Deformation of algebraic cycle classes

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1 Grothendieck's variational Hodge conjecture

Let S be a smooth quasi-projective scheme over a field k of characteristic zero, and $\pi\colon X \to S$ a smooth projective morphism. Fix a point $s \in S$, and set $X_s := X \times_S \{s\}$.

In his thought-provoking paper [8] in 1966, Grothendieck formulated a conjecture now we call the Variational Hodge Conjecture (VHC), that predicts when an algebraic cycle class on X_s lifts to a one on X in terms of the cycle map

$$\mathrm{cl}\colon \mathrm{CH}^p(X_s) \to H^{2p}_{\mathrm{dR}}(X_s/\kappa(s)) = (R^{2p}\pi_*\Omega^\bullet_{X/S})_s \otimes \kappa(s). \tag{1.1}$$

Variational Hodge Conjecture. *For $\xi_s \in \mathrm{CH}^p(X_s)_{\mathbb{Q}}$, the following are equivalent:*

(i) $\mathrm{cl}(\xi_s)$ *lifts to a flat section of $R^{2p}\pi_*\Omega^\bullet_{X/S}$, i.e. a global section killed by the Gauss-Manin connection ∇.*

(ii) There exists a rational cycle class $\xi \in \mathrm{CH}^p(X)_{\mathbb{Q}}$ such that $\mathrm{cl}(\xi|_{X_s}) = \mathrm{cl}(\xi_s)$.

This conjecture is so powerful that, for example, the VHC for abelian schemes X/S implies the Hodge Conjecture for abelian varieties [1].

However, it is probably not possible to deduce the Hodge Conjecture in general from the VHC. The problem is that we can hardly find such a variety in a smooth family that algebraic cycles on it can be controlled. If we can describe a condition when cycle classes on a degenerate fiber lift, this must be helpful, because sometimes it is easier to construct cycles on a degenerate fiber.

In the next section, we propose such a condition, following Grothendieck's VHC.

2 The generalized variational Hodge conjecture

Let X be a smooth quasi-projective scheme over a field k of characteristic zero, and Y a projective scheme over k with a closed immersion $Y \to X$.

We want to describe a condition when cycle classes on Y lift to ones on X.

Since the cycle map (1.1) does not work for singular varieties, we replace it by the *Chern character*

$$\mathrm{ch}\colon K_0(Y) \to \bigoplus_p H^{2p}_{\mathrm{dR}}(Y/k). \tag{2.1}$$

The target of (2.1) is the de Rham cohomology of Y [9] defined to be

$$H^q_{\mathrm{dR}}(Y/k) := H^q(\hat{X}, \hat{\Omega}^\bullet_{X/k}), \tag{2.2}$$

where each $^\wedge$ is the formal completion along Y. This does not depend on the embedding $Y \hookrightarrow X$ [9, Ch. II, Theorem 1.4].

The Chern character ([**?**]) is defined by the composite

$$K_r(Y) \longrightarrow \mathrm{HN}_r(Y) \longrightarrow \mathrm{HP}_r(Y) \simeq \textstyle\bigoplus_p H^{2p-r}_{\mathrm{dR}}(Y/k). \tag{2.3}$$

Here HN (resp. HP) is the negative (resp. periodic) cyclic homology. The first map is the Goodwillie's Chern character (for the definition see [3, 13]), the second map is the canonical one, and the last isomorphism is by the Feigin-Tsygan Theorem theorem [5].

We conjecture:

Conjecture A. *For $\xi_s \in K_0(Y)_{\mathbb{Q}}$, the following are equivalent:*

(i) $\mathrm{ch}(\xi_s) \in \bigoplus_p H^{2p}_{\mathrm{dR}}(Y/k)$ *lifts to* $\bigoplus_p H^{2p}_{\mathrm{dR}}(X/k)$*.*

(ii) There exists $\xi \in K_0(X)_{\mathbb{Q}}$ *such that* $\mathrm{ch}(\xi|_Y) = \mathrm{ch}(\xi_s)$*.*

In case Y is a fiber of a smooth family $X \to S$ as in §1, we can show that the VHC is equivalent to Conjecture A, by Deligne's théorème de la partie fixe [4, §4.1].

3 Infinitesimal deformation

We formulate an infinitesimal version of Conjecture A.

Let k be a field of characteristic zero, $S := \operatorname{Spec} k[[t]]$, X a finite dimensional quasi-compact regular scheme X over k, and Y a proper scheme over k with a closed immersion $Y \to X$. Set $Y_n := \operatorname{Spec} \mathcal{O}_X / I^{n+1}$, where I is the $\mathcal{O}_X$-ideal defining Y.

We define the *de Rham cohomology of* Y by

$$H^q_{\mathrm{dR}}(Y/k) := H^q(\hat{X}, \hat{\Omega}^\bullet_{X/k}), \tag{3.1}$$

where each $^\wedge$ is the formal completion along Y. Again, the definition does not depend on the embedding $Y \hookrightarrow X$. This is a consequence of the generalized version of the Feigin-Tsygan theorem [11].

We can also define the Chern character

$$\mathrm{ch} \colon K_r(Y) \to \bigoplus_p H^{2p-r}_{\mathrm{dR}}(Y/k) \tag{3.2}$$

as above, thanks to the generalized Feigin-Tsygan theorem.

