following presentation with fewer relations

$$W(\Gamma) = \langle S \mid R_W \cup Q_W \rangle.$$

The group $W = W(\Gamma)$ is called the *Coxeter group* of type Γ. We shall omit the reference to Γ if there is no ambiguity. The rank of W is defined to be $\#S$.

Remark 2.2. It is not difficult to see that $W(\Gamma)$ as above has an equivalent presentation

$$W(\Gamma) = \langle S \mid (st)^{m(s,t)} = 1, \forall s, t \in S \text{ with } m(s,t) \neq \infty \rangle.$$

Each generator $s \in S$ of W has order 2. For distinct $s, t \in S$, the order of st is precisely $m(s,t)$ if $m(s,t) \neq \infty$. In case $m(s,t) = \infty$, the element st has infinite order. Therefore, given a pair (W, S) with the above presentation, it uniquely determines a Coxeter matrix, hence a Coxeter graph.

2.1.1. *Parabolic subgroups and Poincaré series.* Let (W, S) be a Coxeter system. For a subset $T \subset S$, let W_T denote the subgroup of W generated by T, called a *parabolic subgroup* of W. In particular, $W_S = W$ and $W_\varnothing = \{1\}$. It is known that (W_T, T) is the Coxeter system (cf. Théorème 2 in Chapter IV of [Bou68]) associated to the Coxeter graph Γ_T (the full subgraph of Γ spanned by T inheriting labels). If $T \subset S$ generates a finite parabolic subgroup, we say that T is a spherical subset. Denote by $\mathcal{S}^f$ the collection of all spherical subsets.

Recall that the length $\ell(w)$ of an element $w \in W$ is the defined as the minimal number k such that w can be written as a word $w = s_1 s_2 \cdots s_k$ with $s_i \in S$, and such a word is called a reduced expression of w. We declare that $\ell(w) = 0$ if and only if $w = 1$. The restriction of the length function to any parabolic subgroup W_T agrees with the length function of the Coxeter system (W_T, T).

Define $W^T := \{w \in W \mid \ell(wt) > \ell(w) \text{ for all } t \in T\}$. Then for $w \in W$, there is a unique element $u \in W^T$ and a unique element $v \in W_T$ such that $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$. Moreover, u is the unique element of minimal length in the coset wW_T. Hence the set W^T is called the set of minimal coset representatives.

For a subset $X \subset W$, we define the Poincaré series of X

$$X(q) = \sum_{x \in X} q^{\ell(x)}.$$

A consequence of the last paragraph is that for a subset $T \subset S$,

$$W(q) = W_T(q) W^T(q).$$

2.1.2. *Geometric representation.* As mentioned in the introduction, each Coxeter group can be geometrically realized as a reflection group acting on a Tits cone. In this subsection, we describe this construction.

Let (W, S) be a Coxeter system. Denote by V the real vector space with basis $\{\alpha_s \mid s \in S\}$. We define a symmetric bilinear form B on V by setting

$$B(\alpha_s, \alpha_t) := -\cos \frac{\pi}{m(s,t)}.$$

We declare that $\cos \frac{\pi}{\infty} = 1$. For each $s \in S$, we define the reflection ρ_s on V by

$$\rho_s(\alpha_t) = \alpha_t - 2B(\alpha_s, \alpha_t)\alpha_t.$$

Remark that $\rho_s(\alpha_s) = -\alpha_s$ and ρ_s fixes pointwise the hyperplane H_s orthogonal to α_s with respect to B. The assignment $s \mapsto \rho_s$ defines a faithful representation of W,

$$\rho : W \to \mathrm{GL}(V).$$

We prefer to consider the contragredient representation

$$\rho^* : W \to \mathrm{GL}(V^*),$$

where V^* is the dual space of V. It is defined by the following

$$\langle \rho^*(w)(f), \alpha \rangle = \langle f, \rho(w^{-1})(\alpha) \rangle,$$

where $f \in V^*, \alpha \in V$ and $\langle \bullet, \bullet \rangle$ is the natural pairing of V^* and V. Let C_0 be the fundamental chamber defined as $C_0 := \{f \in V^* \mid \langle f, \alpha_s \rangle > 0 \text{ for all } s \in S\}$. Define the Tits cone $U := W.\overline{C_0}$ as the W-orbit of the closure of C_0. It is a W-stable subset of V^*. Recall some important properties of U.

