Remarks on Kato's inequality when $\Delta_p u$ is a measure

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1 Introduction

Let Ω be a bounded domain of $\mathbf{R^N}$ $(N \geq 1)$. In this article, we shall study Kato's inequality when $\Delta_p u$ is a measure. By $\Delta_p u$ we denote a a p-Laplace operator:

$$\Delta_p u = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u), \tag{1.1}$$

where $1 < p < \infty$ and $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_N)$.

The classical Kato's inequality for a Laplacian asserts that given any function $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$, then Δu^+ is a Radon measure and the following holds:

$$\Delta u^+ \geq \chi_{[u \geq 0]} \Delta u \qquad \text{in } D'(\Omega), \tag{1.2}$$

where $u^+ = \max[u, 0]$. A similar inequality holds when Δu is replaced by $\Delta_p u$ under additional assumptions on distributional derivatives of $u \in L^1_{\mathrm{loc}}(\Omega)$ (see e.g. [7, 8, 10]).

Our main result (see Theorem 2.1 and Corollary 2.1 below) further extends Kato's inequality involving Δ_p to the case where $\Delta_p u \in M(\Omega)$, where $M(\Omega)$ denotes the space of Radon measures on Ω. In other words, $\mu \in M(\Omega)$ if and only if , for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ such that $|\int_\Omega \varphi\, d\mu| \leq C_\omega ||\varphi||_{L^\infty}$, for any $\varphi \in C_c^\infty(\omega)$.

We begin with recalling that for any $\mu \in M(\Omega)$ can be uniquely decomposed as a sum of two Radon measures on Ω (see e.g. [4, 6]) : $\mu = \mu_d + \mu_c$, where

$$\begin{cases} \mu_d(A) = 0 \quad \text{for any Borel measurable set } A \subset \Omega \text{ such that } C_p(A, \Omega) = 0, \\ \\ |\mu_c|(\Omega \setminus F) = 0 \quad \text{for some Borel measurable set } F \subset \Omega \text{ such that } C_p(F, \Omega) = 0. \end{cases} \tag{1.3}$$

Here by $C_p(K, \Omega)$ we denote a p-capacity of a Borel set K relative to Ω (for the precise definition see Definition 2.3 in §2). We note that $(\mu_d)^+ = (\mu^+)_d$ and $(\mu_c)^+ = (\mu^+)_c$ hold by the definition.

Then we recall an admissible class of functions for the strong maximum principle in [10]:

Definition 1.1. (**Admissible class in** $W^{1,p^*}_{\mathrm{loc}}(\Omega)$) *Let* $1 < p < \infty$ *and* $p^* = \max(1, p-1)$. *A function* $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ *is said to be admissible if and only if* $\Delta_p u \in M(\Omega)$ *and there exists a sequence* $\{u_n\}_{n=1}^\infty \subset W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ *such that:*

1. $u_n \to u$ *a.e. in* Ω, $u_n \to u$ *in* $W^{1,p^*}_{\mathrm{loc}}(\Omega)$ *as* $n \to \infty$.
2. $\Delta_p u_n \in L^1_{\mathrm{loc}}(\Omega)$ $(n = 1, 2, \cdots)$ *and*

$$\sup_n |\Delta_p u_n|(\omega) < \infty \qquad \textit{for every } \omega \subset\subset \Omega. \tag{1.4}$$

Proposition 1.1. *1. Assume that a function $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ is admissible. Then $u^+ = \max[u, 0]$ and $u^- = \max[-u, 0]$ are also admissible.*

2. Assume that $p = 2$. Then a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is admissible if $\Delta u \in M(\Omega)$.

3. A function $u \in W^{1,p}_0(\Omega)$ is admissible if $\Delta_p u \in M(\Omega)$.

2 Main results

Theorem 2.1. *Let $N \geq 1$, $1 < p < \infty$ and Ω be a bounded domain of $\mathbf{R}^N$. Let Φ be a C^1 convex function such that $0 \leq \Phi' < \infty$ on $\mathbf{R}$. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ if $p = 2$ and let $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ if $p \neq 2$. Assume that $\Delta_p u \in M(\Omega)$. Moreover if $p \neq 2$, assume that u is admissible in the sense of Definition 1.1. Then we have*

$$\Delta_p \Phi(u) \geq \Phi'(u)^{p-1}(\Delta_p u)_d - ||\Phi'||_{L^\infty(\mathbf{R})}(\Delta_p u)_c^- \qquad \text{in } D'(\Omega). \tag{2.1}$$