Set

$$H^q_{\mathrm{dR}}(\hat{X}/k) := H^q(\hat{X}, \hat{\Omega}^\bullet_{X/k}), \tag{3.3}$$

$$F^r H^q_{\mathrm{dR}}(\hat{X}/k) := \mathrm{image}(H^q(\hat{X}, \hat{\Omega}^{\geq r}_{X/k}) \to H^q(\hat{X}, \hat{\Omega}^\bullet_{X/k})). \tag{3.4}$$

Consider the diagram

$$\begin{tikzcd} K_r(X) \arrow[r] \arrow[d] & \varprojlim_n K_r(Y_n) \arrow[r] & K_r(Y) \arrow[d, "\mathrm{ch}"] \\ \bigoplus_p H^{2p-r}_{\mathrm{dR}}(\hat{X}/k) \arrow[rr, "\Phi", "\simeq"'] & & \bigoplus_p H^{2p-r}_{\mathrm{dR}}(Y/k). \end{tikzcd} \tag{3.5}$$

We conjecture:

Conjecture B. *Assume $Y \to X$ is the special fiber of a projective family $X \to S := \operatorname{Spec} k[[t_1, \ldots, t_m]]$, i.e. $Y = X \times_S \operatorname{Spec} k$. Then, for $\xi_0 \in K_0(Y)_{\mathbb{Q}}$, the following are equivalent:*

(i) $\Phi^{-1}(\mathrm{ch}(\xi_0))$ *is in* $\bigoplus_p F^p H^{2p}_{\mathrm{dR}}(\hat{X}/k)$.

(ii) There exists $\tilde{\xi} \in (\varprojlim K_0(Y_n))_{\mathbb{Q}}$ *such that* $\mathrm{ch}(\tilde{\xi}|_Y) = \mathrm{ch}(\xi_0)$.

(iii) There exists $\xi \in K_0(X)_{\mathbb{Q}}$ *such that* $\mathrm{ch}(\xi|_Y) = \mathrm{ch}(\xi_0)$.

In case $X \to S$ is smooth, Conjecture B is equivalent to the VHC [2, Appendix A].

In the general case (at least in the case of semistable degeneration), we still have strong relations between Conjecture A and Conjecture B [10, Proposition 3.2.1].

4 Formal deformation

We use the notation in §3.

Theorem 4.1. *Suppose that k is algebraic over $\mathbb{Q}$. Then, for $\xi_0 \in K_r(Y)$, the following are equivalent:*

(i) $\Phi^{-1}(\mathrm{ch}(\xi_0))$ *is in* $\bigoplus_p F^p H^{2p-r}_{\mathrm{dR}}(\hat{X}/k)$.

(ii) There exists $\tilde{\xi} \in \varprojlim K_r(Y_n)$ *such that* $\mathrm{ch}(\tilde{\xi}|_Y) = \mathrm{ch}(\xi_0)$.

In particular, (i)⇔(ii) of Conjecture B holds under the assumption that k is algebraic over $\mathbb{Q}$.

Remark 4.2. *In case $Y \to X$ is the special fiber of a smooth projective family $X \to S := \operatorname{Spec} k[[t_1, \ldots, t_m]]$, Griffith-Green [7] and Morrow [12] proved the theorem, and Bloch-Esnault-Kerz [2] did for general fields k under the Chow-Künneth assumption. The general case was proved in [10].*

Proof. Consider the commutative diagram

$$\begin{tikzcd} K_r(Y_n) \arrow[r] \arrow[d] & K_r(Y) \arrow[r] \arrow[d] & K_{r-1}(Y_n, Y) \arrow[d, "\simeq"] \\ \mathrm{HN}_r(Y_n) \arrow[r] & \mathrm{HN}_r(Y) \arrow[r] & \mathrm{HN}_{r-1}(Y_n, Y) \end{tikzcd} \tag{4.1}$$

with exact rows, where the vertical maps are the Chern characters (3.2). The right vertical map is an isomorphism by the theorem of Goodwillie [6]. By the pro HKR theorem proved by Morrow in [11], we have a pro-isomorphism

$$\varprojlim_n \mathrm{HN}_r(Y_n) \simeq \bigoplus_p H^{2p-r}(\hat{X}, \hat{\Omega}^{\geq p}_{X/k}). \tag{4.2}$$

By taking limits of (4.1) and using (4.2), we have a commutative diagram

$$\begin{tikzcd} \varprojlim_n K_r(Y_n) \arrow[r] \arrow[d] & K_r(Y) \arrow[r] \arrow[d] \arrow[dd, bend left, "\mathrm{ch}"] & \varprojlim_n K_{r-1}(Y_n, Y) \arrow[d, "\simeq"] \\ \varprojlim_n \mathrm{HN}_r(Y_n) \arrow[r] \arrow[d, "\simeq"] & \mathrm{HN}_r(Y) \arrow[r] \arrow[d] & \varprojlim \mathrm{HN}_{r-1}(Y_n, Y) \\ \bigoplus_p H^{2p-r}(\hat{X}, \hat{\Omega}^{\geq p}_{X/k}) \arrow[r] & \bigoplus_p H^{2p-r}_{\mathrm{dR}}(Y/k) & \end{tikzcd} \tag{4.3}$$

By our assumption that Y is proper, we can verify some Mittag-Leffer conditions and show that the upper and middle rows are exact.

Recall that we have an isomorphism

$$\mathrm{HP}_r(Y) \simeq \bigoplus_p H^{2p-r}_{\mathrm{dR}}(Y/k) \tag{4.4}$$

by the generalized Feigin-Tsygan theorem. By a simple diagram chase, it remains to show the following:

Lemma 4.3. *Every* $c \in \ker(\mathrm{HN}_r(Y) \to \mathrm{HP}_r(Y))$ *lifts to* $K_r(Y)$.

This is proved by using the fact that $K^{\mathrm{inf}} := \mathrm{hofib}(K \to \mathrm{HN})$ satisfies cdh-descent ([3]). For the detail of the proof, see [10] □

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