Proposition 2.3 ([Vin71])**.** *Let U be defined as above. Then*
(i) U is a convex cone in V^* with vertex 0.
(ii) $U = V^*$ iff W is finite iff B is positive definite.
(iii) $\mathrm{int}(U)$ is open in V^* and a facet $F \subset \overline{C_0}$ is contained in $\mathrm{int}(U)$ if and only if the stabilizer W_F is finite.

2.1.3. *Finite and affine Coxeter groups.* A Coxeter system (W, S) is said to be *irreducible* if the Coxeter graph Γ is connected. The following proposition allows us to reduce the study of Coxeter systems to that of irreducible ones.

Proposition 2.4. *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ be the connected components of Γ, and $S_i \subset S$ be the vertex set of Γ_i. Then the Coxeter group W is the direct product of its parabolic subgroups $W_{S_1}, W_{S_2}, \ldots, W_{S_m}$ and the Coxeter system (W_{S_i}, S_i) is irreducible.*

The classification of finite irreducible Coxeter groups is well known.

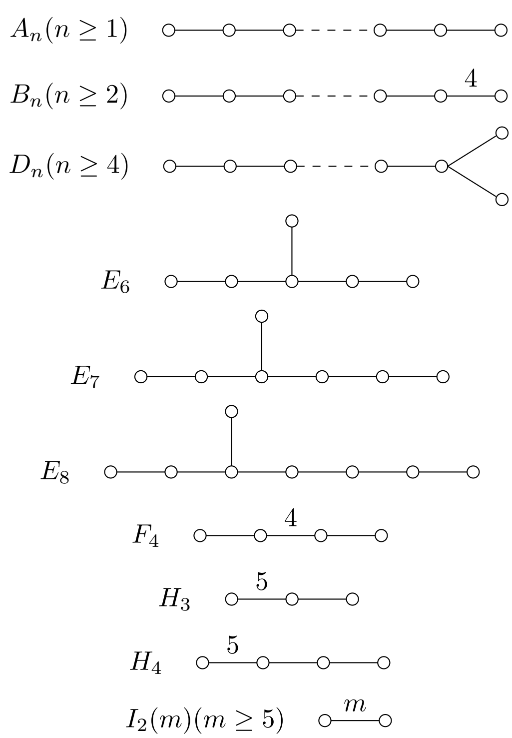

FIGURE 1. Classification of finite irreducible Coxeter groups

Theorem 2.5. *Let (W, S) be a finite irreducible Coxeter system, then the Coxeter graph Γ must be one of the listed graphs in Figure 1, where the subscript denotes the rank of the corresponding Coxeter group.*

As for infinite Coxeter groups, there is an important class called *affine* Coxeter groups. They arise as affine reflection groups in the Euclidean space. Recall that a finite Coxeter group W is called *crystallographic* if W stabilizes a lattice in V where the action is given by the geometric representation $\rho : W \to \mathrm{GL}(V)$, such a group is also known as a Weyl group. They are characterized by the following proposition.

Proposition 2.6. *W is crystallographic if and only if $m(s, t) = 2, 3, 4$ or 6 for each pair of distinct $s, t \in S$.*

We simply rule out H_3, H_4 and $I_2(m)$ for $m = 5, 7, 8, 9, \ldots$ from the list in Figure 1 to get a list of irreducible Weyl groups.

