On Transcendence Theory with little history, new results and open problems

Hyun Seok, Lee

December 27, 2015

Abstract

This talk will be dealt to a survey of transcendence theory, including little history, the state of the art and some of the main conjectures, the limits of the current methods and the obstacles which are preventing from going further.

1 Introudction

Definition 1.1. *An algebraic number is a complex number which is a root of a polynomial with rational coefficients.*

- For example $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ration $(1+\sqrt{5})/2$, the roots of unity $e^{2\pi i a/b}$, the roots of the polynomial $x^5 - 6x + 3 = 0$ are algebraic numbers.

Definition 1.2. *A transcendental number is a complex number which is not algebraic.*

Definition 1.3. *Complex numbers $\alpha_1, \cdots, \alpha_n$ are **algebraically dependent** if there exists a non-zero polynomial $P(x_1, \cdots, x_n)$ in n variables with rational coefficients such that $P(\alpha_1, \cdots, \alpha_n) = 0$. Otherwise, they are called **algebraically independent**.*

The existence of transcendental numbers was proved in 1844 by J. Liouville who gave explicit ad-hoc examples. The transcendence of constants from analysis is harder; the first result was achieved in 1873 by Ch. Hermite who proved the transcendence of e. In 1882, the proof by F. Lindemann of the transcendence of π gave the final (and negative) answer to the Greek problem of squaring the circle. The transcendence of $2^{\sqrt{2}}$ and e^{π}, which was included in Hilbert's seventh problem in 1900, was proved by Gel'fond and Schneider in 1934. During the last century, this theory has been extensively developed, and these developments gave rise to a number of deep applications. In spite of that, most questions are still open. In this article we survey the state of the art on known results and open problems.

2 Irrationality

Definition 2.1. *Given a basis of $b \geq 2$, a real number x is rational if and only if its expansion in basis b is ultimately periodic.*

Let $b \geq 2$ be an integer.

- É. Borel (1909 and 1950) : the b-ary expansion of an algebraic irrational numbers should satisfy some of the laws shared by almost all numbers (with respect to Lebesgue's measure).
- **Remark.** No number satisfies all the laws which are shared by all numbers outside of a set of measure zero, because the intersection of all these sets of full measure is empty!

$$\bigcap_{x \in \mathbb{R}} \mathbb{R} \backslash \{x\} = \emptyset.$$

- More precise statement by B. Adamczewski and Y. Bugeaud.

Conjecture 2.1. *(É. Borel). Let x be an irrational algebraic real number, $b \geq 3$ a positive integer and a an integer in the range $0 \leq a \leq b-1$. Then the digit a occurs at least once in the b-ary expansion of x.*

Corollary 2.1. *Each given sequence of digits should occur infinitely often in the b-ary expansion of any real irrational algebraic number. (consider powers of b).*

- An irrational algebraic number with a regular expansion in some basis b should be transcendental.
- There is no explicitly known example of a triple (b, a, x), where $b \geq 3$ is an integer, a is a digit in $\{0, \cdots, b-1\}$ and x is an algebraic irrational number, for which one claim that the digit a occurs infinitely often in the b-ary expansion of x.

Conjecture 2.2 (The state of the art statement)**.** *A stronger conjecture is due to Borel, is that algebraic irrational real numbers are normal, i.e., each sequence of n digits in basis b should occur with the frequency $1/b^n$, for all b and all n.*

Theorem 2.1 (B. Adamczewski, Y. Bugeaud 2005 ; conjecture of A. Cobham 1968)**.** *If the sequence of digits of a real number x is produced by a finite automaton, then x is either rational or else transcendental.*

3 Irrationality of transcendental numbers

We will first give known and unknown transcendence results

- Known:

$$e, \ \pi, \ \log 2, \ e^{\sqrt{2}}, \ e^{\pi}, \ 2^{\sqrt{2}}, \ \Gamma(1/4)$$

- Not known :

$$e+\pi,\ e\pi, \log\pi, \pi^e,\ \Gamma(1/5),\ \zeta(3),\ \text{Euler-Mascheroni constant}$$

Conjecture 3.1. *Is Catalan's constant*

$$\sum_{n\geq 1}\frac{(-1)^n}{(2n+1)^2} = 0.9159655941772190150\cdots$$

an irrational number?

