ON THE STOKES MATRICES OF THE tt^*-TODA EQUATION

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Abstract. We derive a formula for the signature of the symmetrized Stokes matrix $\mathcal{S}+\mathcal{S}^{\mathrm{T}}$ for the tt^*-Toda equation. As a corollary, we verify a conjecture of Cecotti-Vafa regarding when $\mathcal{S}+\mathcal{S}^{\mathrm{T}}$ is positive definite, reminiscent of a formula of Beukers and Heckmann for the generalized hypergeometric equation. The condition $\mathcal{S}+\mathcal{S}^{\mathrm{T}}>0$ is prominent in the work of Cecotti and Vafa on the tt^* equation; we show that the Stokes matrices $\mathcal{S}$ satisfying this condition are parameterized by a convex polytope.

1. Introduction

The tt^* equation is a system of nonlinear PDEs which appeared in the work of Cecotti and Vafa [2] on the classification of supersymmetric field theories in physics. Dubrovin [3] showed that it admits an isomonodromic deformation interpretation, as well as the standard zero-curvature formulation of harmonic map theory. This leads to a Riemann-Hilbert correspondence between (local) solutions and monodromy data of a meromorphic ODE. Clarifying this correspondence is a subject of current research activity relating several fields of mathematics, including Hodge theory and algebraic geometry.

There are very few examples where solutions can be found. A special case of the tt^* equation, introduced by Cecotti and Vafa, and studied mathematically by Guest-Its-Lin [4, 5] and Mochizuki [7], is the tt^*-Toda equation. This is, essentially, the well-known Toda field equation (2-dimensional Toda lattice), although even in this case the existence of the solutions predicted by Cecotti and Vafa was proved only recently (in the aforementioned references).

This article was motivated by the conjectures of Cecotti and Vafa regarding the symmetrized Stokes matrix $\mathcal{S}+\mathcal{S}^{\mathrm{T}}$, in the case of the tt^*-Toda equation. We shall give a necessary and sufficient condition for $\mathcal{S}+\mathcal{S}^{\mathrm{T}}$ to be positive definite, and a simple formula for the signature of $\mathcal{S}+\mathcal{S}^{\mathrm{T}}$, reminiscent of a formula of Beukers and Heckmann for the generalized hypergeometric equation [1].

Let us now state the tt^*-Toda equations and explain the relevant Stokes matrix. The equations are:

$$2(w_i)_{z\bar{z}} = -e^{2(w_{i+1}-w_i)} + e^{2(w_i-w_{i-1})}\ , \quad w_i : \mathbb{C}^* \to \mathbb{R} \tag{1.1}$$

subject to two further conditions:

(1) the "anti-symmetry" condition: $w_i + w_{n-i} = 0$; and
(2) the radial condition: $w_i = w_i(|z|)$.

We use the convention that $w_i = w_{i+n+1}$ for all $i \in \mathbb{Z}$. This system is the compatibility condition for the linear system:

$$\begin{cases} \Psi_z &= (\mathrm{w}_z + \frac{1}{\lambda}\mathrm{W})\Psi\ , \\ \Psi_{\bar{z}} &= (-\mathrm{w}_{\bar{z}} + \lambda \mathrm{W}^{\mathrm{T}})\Psi\ , \end{cases}$$

where:

$$\mathrm{w} = \mathrm{diag}\,(w_0, \ldots, w_n)\ , \quad \mathrm{W} = \left[\begin{array}{c|c|c|c} & e^{w_1 - w_0} & & \\ \hline & & \ddots & \\ \hline & & & e^{w_n - w_{n-1}} \\ \hline e^{w_0 - w_n} & & & \end{array}\right] .$$

If we write $x = |z|$, then the radial version of (1.1) is the compatibility condition for a linear system, which may then be transformed to (see Equation 1.4 of [5]):

$$\begin{cases} \Psi_\zeta &= \left(-\frac{1}{\zeta^2}\mathrm{W} - \frac{1}{\zeta}x\mathrm{w}_x + x^2\mathrm{W}^{\mathrm{T}}\right)\Psi\ , \\ \Psi_x &= \left(\frac{1}{x\zeta}\mathrm{W} + x\zeta\mathrm{W}^{\mathrm{T}}\right)\Psi\ , \end{cases} \tag{1.2}$$

where $\zeta = \frac{\lambda}{z}$.

The ζ-system (1.2) is a meromorphic linear ODE in the complex variable ζ, with poles of order two at both $\zeta = 0$ and $\zeta = \infty$. The Stokes matrices at these two poles are equivalent, so we shall only consider the Stokes matrix at $\zeta = \infty$, and denote it by $\mathcal{S}$. By the general theory of isomonodromic deformations, Stokes matrices $\mathcal{S}$ correspond to local solutions near 0 (i.e. defined on intervals of the form $(0, \varepsilon)$) of the tt^*-Toda equation. Further details and explanation may be found in [4, 5], where $\mathcal{S}$ is computed in terms of the asymptotic behaviour of the functions w_i.

It was conjectured by Cecotti and Vafa, and confirmed in [4, 5, 6, 7], that the condition $\mathcal{S} + \mathcal{S}^{\mathrm{T}} > 0$ implies that the corresponding local solution of the tt^*-Toda equation is globally defined on $\mathbb{C}^*$ (i.e. such that $\varepsilon = \infty$). It is, therefore, of interest to describe the set of such Stokes matrices explicitly, and this is our first main result.