Given a Weyl group W acting on V, let $\mathrm{Aff}(V)$ be the affine group of V, which is the semidirect product of $\mathrm{GL}(V)$ with the group of translations in V. We define the affine Weyl group W_a as the subgroup of $\mathrm{Aff}(V)$ generated by affine reflections along affine hyperplanes $H_{\alpha,k} = \{v \in V \mid (v, \alpha) = k\}$ for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, where $\Phi = \{w(\alpha_s) \mid s \in S, w \in W\}$ is the root system of W and $(\bullet, \bullet)$ is the scalar product on V induced by the positive definite form B.

We list the Coxeter graphs of irreducible affine Weyl groups in Figure 2. In each graph, the number of vertices is equal to the subscript plus 1.

2.2. **Artin groups.** Associated to a Coxeter graph Γ, the Artin group $A(\Gamma)$ is obtained from the presentation of $W(\Gamma)$ by dropping the relation set Q_W.

Definition 2.7. Given a Coxeter graph Γ (hence a Coxeter system (W, S)), we introduce a set $\Sigma = \{a_s \mid s \in S\}$ in one-to-one correspondence with S. Then the Artin system associated to Γ is the pair $(A(\Gamma), \Sigma)$, where $A(\Gamma)$ is the *Artin group* of type Γ defined by the following presentation:

$$A(\Gamma) = \langle \Sigma \mid \overline{R_A} \rangle,$$

where $\overline{R_A} = \{R(a_s, a_t) \mid m(s, t) < \infty\}$ and $R(a_s, a_t) = (a_s a_t)_{m(s,t)} (a_t a_s)^{-1}_{m(s,t)}$.

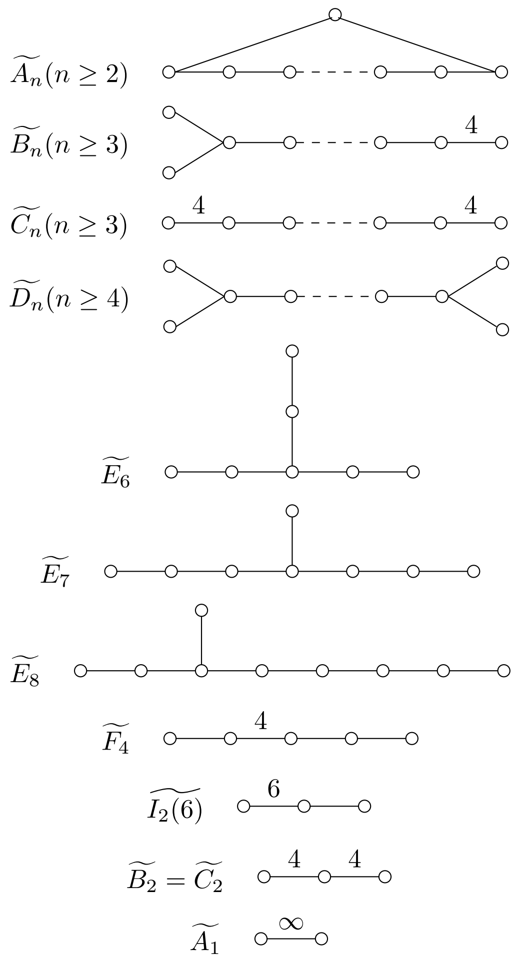

FIGURE 2. Coxeter graphs for affine Coxeter groups

Note that since $R(a_s, a_t) = R(a_t, a_s)^{-1}$, we may reduce the relation set R_A by introducing a total order on S and put $R_A := \{R(a_s, a_t) \mid m(s,t) < \infty, s < t\}$. We have the following presentation with fewer relations

$$A(\Gamma) = \langle \Sigma \mid R_A \rangle.$$

There is a canonical projection $p : A(\Gamma) \to W(\Gamma)$, $a_s \mapsto s (s \in S)$, the kernel is called the *pure Artin group* of type Γ, denoted by $PA(\Gamma)$. The three groups fit into the exact sequence