- Cataln's constant is the value at $s=2$ of the Dirichlet L-function $L(s,\chi_{-4})$ associated with the Kronecker character

$$\chi_{-4}(n) = \left(\frac{n}{4}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

- The Dirichlet L-function $L(s,\chi_{-4})$ associated with the Kronecker character χ_{-4} is the quotient of the Dedekind zeta function of $\mathbb{Q}(i)$ and the Riemann zeta function:

$$\zeta_{\mathbb{Q}(i)}(s) = L(s,\chi_{-4})\zeta(s)$$

Theorem 3.1 (Basel Problem, Euler 1735)**.** *For any even integer value of $s\geq 2$, the number $\zeta(s)$ is a rational multiple of π^s.*

- $\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$, $\zeta(6)=\pi^6/945$, $\zeta(8)=\pi^8/9450\ \cdots$.

Theorem 3.2 (Apéry 1978)**.** *The number*

$$\zeta(3)=\sum_{n\geq 1}\frac{1}{n^3} = 1.2020569031595942853997381615 11\cdots$$

is irrational.

Conjecture 3.2. *Is the number $\zeta(3)/\pi^3$ irrational?*

Conjecture 3.3. *Is the number*

$$\zeta(5)=\sum_{n\geq 1}\frac{1}{n^5} = 1.0369277551433699263313654864 57\cdots$$

irrational.

Theorem 3.3 (Tanguy Rivoal, 2000)**.** *Let $\epsilon>0$. For any sufficiently large odd integer a, the dimension of the $\mathbb{Q}$-vector space spanned by the numbers $1,\zeta(3),\zeta(5),\cdots,\zeta(a)$ is at least*

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

Definition 3.1. ***Euler-Mascheroni constant*** *defined by*

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$

Conjecture 3.4. *Is it a rational number?*

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right) = \int_1^{\infty}\left(\frac{1}{[x]}-\frac{1}{x}\right)dx = -\int_0^1\int_0^1 \frac{(1-x)dxdy}{(1-xy)\log(xy)}.$$

Theorem 3.4 (S.D. Adihikari, N. Saradha, T.N. Shorey and R. Tijdemann 2001)**.** *Let* $f:\mathbb{Z}\to\overline{\mathbb{Q}} \mod q$ *and such that*

$$S = \sum_{n=1}^{\infty}\frac{f(n)}{n} \tag{1}$$

converges. Then $S=0$ *or* $S\notin\overline{\mathbb{Q}}$*. In the latter case we have*

$$\log|S-\alpha| \geq -c^q q^{3q}(d_\alpha d_f)^{q+3}\max(h_\alpha, h_f)$$

for any algebraic number α*, where c is some computable absolute constant.*

Corollary 3.1. *Let* $q\geq 2$ *be an integer and* χ *non-principal Dirichlet character* mod q*. Then* $L(1,\chi)$ *is transcendental.*

Theorem 3.5 ((N. Saradha, R. Tijdeman, 2003))**.** *Let*

$$T = \sum_{n=0}^{\infty}\frac{(-1)^n(\alpha n+\beta)}{(qn+s_1)(qn+s_2)}$$

with $\alpha,\beta\in\overline{\mathbb{Q}}$*,* $s_1,s_2\in\mathbb{Z}$ *and* $|\alpha|+|\beta|>0$*. Let* Φ_{2q} *be irreducible over* $\mathbb{Q}(\alpha,\beta)$ *and* s_1,s_2 *distinct integers such that* qn_1+s_1*,* qn_2+s_2 *do not vanish for* $n\geq 0$*. Assume that* $\alpha\neq 0$ *if* $s_1\equiv s_2 \pmod q$*. Then* T *is transcendental.*

- For example,

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{n^2+n+1} = \frac{32\pi}{e^{\frac{1}{2}\pi\sqrt{3}}+e^{-\frac{1}{2}\pi\sqrt{3}}}, \quad \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1} = \frac{\pi}{e^{2\pi}-1}+\frac{\pi+1}{2}.$$

Definition 3.2. *The Fibonacci sequence* $(F_n)_{n\geq 0}$ *is defined by*

$$F_0=0,\ F_1=1,\quad F_n=F_{n-1}+F_{n-2}\quad (n\geq 2).$$

We note that the following series involving Fibonacci numbers

- The number

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+2}} = 1$$

  is rational.