We show that they form a convex polytope described by simple explicit equations. It follows from the previous references that a necessary and sufficient condition for the local solution of the tt^*-Toda equation to be globally defined on $\mathbb{C}^*$ is that the eigenvalues of the monodromy $\mathcal{S}\mathcal{S}^{-\mathrm{T}}$ are unimodular. (Here, $\mathcal{S}^{-\mathrm{T}}$ denotes the transpose of $\mathcal{S}^{-1}$.) For such Stokes matrices, we prove the following explicit characterization of the signature of $\mathcal{S} + \mathcal{S}^{\mathrm{T}}$, which we expect to be of use in future investigations of the tt^*-Toda equation:

Theorem: *Let $\sigma = (N_+, N_-, N_0)$ denote the signature triple of $\mathcal{S} + \mathcal{S}^{\mathrm{T}}$, where N_+, N_-, and N_0 are the number of positive, negative, and zero eigenvalues of $\mathcal{S} + \mathcal{S}^{\mathrm{T}}$, respectively. Then σ is equal to the signature triple of the diagonal matrix:*

$$\mathrm{diag}\,((-1)^{n+1}p(\pi_0), \ldots, (-1)^{n+1}p(\pi_n))\ .$$

Here, $\pi_0, \ldots, \pi_n$ are the $n+1$ roots of $x^{n+1} - (-1)^{n+1}$, and the real polynomial $p(x)$ is the characteristic polynomial of a certain matrix $\mathcal{R}$ satisfying $(-1)^n\mathcal{R}^{n+1} = \mathcal{S}\mathcal{S}^{-\mathrm{T}}$.

Corollary: $\mathcal{S} + \mathcal{S}^{\mathrm{T}} > 0$ iff $(-1)^{n+1}p(\pi_k) > 0$ *for all k.*

Let us explain what this means in terms of solutions to the tt^*-Toda equation. (For ease of notation, we only describe the case $n + 1 = 2m$.) It was shown in [4, 5, 6, 7] that solutions $w_i : \mathbb{C}^* \to \mathbb{R}$ are in one-to-one correspondence with real numbers γ_i satisfying $\gamma_i - \gamma_{i-1} \geq -2$ for all i, where $2w_i(z) \sim \gamma_i \log|z|$ as $|z| \to 0$. The corresponding eigenvalues of $\mathcal{R}$ are $exp\big(\pm \frac{i\pi}{n+1}(\gamma_j + 2j + 1)\big)$, $0 \leq j \leq m-1$, with

$$0 \leq \tfrac{\pi}{2m}(\gamma_0 + 1) \leq \tfrac{\pi}{2m}(\gamma_1 + 3) \leq \cdots \leq \tfrac{\pi}{2m}(\gamma_{m-1} + 2m - 1) \leq \pi\ .$$

The condition $\mathcal{S}+\mathcal{S}^{\mathrm{T}}>0$ means that these points must interlace with the $(n+1)^{\mathrm{th}}$ roots of unity, implying that:

$$0<\gamma_0+1<2<\gamma_1+3<4<\cdots<2m-2<\gamma_{m-1}+2m-1<2m\ ,$$

and this means that $0<\gamma_j<1$ for all $j=0,\ldots,m-1$.

Our second main result is a formula for the sign of $p(\pi_k)$ when the eigenvalues of $\mathcal{S}\mathcal{S}^{-\mathrm{T}}$ are unimodular and $N_0=0$ (in the notation of the Theorem). If $n+1=2m$ or $n+1=2m+1$, then this gives a characterization of the open subsets of $\mathbb{R}^m$ corresponding to each possible signature pair (N_+,N_-) in terms of configurations of points on the unit circle which interlace with the roots π_k of $x^{n+1}-(-1)^{n+1}$. To prove this, we show that $p(x)$ satisfies the signed-palindromicity condition $p(x)=(-x)^n p(\frac{1}{x})$, and hence, it may be uniquely factored as:

$$p(x)=\begin{cases} x^m\tilde{p}(x+\frac{1}{x})\ , & n+1=2m\ , \\ (x-1)x^m\tilde{p}(x+\frac{1}{x})\ , & n+1=2m+1\ . \end{cases}$$

Defining the sequence $\{\tilde{p}^{[k]}(x)\}_k$ of shifted polynomials by $\tilde{p}^{[0]}(x):=\tilde{p}(x+2)$ and:

$$\tilde{p}^{[k]}(x):=\begin{cases} \tilde{p}(x+2\cos\frac{k\pi}{m})\ , & 1\le k\le m\ ,\ n+1=2m\ , \\ \tilde{p}(x+2\cos\frac{(2k-1)\pi}{2m+1})\ , & 1\le k\le m+1\ ,\ n+1=2m+1\ , \end{cases}$$

we prove the following, which amounts to a special case of Theorem 4.5 of [1]: **Proposition:** *If ν_k is the number of sign changes in the sequence of non-zero coefficients of $\tilde{p}^{[k]}(x)$, then:*

$$sgn(p(\pi_k))=\begin{cases} (-1)^{\nu_k-k}\ , & n+1=2m\ , \\ (-1)^{\nu_{k+1}-(k+1)}\ , & n+1=2m+1\ . \end{cases}$$

Since $p(x)$ is the characteristic polynomial of $\mathcal{R}$, and the entries of $\mathcal{R}$ satisfy an explicit relation with the entries of the Stokes matrix $\mathcal{S}$, the proposition gives a simple way to check positivity of $\mathcal{S}+\mathcal{S}^{\mathrm{T}}$.

All of the above results are first proved for a conveniently defined, "abstract Stokes matrix" S. We then explain the precise relation between this "abstract Stokes matrix", and the Stokes matrix $\mathcal{S}$ of (1.2).

References

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