From this theorem it follows that we have:

Corollary 2.1. *Assume the same assumptions in Theorem 2.1. Then it holds that*

$$\Delta_p(u^+) \geq \chi_{[u \geq 0]}(\Delta_p u)_d - (\Delta_p u)_c^- \qquad \text{in } D'(\Omega), \tag{2.2}$$

$$\Delta_p|u| \geq \mathrm{sgn}(u)(\Delta_p u)_d - |\Delta_p u|_c \qquad \text{in } D'(\Omega), \tag{2.3}$$

where $\mathrm{sgn}(t) = 1$ for $t > 0$, $\mathrm{sgn}(t) = -1$ for $t < 0$, and $\mathrm{sgn}(0) = 0$.

Theorem 2.2. (**Inverse maximum principle**) *Let $N \geq 1$, $1 < p < \infty$ and let Ω be a bounded domain of $\mathbf{R}^N$. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ if $p = 2$ and let $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ if $p \neq 2$. Assume that $u \geq 0$ a.e. in Ω and $\Delta_p u \in M(\Omega)$. Moreover if $p \neq 2$, assume that u is admissible in the sense of Definition 1.1. Then we have*

$$(-\Delta_p u)_c \geq 0 \qquad \text{in } \Omega. \tag{2.4}$$

Example 1. *Let $u = |x|^\alpha$ for $\alpha = (p - N)/(p - 1)$.*

1. *u satisfies*
$$\Delta_p u = \alpha|\alpha|^{p-2} c_N \delta,$$
where δ denotes a Dirac mass and c_N denotes the surface area of the N-dimensional unit ball B_1. It is easy to see that $|\nabla u| \in L^1_{loc}(\Omega)$ if and only if $p > 2 - 1/N$. When $1 < p \leq 2 - 1/N$ holds, we can consider u as a renormalized solution. We recall that if $p \leq 2 - 1/N$, then we cannot expect the solution of an equation of the form $\Delta_p u = f$ (a Radon measure) to be in $W^{1,1}_{loc}(\Omega)$. For the detail, see e.g. [1, 2, 5, 11, 12].

2. *If $2 - 1/N \leq p \leq N$, then $(-\Delta_p(u^+))_c = (-\Delta_p u)_c^+ = -\alpha|\alpha|^{p-2} c_N \delta \geq 0$. If $p > N$, then $(-\Delta_p(u^+))_c = (-\Delta_p u)_c^+ = 0$ and $\Delta_p(u^+) \geq \chi_{[u \geq 0]}(\Delta_p u)_d = \alpha|\alpha|^{p-2} c_N \delta \geq 0$.*

3. *When $p > 2 - 1/N$, u is admissible in $W^{1,p^*}(B_1)$. In fact, $u = |x|^\alpha$ is approximated by a sequence of admissible functions $v_{\alpha(n)} = |x|^{\alpha(n)} \in L^1(B_1)$ where $\alpha(n) = \alpha + 1/(n(p - 1))$ $(n = 1, 2, \cdots)$. Then, in the sense of measures we have*
$$\Delta_p v_{\alpha(n)} = \frac{1}{n}|\alpha(n)|^{p-2}\alpha(n)|x|^{1/n-N} \to \Delta_p u \qquad \text{as } n \to \infty.$$
Therefore there exits a sequence $\{n_{\alpha(n)}\}$ such that $\{n_{\alpha(n)}\} \to \infty$ as $n \to \infty$ and a sequence of mollification $\{(v_{\alpha(n)})_\rho^{n_{\alpha(n)}}\}$ satisfies the conditions in Definition 1.1

3 Lemmas

Let us describe lemmas which are useful for the proof of the main results. Given $k > 0$, we denote by T_k:$\mathbf{R} \to \mathbf{R}$ a truncation function

$$T_k(s) := \begin{cases} k & if \ s \geq k, \\ s & if \ -k < s < k, \\ -k & if \ \ s \leq -k. \end{cases} \tag{3.1}$$

Since $T_k|_{\mathbf{R}_+}$ is concave, we have the following lemma in the spirit of the standard L^1-version of Kato's inequality (see [9]).