- While

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1-\sqrt{5}}{2}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}+1} = \frac{\sqrt{5}}{2}$$

  are irrational algebraic numbers.

- The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2i}},\ i=1,2,3, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n^2}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2n}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2^n+1}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2^n-1}+F_{2^n+1}}$$

  are all transcendentals.

- Each of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n+F_{n+2}}, \quad \sum_{n\geq 1} \frac{1}{F_1 F_2 \cdots F_n}$$

  is irrational, but it is not known whether they are algebraic or transcendental.

The first challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.

Theorem 3.6 (Iekata Shiokawa, Carsten Elsner and Shun Shimomura, 2006)**.** *For $\Re(s) > 0$,*

$$\zeta_F(s) = \sum_{n\geq 1} \frac{1}{F_n^s}$$

$\zeta_F(2), \zeta_F(4), \zeta_F(6)$ *are algebraically independent.*

4 Transcendental Numbers

Theorem 4.1 (Liouville, 1844)**.** *If α is real and algebraic of degree d, then there is a positive constant $C(\alpha)$, i.e. depending only on α, such that for every rational $\frac{q}{p}$,*

$$\left|\alpha - \frac{q}{p}\right| > \frac{C(\alpha)}{p^d}.$$

J. Liouville gave the first example of transcendental numbers.

Corollary 4.1. $\alpha := \sum_{n=1}^{\infty} 10^{-n!}$ *is transcendental.*

- Charles Hermite(1873): Transcendence of e
- Ferdinand Lindemann(1882): Transcendence of π

Theorem 4.2 (Hermite-Lindemann, 1882)**.** *For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.*

Corollary 4.2. *Transcendence of* $\log \alpha$ *and of* e^{β} *for* α *and* β *non-zero algebraic complex numbers, provided* $\log \alpha \neq 0$

Definition 4.1. *A complex function is called transcendental if it is transcendental over the field* $\mathbb{C}(z)$*, which means that the function* z *and* $f(z)$ *are algebraically independent: if* $P \in \mathbb{C}[X, Y]$ *is a non-zero polynomial, then the function* $P(z, f(z))$ *is not 0.*

Definition 4.2. *A* ***exceptional set*** *defined by the transcendental function* f *given, that is the set*

$$\mathcal{E}(f) = \{\alpha \in \overline{\mathbb{Q}} : f(\alpha) \in \overline{\mathbb{Q}}\}$$

Question 4.1 (Weierstrass)**.** *Is it true that a transcendental entire function* f *takes usually transcendental values at algebraic point.*

Example 4.1.

1. *For* $f(z) = e^z$*, there is a single exceptional point* α *algebraic with* e^{α} *also algebraic, namely* $\alpha = 0$*.*
2. *For* $f(z) = e^{P(z)}$ *where* $P \in \mathbb{Z}[z]$ *is a non-constant polynomial, there are finitely many exceptional points* α*, namely the roots of* P*.*
3. *The exceptional set of* $e^z + e^{1+z}$ *is empty.*
4. *The exceptional set of functions like* $e^z + e^{1+z}$ *is empty (Lindemann-Weierstrass).*
5. *The exceptional set of functions like* 2^z *or* $e^{i\pi z}$ *is* $\mathbb{Q}$*, (Gel'fond and Schneider)*

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain,

Theorem 4.3. *If* S *is a countable subset of* $\mathbb{C}$ *and* T *is a dense subset of* $\mathbb{C}$*, there exist transcendental entire functions* f *mapping* S *into* T*, as well as its derivative.*

Theorem 4.4 (van der Poorten)**.** *There are transcendental entire functions* f *such that* $D^k f(\alpha) \in \mathbb{Q}(\alpha)$ *for all* $k \geq 0$ *and all algebraic* α*.*

