Ruelle-Perron-Frovenius の定理による Gibbs 測度の存在証明

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1 Introduction

We study the Ruelle Perron Frobenius theorem, which are defined for continuous functions on the space, $\sum_n^+ = \{1, \ldots, n\}^{\mathbb{N}}$, of sequences $\underline{x} = \{x_i\}_{i=0}^{\infty}$ of 1 to n. The operators act on the Banach space $C(\sum_n^+)$ of continuous functions on $\sum_n^+$ with the sup $\|\phi\| = \sup\{|\phi(x)| : \underline{x} \in \sum_n^+\}$. The set $M(\sum_n^+, \sigma)$ denotes the set of all σ-invariant Borel probability measures. For Hölder continuous function ϕ, there is a unique measure called equilibrium state for ϕ , which attains the supremum in the Variational Principle

$$P(\phi) = \sup\left\{ h_\mu + \int \phi d\mu : \mu \in M(\textstyle\sum_n^+, \sigma) \right\}$$

where $P(\phi)$ is the pressure of ϕ and h_μ is the entropy of the μ.

The Ruelle Perron Frobenius theoem says the Ruell Perron Frobenius operators for Hölder continuous functions have the eigenfunction h and the eigenmeasure ν with the same positive eigenvalue λ. This is an extension of the Perron Frobenius theorem to the infinite dimensional space $C(\sum_n^+)$. The theorem is important since the unique equilibrium state for a Hölder continuous function is given by the eigenfunction and the eigenmeasure of the Ruelle Perron Frobenius operator for ϕ.

2 Definition of Symbolic Dynamical System

Endow $\sum_n^+ = \{1, \ldots, n\}$ with the product topology of the discrete topology of $\{1, \ldots, n\}$. We can define a metric which coincides with the topology. Take $\beta \in (0, 1)$. Define a metric d_β on $\sum_n^+$ by

$$d_\beta(\underline{x}, \underline{y}) = \begin{cases} 0 & \text{if} \quad \underline{x} = \underline{y}, \\ \beta^N & \text{otherwise,} \end{cases}$$

where $N = \min\{i \geq 0 : x_i \neq y_i\}$. Observe that $\sum_n^+$ is a compact metric space. Let $\mathcal{B}(\sum_n^+)$ denote the set of all Borel sets of $\sum_n^+$ and the set

$$[x_0, \ldots, x_{m-1}] = \{\underline{y} \in \textstyle\sum_n^+ : x_i = y_i \text{ for all } i = 0, \ldots, m-1\}$$

is called a cylinder set starting 0 with length m. Note that the set of all cylinder sets is a basis of the topology of $\sum_n^+$.

Define a continuous map $\sigma : \sum_n^+ \to \sum_n^+$ by $\sigma(\{x_i\}_{i=0}^{\infty}) = \sigma(\{x_{i+1}\}_{i=0}^{\infty})$, which is called the shift map.

$\mu \in M(\sum_n^+)$ is σ-invariant if

$$\mu(\sigma^{-1}B) = \mu(B)$$

for all $B \in \mathcal{B}(X)$. The set $M(\sum_n^+, \sigma)$ denotes the set of all σ-invariant measures on $\sum_n^+$. For $\phi \in C(\sum_n^+)$ and $k \in \mathbb{N}$ define

$$\mathrm{var}_k\phi = \sup\{\|\phi(\underline{x}) - \phi(\underline{y})\| : x_i = y_i \text{ for } 0 \leq^{\forall} i \leq k\}$$

and define

$$\mathcal{F} = \{\phi \in C(\textstyle\sum_n^+) : \ {}^{\exists}b > 0,\ {}^{\exists}\alpha \in (0,1),\ \mathrm{var}_k\phi \leq b\alpha^k \text{ for } {}^{\forall}k \geq 0\}.$$

Then $\phi \in \mathcal{F}$ is Hölder continuous with regard to d_β, i.e., there is $C > 0$ and $\gamma \in (0, 1]$ such that

$$\|\phi(\underline{x}) - \phi(\underline{y})\| \leq C d_\beta(\underline{x}, \underline{y})^\gamma$$

for all $\underline{x}, \underline{y} \in \sum_n^+$. The set $\mathcal{F}$ is the set of all Hölder continuous functions.

We can introduce weak*-topology on the dual space $C(\sum_n^+)^*$ of the Banach space $C(\sum_n^+)$ to $M(\sum_n^+)$ by the Riesz Representaion Theorem. For $\mu \in M(\sum_n^+)$, define $F_\mu \in C(\sum_n^+)^*$ by

$$F_\mu(\phi) = \int \phi d\mu$$

for every $\phi \in C(\sum_n^+)$. Then, by the Riesz Representation Theorem, this correspondence becomes a bijection between $M(\sum_n^+)$ and $\{F \in C(\sum_n^+)^* : F$ is linear, normalized, continuous and positive$\}$, a subset of $C(\sum_n^+)^*$. We induce the weak*-topology on $C(\sum_n^+)^*$ to $M(\sum_n^+)$ by this correspondence. Note that $M(\sum_n^+)$ and $M(\sum_n^+, \sigma)$ are non-empty and compact in weak*-topology.