$$1 \to PA(\Gamma) \to A(\Gamma) \xrightarrow{p} W(\Gamma) \to 1.$$

The projection p has a canonical (set) section $\psi : W(\Gamma) \to A(\Gamma)$ given by

$$\psi(w) = \psi(s_{i_1} s_{i_2} \cdots s_{i_k}) = s_{i_1} s_{i_2} \cdots s_{i_k},$$

where $s_{i_1} s_{i_2} \cdots s_{i_k}$ ($s_{i_j} \in S$) is a reduced expression of $w \in W(\Gamma)$. As a consequence of a theorem of Matsumoto [Mat64], ψ is well-defined.

We say that an Artin group $A(\Gamma)$ is of finite type (or spherical type) if the associated Coxeter group $W(\Gamma)$ is finite. $A(\Gamma)$ is of infinite type (or non-spherical type) if $W(\Gamma)$ is infinite.

2.3. $K(\pi, 1)$ **conjecture.** Consider a Coxeter graph Γ and the associated Coxeter system (W, S) with W finite and rank $\#S = n$. Recall that W can be realized as a reflection group acting on $\mathbb{R}^n$. Let $\mathcal{A}$ be the collection of the reflection hyperplanes, known as the Coxeter arrangement associated to W. Let

$$M(\Gamma) := \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H \otimes \mathbb{C}$$

be the complement to the complexified arrangement $\mathcal{A}_{\mathbb{C}} = \{H \otimes \mathbb{C} \mid H \in \mathcal{A}\}$ in $\mathbb{C}^n$. The Coxeter group W acts freely and properly discontinuously on $M(\Gamma)$. Denote the orbit space by $N(\Gamma) = M(\Gamma)/W$.

Theorem 2.8 ([Bri71]). *For an Artin group $A(\Gamma)$ of finite type, the fundamental group of $N(\Gamma)$ is isomorphic to $A(\Gamma)$.*

The classical case $\Gamma = A_n$ is previously proved in [FN62].

The following theorem of Deligne shows that $N(\Gamma)$ is a $K(\pi, 1)$ space.

Theorem 2.9 ([Del72]). *Let $\mathcal{A}$ be a finite real central* simplicial *arrangement, then the complement $M(\mathcal{A})$ to the complexification of $\mathcal{A}$ is a $K(\pi, 1)$ space.*

Here a real arrangement is called simplicial if any chamber (a connected component of the complement) is a simplicial cone. Since Coxeter arrangements are simplicial ([Bou68]), then we have

Corollary 2.10. *The complement $M(\Gamma)$ is a $K(\pi, 1)$ space and so is $N(\Gamma)$.*

An Artin group $A(\Gamma)$ of finite type thus has a classifying space $N(\Gamma)$. As for Artin groups of infinite type, the above construction can be mildly modified. Suppose now the Coxeter group $W(\Gamma)$ is infinite of rank n, realized as a reflection group acting on a Tits cone $U \subset \mathbb{R}^n$. $\mathcal{A}$ is the collection of reflection hyperplanes. Let

$$M(\Gamma) := \big(\mathrm{int}(U) + \sqrt{-1}\mathbb{R}^n\big) \setminus \bigcup_{H \in \mathcal{A}} H \otimes \mathbb{C}.$$

Then W acts on $M(\Gamma)$ freely and properly discontinuously. Denote the orbit space by

$$N(\Gamma) := M(\Gamma)/W.$$

It is known that

Theorem 2.11 ([vdL83]). *The fundamental group of $N(\Gamma)$ is isomorphic to the Artin group $A(\Gamma)$.*

In general, $N(\Gamma)$ is only conjectured to be a classifying space of $A(\Gamma)$.

