1 Grothendieck’s variational Hodge conjecture

Let $S$ be a smooth quasi-projective scheme over a field $k$ of characteristic zero, and $\pi : X \to S$ a smooth projective morphism. Fix a point $s \in S$, and set $X_s := X \times_S \{s\}$.

In his thought-provoking paper [5] in 1966, Grothendieck formulated a conjecture now we call the Variational Hodge Conjecture (VHC), that predicts when an algebraic cycle class on $X_s$ lifts to a one on $X$ in terms of the cycle map

$$\text{cl} : CH^p(X_s) \to H^{2p}_{\text{dR}}(X_s/\kappa(s)) = (R^{2p}\pi_*\Omega_{X/S}^\bullet)_s \otimes \kappa(s).$$

(1.1)

Variational Hodge Conjecture. For $\xi_s \in CH^p(X_s)_\mathbb{Q}$, the following are equivalent:

(i) $\text{cl}(\xi_s)$ lifts to a flat section of $R^{2p}\pi_*\Omega_{X/S}^\bullet$, i.e. a global section killed by the Gauss-Manin connection $\nabla$.

(ii) There exists a rational cycle class $\xi \in CH^p(X)_\mathbb{Q}$ such that $\text{cl}(\xi|_{X_s}) = \text{cl}(\xi_s)$.

This conjecture is so powerful that, for example, the VHC for abelian schemes $X/S$ implies the Hodge Conjecture for abelian varieties [4].

However, it is probably not possible to deduce the Hodge Conjecture in general from the VHC. The problem is that we can hardly find such a variety in a smooth family that algebraic cycles on it can be controlled. If we can describe
a condition when cycle classes on a degenerate fiber lift, this must be helpful, because sometimes it is easier to construct cycles on a degenerate fiber.

In the next section, we propose such a condition, following Grothendieck’s VHC.

## 2 The generalized variational Hodge conjecture

Let $X$ be a smooth quasi-projective scheme over a field $k$ of characteristic zero, and $Y$ a projective scheme over $k$ with a closed immersion $Y \to X$.

We want to describe a condition when cycle classes on $Y$ lift to ones on $X$.

Since the cycle map (1.1) does not work for singular varieties, we replace it by the Chern character

$$\text{ch}: K_0(Y) \to \bigoplus_p H_{\text{dR}}^{2p}(Y/k).$$

(2.1)

The target of (2.1) is the de Rham cohomology of $Y$ [9] defined to be

$$H_q^{\text{dR}}(Y/k) := H_q(^X\Omega^*_{X/k}),$$

(2.2)

where each $^\wedge$ is the formal completion along $Y$. This does not depend on the embedding $Y \to X$ [9] Ch. II, Theorem 1.4.

The Chern character ([?]) is defined by the composite

$$K_r(Y) \xrightarrow{\text{HN}_r} H_{\text{dR}}^r(Y/k) \xrightarrow{\text{HP}_r} \bigoplus_p H_{\text{dR}}^{2p-r}(Y/k).$$

(2.3)

Here HN (resp. HP) is the negative (resp. periodic) cyclic homology. The first map is the Goodwillie’s Chern character (for the definition see [3] [13]), the second map is the canonical one, and the last isomorphism is by the Feigin-Tsygan Theorem theorem [5].

We conjecture:

**Conjecture A.** For $\xi_s \in K_0(Y)\mathbb{Q}$, the following are equivalent:

(i) $\text{ch}(\xi_s) \in \bigoplus_p H_{\text{dR}}^{2p}(Y/k)$ lifts to $\bigoplus_p H_{\text{dR}}^{2p}(X/k)$.

(ii) There exists $\xi \in K_0(X)\mathbb{Q}$ such that $\text{ch}(\xi|_Y) = \text{ch}(\xi_s)$.

In case $Y$ is a fiber of a smooth family $X \to S$ as in §1, we can show that the VHC is equivalent to Conjecture A, by Deligne’s théorème de la partie fixe [4] §4.1.

## 3 Infinitesimal deformation

We formulate an infinitesimal version of Conjecture A.

Let $k$ be a field of characteristic zero, $S := \text{Spec} k[[t]]$, $X$ a finite dimensional quasi-compact regular scheme $X$ over $k$, and $Y$ a proper scheme over $k$ with a
closed immersion $Y \to X$. Set $Y_n := \text{Spec } \mathcal{O}_X/I^{n+1}$, where $I$ is the $\mathcal{O}_X$-ideal defining $Y$.

We define the de Rham cohomology of $Y$ by

$$H^q_{dR}(Y/k) := H^q(\hat{\mathcal{X}}, \hat{\Omega}^*_X/k),$$

where each $\hat{\cdot}$ is the formal completion along $Y$. Again, the definition does not depend on the embedding $Y \hookrightarrow X$. This is a consequence of the generalized version of the Feigin-Tsygan theorem \[11\].