3 The existence and the uniqueness of equilibrium states

For any Hölder continuous function there is a unique Gibbs measure, which gives quantitative information on the measure of cylinder sets. The Gibbs measures for Hölder continuous functions are also the unique equilibrium state.

Definition 3.1 *Gibbs measures[2]*
Let ϕ be a continuous function on $\sum_n^+$. A measure $\mu \in M(\sum_n^+)$ is called a Gibbs measure for ϕ if there are $c_1,\ c_2 > 0$ and $P \in \mathbb{R}$ such that

$$c_1 \leq \frac{\mu([x_0, \ldots, x_{m-1}])}{\exp(-mP + \sum_{k=0}^{m-1} \phi(\sigma^k \underline{x}))} \leq c_2$$

for all $\underline{x} \in \sum_n^+$ and $m \geq 1$.

if μ is a Gibbs measure for a Hölder continuous function, the value P in the above definition is actually the pressure for ϕ. We define the Ruelle Perron Frobenius operators whose eigenfunctions and eigenmeasures are used to construct Gibbs measures.

Definition 3.2 *Ruelle Perron Frobenius operator[2]*
For $\phi \in C(\sum_n^+)$ define an operator $\mathcal{L} = \mathcal{L}_\phi : C(\sum_n^+) \to C(\sum_n^+)$ by

$$\mathcal{L}f(\underline{x}) = \sum_{\underline{y} \in \sigma^{-1}\underline{x}} e^{\phi(\underline{y})} f(\underline{y})$$

for every $f \in C(\sum_n^+)$ where $\underline{x} \in \sum_n^+$.

The dual operator $\mathcal{L}^* : M(\sum_n^+) \to M(\sum_n^+)$ is

$$\mathcal{L}^*\nu(f) = \int \mathcal{L} f d\nu$$

for every $\nu \in M(\sum_n^+)$ where $f \in C(\sum_n^+)$.

We have the following theorem.

Theorem 1 *Ruelle Perron Frobenius Theorem [2, Theorem 1.7.]*
Suppose $\phi \in \mathcal{F}$. Then there are $\lambda > 0$, $h \in C(\sum_n^+)$ with $h > 0$ and $\nu \in M(\sum_n^+)$ such that

$$\mathcal{L}h = \lambda h, \ \ \mathcal{L}^*\mu = \lambda\mu, \ \ \nu(h) = 1, \tag{3.3}$$

and

$$\|\lambda^{-m}\mathcal{L}^m g - \nu(g)h\| \to 0 \quad \text{as} \quad n \to \infty \tag{3.4}$$

for every $g \in C(\sum_n^+)$.

This theorem can be considered as the extension of the Perron Frobenius Theorem. The theorem says that for a $d \times d$ matrix $B = (b_{ij})$ with positive entries there are right and left eigenvectors with the same positive eigenvalue λ such that thier elements are all positive. Define a linear functional $T : C(\sum_n^+) \to \mathbb{R}$ by $T(g) = \nu(g)$. Note that the space $C(\sum_n^+)$ can be decomposed into the eigenspace $E(\lambda)$ of λ and $\mathrm{Ker}(T)$. (3.4) implies λ is the spectral radius of $\mathcal{L}$.

We can get a Gibbs measure, which is actually an equilibrium state, for a Hölder continuous function. Let $\phi \in \mathcal{F}$. We get $\lambda > 0$, $h \in C(\sum_n^+)$ and $\nu \in M(\sum_n^+)$ as in the above theorem. Define $\mu \in M(\sum_n^+)$ by

$$\mu(f) = \nu(hf) = \int f(\underline{x})h(\underline{x})d\nu \tag{3.5}$$

for all $f \in C(\sum_n^+)$. The measure μ becomes σ-invariant and the unique Gibbs measure for ϕ. The measure μ is also the unique equilibrium state for ϕ.

Theorem 2 *The existence and uniqueness of Gibbs measures [2, Theorem 1.2. and Theorem 1.22]*
Suppose $\phi \in \mathcal{F}$. Then there is a unique Gibbs measure μ with the constant

$$P = P(\phi).$$

Theorem 3 *The existence and uniqueness of equilibrium states [2, Theorem 1.22.]*
Suppose $\phi \in \mathcal{F}$. Then the Gibbs measure μ_ϕ for ϕ is the unique equilibrium state for ϕ, i.e.,

$$h_{\mu_\phi} + \int \phi \; d\mu_\phi = P(\phi).$$

4 Phase transition

In the previous section we saw the case where uniqueness of equilibrium states holds. On the other hand, it is not clear whether uniqueness holds for continuous functions which is not Hölder continuous. Indeed there are examples of functions which have more than two equilibrium states.[3] The coexistence of multiple equilibrium states is called a phase transition with the analogy of the statistical mechanics.

It is interesting to find functions which have more than two equilibrium states and the condition on functions that phase transition happens. It is also interesting to search how common phase transition happens, which means whether functions with more than two equilibrium states are dense in the set of all continuous functions.

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