Conjecture 2.12. *Let Γ be an arbitrary Coxeter graph, then the orbit space $N(\Gamma)$ is a $K(\pi, 1)$ space, hence is a classifying space of the Artin group $A(\Gamma)$.*

However, this conjecture is proved to hold for a few classes of Artin groups besides the finite types. Here is a list of such classes known so far.

- Artin groups of large type ([Hen85]).
- 2-dimensional Artin groups ([CD95]).
- Artin groups of FC type ([CD95]).
- Artin groups of affine types $\widetilde{A_n}, \widetilde{C_n}$ ([Oko79]).
- Artin groups of affine type $\widetilde{B_n}$. [CMS10]

3. Existing results about (co)homology of Artin groups

3.1. **Salvetti complex.** We briefly recall the central construction, the so-called Salvetti complex $Sal(\Gamma)$ associated to a Coxeter graph Γ. This complex will have the homotopy type of the complement $M(\Gamma)$. The Salvetti complex was first introduce by M. Salvetti in [Sal87] for real hyperplane arrangements. However, we shall follow the construction used in [Par12].

Let Γ be a Coxeter graph and (W, S) the associated Coxeter system. Recall that $\mathcal{S}^f = \{T \subset S \mid W_T \text{ is finite}\}$ is the set of spherical subsets. Define a partial order $\preceq$ on the set $W \times \mathcal{S}^f$ by declaring $(w, T) \preceq (w', T')$ if

$$T \subset T', (w')^{-1}w \in W_{T'}, (w')^{-1}w \text{ is } (\varnothing, T)\text{-minimal}.$$

where an element $u \in W$ is called $(\varnothing, T)$-minimal if u is of minimal length in the coset uW_T (such element of minimal length is unique, see [Bou68] or [Hum90]). This $\preceq$ is indeed a partial order (Lemma 3.2 of [Par12]).

Definition 3.1. For a Coxeter graph Γ, the associated Salvetti complex $Sal(\Gamma)$ is defined as the geometric realization of the derived complex of the poset $(W \times \mathcal{S}^f, \preceq)$.

We shall not distinguish a complex with its geometric realization. Note that W acts on $Sal(\Gamma)$ by $u(w, T) := (uw, T)$ for $u, w \in W, T \in \mathcal{S}^f$. We denote the orbit space by $Z(\Gamma) := Sal(\Gamma)/W$.

Theorem 3.2. *There is a homotopy equivalence $f : Sal(\Gamma) \to M(\Gamma)$, which is W-equivariant. Thus f induces a homotopy equivalence $\bar{f} : Z(\Gamma) \to N(\Gamma)$.*

3.2. **Salvetti's algebraic complex for Artin groups.** In this subsection, we recall an algebraic complex introduced by Salvetti that is useful in computation. The contents here could be found in [MSV12], see also [Sal94, DCS96].

Consider a Coxeter system (W, S) with rank $\#S = n$ and its Coxeter graph Γ. The cellular structure of $Z(\Gamma) = Sal(\Gamma)/W$ can be described combinatorially (cf. [Sal94]). Each k-cell of $Z(\Gamma)$ is dual to a unique k-codimensional facet of the fundamental chamber of the arrangement $\mathcal{A}$ and such a facet corresponds to a unique intersection of k hyperplanes of the fundamental chamber. Hence each k-cell of $Z(\Gamma)$ is indexed by a unique spherical subset of S of cardinality k. A total ordering of S then determines an orientation of each cell. The cellular complex of $Z(\Gamma)$ can be identified with an algebraic complex $(\bar{C}_*, \bar{\partial}_*)$ obtained from the Salvetti's algebraic complex which we now describe.