Definition 4.3. *An* ***integer valued entire function*** *is a function* f*, which is analytic in* $\mathbb{C}$*, and maps* $\mathbb{N}$ *into* $\mathbb{Z}$*, e.g.* 2^z *is an integer valued entire function, not a polynomial.*

Question 4.2. *Are there integer valued entire function growing slower than* 2^z *without being a polynomial?*

Definition 4.4. *Let* f *be a transcendental entire function in* $\mathbb{C}$*. For* $R > 0$ *set*

$$|f|_R = \sup_{|z|=R} |f(z)|.$$

Theorem 4.5 (G. Pólya, 1914)**.** *If f is not a polynomial and $f(n) \in \mathbb{Z}$ for $n \in \mathbb{Z}_{\geq 0}$, then*

$$\limsup_{R\to\infty} 2^{-R}|f|_R \geq 1.$$

Theorem 4.6 (A.O. Gel'fond 1929)**.** *Growth of entire functions mapping the Gaussian integers into themselves. Newton interpolation series at the points in $\mathbb{Z}[i]$.*

Theorem 4.7. *An entire function f which is not a polynomial and satisfies $f(a+bi) \in \mathbb{Z}[i]$ for all $a+bi \in \mathbb{Z}[i]$ satisfies*

$$\limsup_{R\to\infty} \frac{1}{R^2} \log|f|_R \geq \delta.$$

Theorem 4.8 (F. Gramin, 1981)**.** $\delta = \frac{\pi}{2e} = 0.5888636748\cdots$

- This is best possible proved by D.W. Masser (1980).

Theorem 4.9 (A.O. Gel'fond, 1929)**.** *If e^π is rational, then the function $e^{\pi z}$ takes values $\mathbb{Q}(i)$ when the argument z is in $\mathbb{Z}[i]$.*

Theorem 4.10 (A.O. Gel'fond and Th. Schneider, 1934)**.** *Transcendence of α^β and of $(\log\alpha_1)/(\log\alpha_2)$ for algebraic α, β, α_1 and α_2.*

Theorem 4.11 (A.O. Gel'fond, 1948)**.** *The two number $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent. More generally, if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an algebraic number of degree $d \geq 3$, then two at least of the number*

$$\alpha^\beta, \alpha^{\beta^2}, \cdots, \alpha^{\beta^{d-1}}$$

are algebraically independent.

Theorem 4.12 (Six Exponentials)**.** *Let $x_1, \cdots, x_d$ be complex numbers which are linearly independent over $\mathbb{Q}$ and let $y_1, \cdots, y_l$ also be complex numbers which are linearly independent over $\mathbb{Q}$. Assume $dl > d+l$. Then one at least of the dl numbers*

$$\exp(x_i y_j), \qquad (1 \leq i \leq d,\ 1 \leq j \leq l)$$

is transcendental.

- The theorem was first explicitly stated and proved in its complete form independently by Serge Lang and Kanakanahalli Ramachandra in the 1960s.
- It is clear that the interesting case is $d = 3$, $l = 2$ (or $d = 2$, $l = 3$), and this explains the name of result.
- One conjectures that the conclusion is still valid under the weaker hypothesis $dl \geq d + l$ ($d = 2$, $l = 2$).

Theorem 4.13 (A. Baker, 1966)**.** *Let* $\alpha_1, \cdots, \alpha_n$ *be non-zero algebraic numbers for which* $\log \alpha_1, \cdots, \log \alpha_n, 2\pi i$ *are linearly independent over* $\mathbb{Q}$*. Then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \neq 0$$

for any algebraic numbers $\beta_1, \cdots, \beta_n$ *that are not all zero.*

Corollary 4.3. *Transcendence of numbers like*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \text{ or } e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$$

for algebraics α_i*'s and* β_j*'s, e.g.* $2^{\sqrt{2}} \cdot 3^{\sqrt{5}} \cdot 5^{\sqrt{7}}$*.*

Theorem 4.14 (G.V. Chudnovsky, 1976)**.** *Algebraic independence of the numbers* π *and* $\Gamma(1/4)$*, also algebraic independence of the numbers* π *and* $\Gamma(1/3)$*.*