We can also define the Chern character

$$\text{ch}: K_r(Y) \to \bigoplus_p H^{2p-r}_{dR}(Y/k)$$

as above, thanks to the generalized Feigin-Tsygan theorem.

Consider the diagram

$$
\begin{array}{ccc}
K_r(X) & \longrightarrow & \lim_{\longleftarrow n} K_r(Y_n) \\
\downarrow & & \downarrow \text{ch} \\
\bigoplus_p H^{2p-r}_{dR}(\hat{\mathcal{X}}/k) & \xrightarrow{\Phi} & \bigoplus_p H^{2p-r}_{dR}(Y/k).
\end{array}
$$

We conjecture:

**Conjecture B.** Assume $Y \to X$ is the special fiber of a projective family $X \to S := \text{Spec } k[[t_1, \ldots, t_m]]$, i.e. $Y = X \times_S \text{Spec } k$. Then, for $\xi_0 \in K_0(Y)_Q$, the following are equivalent:

(i) $\Phi^{-1}(\text{ch}(\xi_0))$ is in $\bigoplus_p F^p H^{2p}_{dR}(\hat{\mathcal{X}}/k)$.

(ii) There exists $\xi \in (\lim_{\longleftarrow n} K_0(Y_n))_Q$ such that $\text{ch}(\xi|_Y) = \text{ch}(\xi_0)$.

(iii) There exists $\xi \in K_0(X)_Q$ such that $\text{ch}(\xi|_Y) = \text{ch}(\xi_0)$.

In case $X \to S$ is smooth, Conjecture B is equivalent to the VHC \[2\] Appendix A.

In the general case (at least in the case of semistable degeneration), we still have strong relations between Conjecture A and Conjecture B \[10\] Proposition 3.2.1].
4 Formal deformation

We use the notation in §3.

**Theorem 4.1.** Suppose that $k$ is algebraic over $\mathbb{Q}$. Then, for $\xi_0 \in K_r(Y)$, the following are equivalent:

(i) $\Phi^{-1}(ch(\xi_0))$ is in $\bigoplus_p F_p H_{dR}^{2p-r}(\hat{X}/k)$.

(ii) There exists $\xi \in \varprojlim K_r(Y_n)$ such that $ch(\xi|_Y) = ch(\xi_0)$.

In particular, (i)$\Leftrightarrow$(ii) of Conjecture B holds under the assumption that $k$ is algebraic over $\mathbb{Q}$.

**Remark 4.2.** In case $Y \to X$ is the special fiber of a smooth projective family $X \to S := \text{Spec}(k[[t_1, \ldots, t_m]])$, Griffith-Green [7] and Morrow [12] proved the theorem, and Bloch-Esnault-Kerz [2] did for general fields $k$ under the Chow-Künneth assumption. The general case was proved in [10].

**Proof.** Consider the commutative diagram

$$
\begin{array}{cccccc}
K_r(Y_n) & \to & K_r(Y) & \to & K_{r-1}(Y_n, Y) \\
\downarrow & & \downarrow & & \downarrow \cong \\
HN_r(Y_n) & \to & HN_r(Y) & \to & HN_{r-1}(Y_n, Y)
\end{array}
$$

(4.1)

with exact rows, where the vertical maps are the Chern characters (3.2). The right vertical map is an isomorphism by the theorem of Goodwillie [6]. By the pro HKR theorem proved by Morrow in [11], we have a pro-isomorphism

$$
\varprojlim_n HN_r(Y_n) \simeq \bigoplus_p H^{2p-r}(\hat{X}, \hat{\Omega}_{X/k}^{\geq p}).
$$

(4.2)

By taking limits of (4.1) and using (4.2), we have a commutative diagram

$$
\begin{array}{cccccc}
\varprojlim_n K_r(Y_n) & \to & K_r(Y) & \to & \varprojlim_n K_{r-1}(Y_n, Y) \\
\downarrow & & \downarrow & & \downarrow \cong \\
\varprojlim_n HN_r(Y_n) & \to & HN_r(Y) & \to & \varprojlim_n HN_{r-1}(Y_n, Y) \\
\downarrow \cong & & & & \downarrow ch \\
\bigoplus_p H^{2p-r}(\hat{X}, \hat{\Omega}_{X/k}^{\geq p}) & \to & \bigoplus_p H_{dR}^{2p-r}(Y/k)
\end{array}
$$

(4.3)

By our assumption that $Y$ is proper, we can verify some Mittag-Leffler conditions and show that the upper and middle rows are exact.

Recall that we have an isomorphism

$$
HP_r(Y) \simeq \bigoplus_p H_{dR}^{2p-r}(Y/k)
$$

(4.4)
by the generalized Feigin-Tsygan theorem. By a simple diagram chase, it remains to show the following:

**Lemma 4.3.** Every $c \in \ker(HN_r(Y) \to HP_r(Y))$ lifts to $K_r(Y)$.

This is proved by using the fact that $K_{\inf} := \text{hofib}(K \to HN)$ satisfies cdh-descent ([3]). For the detail of the proof, see [10]. □

**References**