Salvetti introduced an algebraic complex (C_*, ∂_*) for $A(\Gamma)$ which computes the (co)homology groups of $Z(\Gamma)$. Consider a representation of $A(\Gamma)$

$$\lambda : A(\Gamma) \to \mathrm{Aut}(M)$$

where M is a $\mathbb{Z}$-module. Let $\mathcal{L}_\lambda$ be the local system on $Z(\Gamma)$ determined by λ. Define the complex C_* as follows

$$C_k = \bigoplus_{\substack{J \in \mathcal{S}^f \\ \#J = k}} M.e_J,$$

and the boundary map could be written as

$$\partial(a.e_J) = \sum_{s \in J} \left((-1)^{\#\{t \in J | t \leq s\}} \sum_{\beta \in W_J^{J-\{s\}}} (-1)^{\ell(\beta)} \lambda(\psi(\beta))(a) \right) e_{J-\{s\}},$$

where $W_J^{J-\{s\}} = \{\beta \in W_J \mid \ell(\beta t) > \ell(\beta), \forall t \in J - \{s\}\}$ is the set of *minimal coset representatives* of $W_J/W_{J-\{s\}}$, i.e. the collection of the unique element of minimal length in each coset of $W_J/W_{J-\{s\}}$ (see Subsection 2.1.1) and $\psi : W \to A(\Gamma)$ is the canonical section (Subsection 2.2).

Theorem 3.3 ([Sal94]). *In the above situation,*

$$H_*(C_*) \cong H_*(Z(\Gamma); \mathcal{L}_\lambda).$$

Remark 3.4. The complex (C^*, δ^*) for cohomology is similar. Precisely, $C^* = C_*$ and

$$\delta(a.e_J) = \sum_{\substack{s \in S \setminus J \\ |W_{J \cup \{s\}}| < \infty}} \left((-1)^{\#\{t \in J | t \leq s\}} \sum_{\beta \in W_J^{J-\{s\}}} (-1)^{\ell(\beta)} \lambda(\psi(\beta))(a) \right) e_{J \cup \{s\}}.$$

Also, $H^*(C^*) \cong H^*(Z(\Gamma); \mathcal{L}_\lambda)$.

The complex will become much simpler if we restrict ourselves to the case $M = \mathbb{Z}$ and λ is trivial. We shall denote this specific complex by $(\bar{C}_*, \bar{\partial}_*)$

$$\bar{C}_k = \bigoplus_{\substack{J \in \mathcal{S}^f \\ \#J = k}} \mathbb{Z}e(J),$$

where we write instead $e(J)$ for the generator. Let $X(q)$ be the Poincaré series of a subset $X \subset W$ (Subsection 2.1.1), then the boundary can be written as

$$\bar{\partial} e(J) = \sum_{s \in J} \left((-1)^{\#\{t \in J | t \leq s\}} W_J^{J-\{s\}}(-1) \right) e(J - \{s\}), \tag{3.1}$$

where $W_J^{J-\{s\}}(-1)$ is the number obtained by substituting -1 for q in the Poincaré series (actually a polynomial here) (cf. Section 1.11 of [Hum90])

$$W_J^{J-\{s\}}(q) = \frac{W_J(q)}{W_{J-\{s\}}(q)}.$$

3.3. **Summary of existing results.** The Salvetti's algebraic complex provides a relatively simple model for computations. We describe very briefly the standard method to compute cohomology of $Z(\Gamma)$ (see [Cal14]). Let Γ be a Coxeter graph and (W, S) the associated Coxeter system with rank n. Fix a total order on $S = \{s_1, s_2, \ldots, s_n\}$ by setting $s_1 < s_2 < \cdots < s_n$. Let M be a $\mathbb{Z}$-module and $\lambda : A(\Gamma) \to \mathrm{Aut}(M)$ a representation. (C^*, δ^*) is the Salvetti's algebraic complex defined in the previous subsection. Then there is a natural decreasing filtration on C^*, define

$$F^k C^* := \langle e_J \mid s_{n-k+1}, \ldots, s_n \in J \rangle$$

as the subcomplex of C^* generated by e_J where J contains the largest k standard Coxeter generators. This leads to a sequence of inclusions

$$0 = F^{n+1}C^* \subset F^n C^* \subset \cdots F^1 C^* \subset F^0 C^* = C^*.$$

There is then a spectral sequence $\{E_r, d_r\}$ with $E_0^{i,j} = F^i C^j / F^{i+1} C^j$ abuts to $H^{i+j}(C^*)$.