Corollary 4.4. *Transcendece of* $\Gamma(1/4)$ *and* $\Gamma(1/3)$*.*

Theorem 4.15 (Yu. V. Nesternko, 1996)**.** *Algebraic independence of* $\Gamma(1/4), \pi$ *and* e^{π}*, also algebraic independence of* $\Gamma(1/3), \pi$ *and* $e^{\pi\sqrt{3}}$*.*

Corollary 4.5. *The numbers* π *and* e^{π} *are algebraically independent.*

Definition 4.5. *Let* $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ *be a lattice in* $\mathbb{C}$*. The canonical product attached to* Ω *is the Weiersraßsigma function*

$$\sigma(z) = \sigma_{\Omega}(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{(z/\omega) + (z^2/2\omega^2)}.$$

- The number
$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4} \pi^{1/2} e^{\pi/8} \Gamma(1/4)^{-2}$$
is transcendental.

5 Conjectures

Conjecture 5.1 (Four Exponentials Conjecture)**.** *Let* x_1, x_2 *be two* $\mathbb{Q}$*-linearly independent complex numbers and* y_1, y_2 *also two* $\mathbb{Q}$*-linearly independent complex numbers. Then one at least of the four numbers*

$$e^{x_1 y_1}, \ e^{x_1 y_2}, \ e^{x_2 y_1}, \ e^{x_2 y_2}$$

is transcendental.

Definition 5.1. *A* ***period*** *is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational function with rational coefficients, over domains in* $\mathbb{R}^n$ *given by polynomial inequalities with rational coefficients.*

Example 5.1. *Basic example of a period:*

$$e^{z+2\pi i} = e^z, \quad 2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\pi = \iint_{x^2+y^2\leq 1} dxdy = 2\int_{-1}^{1} \sqrt{1-x^2}dx = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{1-x^2}$$

Further examples

$$\sqrt{2} = \int_{2x^2\geq 1} dx \quad \textit{and all algebraic numbers.}$$

$$\log 2 = \int_{1<x<2} \frac{dx}{x} \quad \textit{and all logarithms of algebraic numbers.}$$

A product of periods is a period (subalgebra of $\mathbb{C}$), but $1/\pi$ is expected not to be a period.

A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following

Conjecture 5.2 (Kontsevich-Zagier)**.** *If a period has two integral representations, then one can pass from one formula to another in which all functions and domains of integration are algebraic with algebraic coefficients.*

In other words, we do not expect any miraculous conicidene of two integrals of algebraic functions which will not be possible to prove using three simple rules. This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.

Conjecture 5.3 (Scahnuel)**.** *If $x_1, \cdots, x_n$ are $\mathbb{Q}$-linearly independent complex numbers, then n at least of the $2n$ numbers $x_1, \cdots, x_n$, $e^{x_1}, \cdots, e^{x_n}$ are algebraically independent.*

A simple geometric construction on the moduli spaces $\mathcal{M}_{0,n}$ of curves of genus 0 with n ordered marked points is described which gives a common framework for many irrationality proofs for zeta values. This construction yields Apéry's approximations to $\zeta(2)$ and $\zeta(3)$, and for larger n, an infinite family of small linear forms in multiple zeta values with an interesting algebraic structure. It also contains a generalization of the linear forms used by Ball and Rivoal to prove that infinitely many odd zeta values are irrational.

Theorem 5.1 (Francis Brown, 19 Dec 2014)**.** *For k, $s_1, \cdots, s_k$ positive integers with $s_1 \geq 2$, we set $\underline{s} = (s_1, \cdots, s_k)$ and*

$$\zeta(\underline{s}) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}.$$

The $\mathbb{Q}$-vector space $\mathcal{E}$ spanned by the number $\zeta(\underline{s})$ is also a $\mathbb{Q}$-algebra. For $n \geq 2$, denote by $\mathcal{E}_n$ the $\mathbb{Q}$-subspace of $\mathcal{E}$ spanned by the real numbers $\zeta(\underline{s})$ where $\underline{s}$ has weight $s_1 + \cdots + s_k = n$.
The number $\zeta(s_1, \cdots, s_k)$ $s_1 + \cdots + s_k = n$, where each s_i is 2 or 3, span $\mathcal{E}_n$ over $\mathbb{Q}$.

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