Lemma 1. *Assume that* $v \in W^{1,p}_{\rm loc}(\Omega)$, $\Delta_p v \in L^1_{\rm loc}(\Omega)$ *and* $v \geq 0$ *a.e. in* Ω. *Then, for any* $k \geq 0$ *we have*

$$\Delta_p(T_k(v)) \leq t_k(v)\Delta_p v \quad \ in \ \ D'(\Omega), \tag{3.2}$$

where the function $t_k : \mathbf{R}_+ \to \mathbf{R}$ *is given by*

$$t_k(s) := \begin{cases} 1 & if \ 0 \leq s \leq k, \\ 0 & if \ \ s > k. \end{cases}$$

Lemma 2. *Let* $N \geq 1$, $1 < p < \infty$, $p \neq 2$ *and* Ω *be a bounded domain of* $\mathbf{R}^N$. *Let* Φ *be a* C^1 *convex function such that* $0 \leq \Phi' < \infty$ *on* $\mathbf{R}$. *Let* $u \in W^{1,p^*}_{\rm loc}(\Omega)$. *Assume that* u *is admissible in the sense of Definition 1.1. Then we have the followings:*

1. $T_k(u) \in W^{1,p}_{loc}(\Omega)$ *for every* $k > 0$. *Moreover, given* $\omega \subset\subset \omega' \subset\subset \Omega$, *there exist positive constant* C *such that*

$$\int_\omega |\nabla T_k(u)|^p \leq Ck \left(\int_{\omega'} |\Delta_p u| + \int_{\omega'} |\nabla u|^{p-1} \right), \tag{3.3}$$

where positive constant C *are independent on each* u.

2. *If* $u \geq 0$ *a.e. in* Ω, *then* $\Delta_p(T_k(u))$ *is a Radon measure for every* $k > 0$ *and we have*

$$\Delta_p(T_k(u)) \leq (\Delta_p u)^+ \qquad in \ \Omega. \tag{3.4}$$

The next lemma is seen in [3]; Theorem 2.1.

Lemma 3. *Let* $1 < p < \infty$. *Let* $\nu \in M(\Omega)$. *Then* $\nu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ *if and only if* ν *is a diffuse measure. Here* $p' = p/(p-1)$ *and* $W^{-1,p'}(\Omega)$ *is the dual space of* $W^{1,p}_0(\Omega)$.

When $p = 2$, the next lemma is seen in [4]; Lemma 2.1.

Lemma 4. *Assume that* $\nu \in M(\Omega)$ *is a diffuse measure with respect to p-capacity (i.e.* $\nu_c = 0$*). Let* $\{v_n\}$ *be a sequence in* $L^\infty(\Omega) \cap W^{1,p}_0(\Omega)$ *such that* $||v_n||_\infty \leq C$ *and* $v_n \to v$ *weakly in* $W^{1,p}_0(\Omega)$. *Then*

$$v_n \to v \ \ in \ \ L^1_{loc}(\Omega; d\nu).$$

Equivalently, there exists a subsequence $\{v_{n_k}\}$ *converging to* v $|\nu|$*-a.e. in* Ω. *Here* $L^1_{loc}(\Omega; d\nu) = \{f : \int_\omega |f| \, d\nu < \infty, \forall \omega \subset\subset \Omega\}$

Without loss of generality in the proof of Theorem 2.1, we may assume that $\Phi \in C^2(\mathbf{R})$, $0 \leq \Phi' \leq 1$ and Φ'' has compact support in $\mathbf{R}$. Since Φ is convex and Φ' is uniformly bounded, we see that both limit $\lim_{t \to \pm\infty} \Phi'(t)$ exist. Then we prepare the following lemma.

Lemma 5. *Assume* $u \in W^{1,p^*}_{\rm loc}(\Omega)$ *is admissible. Let* $\varphi \in C^\infty_0(\Omega)$. *Let* Φ *be a* C^2 *convex function such that* $\operatorname{supp} \Phi''$ *has compact support and*

$$\sup_{t \in \mathbf{R}} (\Phi'(t))^{p-2} \Phi''(t) < \infty. \tag{3.5}$$

Then $\Phi'(u_n)^{p-1}\varphi \to \Phi'(u)^{p-1}\varphi$ *weakly in* $W^{1,p}_0(\Omega)$ *as* $n \to \infty$.

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