Many computations for specific types Arin groups exist in the literature. Beside those listed in the introduction, see [DCPSS99, DCPS01] for the cohomology of Artin groups of finite type with coefficient the Laurent polynomial ring $\mathbb{Q}[q^{\pm}]$ on which each standard generator acts as multiplication by $-q$, [CMS08] for type B_n with coefficient the 2-parameter Laurent polynomial ring $\mathbb{Q}[q^{\pm}, t^{\pm}]$ on which the first $n-1$ standard generators act as multiplication by $-q$ and the last acts as multiplication by $-t$, as well as type $\widetilde{A_n}$ with trivial $\mathbb{Q}$ coefficient, [CMS10] for type $\widetilde{B_n}$ with coefficient $\mathbb{Q}[q^{\pm}, t^{\pm}]$ on which the first n standard generators act as multiplication by $-q$ and the last acts as multiplication by $-t$. See also [SV13] for twisted cohomology of Artin groups of exceptional affine types using discrete Morse theory.

Next we state Clancy-Ellis' theorem. Let us first fix some notations associated to a Coxeter graph. Let Γ be a Coxeter graph with vertex set S. Denote by $P(\Gamma)$ the set of pairs of non-adjacent vertices of Γ, namely $P(\Gamma) = \{\{s,t\} \subset S \mid m(s,t) = 2\}$. Write $\{s,t\} \equiv \{s', t'\}$ if two such pairs in $P(\Gamma)$ satisfy $s = s'$ and $m(t,t')$ is odd. This generates an equivalence relation on $P(\Gamma)$, denoted by $\sim$. Let $P(\Gamma)/\sim$ be the set of equivalence classes. An equivalence class is called a *torsion* class if it is represented by a pair $\{s,t\} \in P(\Gamma)$ such that there exists a vertex $v \in S$ with $m(s,v) = m(t,v) = 3$. In the above situation, we have the following theorem.

Theorem 3.5 ([CE10])**.** *The second integral homology of the quotient Salvetti complex $Z(\Gamma) = Sal(\Gamma)/W(\Gamma)$ associated to a Coxeter graph Γ is*

$$H_2(Z(\Gamma); \mathbb{Z}) = \mathbb{Z}_2^{p(\Gamma)} \oplus \mathbb{Z}^{q(\Gamma)},$$

where

$$\begin{aligned} p(\Gamma) &:= \textit{ number of torsion classes in } P(\Gamma)/\sim, \\ q_1(\Gamma) &:= \textit{ number of non-torsion classes in } P(\Gamma)/\sim, \\ q_2(\Gamma) &:= \#(Q(\Gamma) - P(\Gamma)) = \#\{\{s,t\} \subset S \mid m(s,t) \geq 4 \textit{ is even}\}, \\ q_3(\Gamma) &:= \mathrm{rank} H_1(\Gamma_{odd}; \mathbb{Z}), \\ q(\Gamma) &:= q_1(\Gamma) + q_2(\Gamma) + q_3(\Gamma). \end{aligned}$$

Remark 3.6. Whenever $K(\pi, 1)$ conjecture holds for the Artin group $A(\Gamma)$, Theorem 3.5 gives the second integral homology of $A(\Gamma)$.

4. Second mod 2 homology of Artin groups

As we see in the previous section, almost all existing results about (co)homology of Artin groups are based on the truth of the $K(\pi, 1)$ conjecture. In this section, nevertheless, we present our main results which hold for arbitrary Artin groups, regardless of the $K(\pi, 1)$ conjecture.

Theorem 4.1 ([AL])**.** *Let Γ be an arbitrary Coxeter graph and $A(\Gamma)$ the associated Artin group. Then the second mod 2 homology of $A(\Gamma)$ is*

$$H_2(A(\Gamma); \mathbb{Z}_2) \cong \mathbb{Z}_2^{p(\Gamma) + q(\Gamma)}.$$

The spirit of the proof is a combination of the classical Hopf formula for the second homology of a group with a finite presentation and the fact $H_2(W(\Gamma); \mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)+q(\Gamma)}$ by Howlett ([How88]).

The proof of Theorem 4.1 also implies the following.

Corollary 4.2. *The projection $p : A(\Gamma) \to W(\Gamma)$ induces an epimorphism $p_* : H_2(A(\Gamma); \mathbb{Z}) \to H_2(W(\Gamma); \mathbb{Z})$.*

Corollary 4.3. *Let $M(\Gamma)$ be the complement of the complexified arrangement of reflection hyperplanes associated to the Coxeter group $W(\Gamma)$ and $N(\Gamma) = M(\Gamma)/W(\Gamma)$. If Γ satisfies the following conditions*

- *$P(\Gamma)/\sim$ consists of torsion classes.*
- *$\Gamma = \Gamma_{odd}$.*
- *Γ is a tree.*

Then $H_2(A(\Gamma);\mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)}$. Hence the Hurewicz homomorphism $h_2 : \pi_2(N(\Gamma)) \to H_2(N(\Gamma);\mathbb{Z})$ is trivial. Furthermore, for any Coxeter graph, the Hurecicz homomorphism becomes trivial after taking tensor product with $\mathbb{Z}_2$.

Proof. Since $N(\Gamma)$ is path-connected and has fundamental group $\pi_1(N(\Gamma)) = A(\Gamma)$, there is an exact sequence

$$\pi_2(N(\Gamma)) \xrightarrow{h_2} H_2(N(\Gamma);\mathbb{Z}) \xrightarrow{f} H_2(A(\Gamma);\mathbb{Z}) \to 0. \tag{4.1}$$

Suppose that Γ satisfies the three conditions, then $q_1(\Gamma) = q_2(\Gamma) = q_3(\Gamma) = 0$. Theorem 3.5 implies that $H_2(N(\Gamma);\mathbb{Z}) = \mathbb{Z}_2^{p(\Gamma)}$. Then by Corollary 4.2, $H_2(A(\Gamma);\mathbb{Z})$ sits in the following sequence

$$\mathbb{Z}_2^{p(\Gamma)} \twoheadrightarrow H_2(A(\Gamma);\mathbb{Z}) \twoheadrightarrow \mathbb{Z}_2^{p(\Gamma)},$$

the composition must be an isomorphism, hence $H_2(A(\Gamma);\mathbb{Z}) \cong \mathbb{Z}_2^{p(\Gamma)}$. As a result, f must be an isomorphism and h_2 must be trivial.

Now suppose Γ is arbitrary. By right-exactness of tensor functor, taking tensor product with $\mathbb{Z}_2$ preserves the exactness of 4.1,

$$\pi_2(N(\Gamma)) \otimes \mathbb{Z}_2 \xrightarrow{h_2 \otimes \mathrm{id}_{\mathbb{Z}_2}} H_2(N(\Gamma);\mathbb{Z}) \otimes \mathbb{Z}_2 \xrightarrow{f \otimes \mathrm{id}_{\mathbb{Z}_2}} H_2(A(\Gamma);\mathbb{Z}) \otimes \mathbb{Z}_2 \to 0.$$

Note that $f \otimes \mathrm{id}_{\mathbb{Z}_2}$ is an isomorphism as a consequence of Theorem 4.1 and Clancy-Ellis' Theorem 3.5. Hence $h_2 \otimes \mathrm{id}_{\mathbb{Z}_2}$ must be trivial. □

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