STRICTLY CONVEX WULFF SHAPES AND C^1 CONVEX INTEGRANDS

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ABSTRACT. Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function and let $\mathcal{W}_\gamma$ be the Wulff shape associated with γ. We show that Wulff shape $\mathcal{W}_\gamma$ is strictly convex if and only if convex integrand of $\mathcal{W}_\gamma$ is of class C^1. We also show that if the boundary of $\mathcal{W}_\gamma$ is a C^1 submanifold, then γ must be the convex integrand of $\mathcal{W}_\gamma$.

1. INTRODUCTION

Let n be a positive integer. Given a continuous function $\gamma : S^n \to \mathbb{R}_+$ where $S^n \subset \mathbb{R}^{n+1}$ is the unit sphere and $\mathbb{R}_+$ is the set consisting of positive real numbers, the *Wulff shape associated with* γ, denoted by $\mathcal{W}_\gamma$, is the following intersection

$$\mathcal{W}_\gamma = \bigcap_{\theta \in S^n} \Gamma_{\gamma,\theta}.$$

Here, $\Gamma_{\gamma,\theta}$ is the following half-space:

$$\Gamma_{\gamma,\theta} = \{x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta)\}.$$

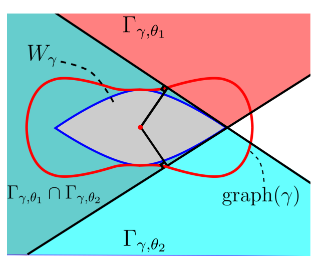

FIGURE 1. A Wulff shape $\mathcal{W}_\gamma$.

By definition, the Wulff shape $\mathcal{W}_\gamma$ is a convex body such that the origin of $\mathbb{R}^{n+1}$ is an interior point of $\mathcal{W}_\gamma$. The notion of Wulff shape was first introduced by G. Wulff in [9]. Let $Id : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \{1\}$ be the map defined by $Id(x) = (x, 1)$. Denote the point $(0, \ldots, 0, 1) \in \mathbb{R}^{n+2}$ by N. The set $S^{n+1} - H(-N)$ is denoted by

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$S^{n+1}_{N,+}$. Let $\alpha_N : S^{n+1}_{N,+} \to \mathbb{R}^{n+1} \times \{1\}$ be the central projection relative to N, namely, α_N is defined as follows for any $P = (P_1, \ldots, P_{n+1}, P_{n+2}) \in S^{n+1}_{N,+}$ (see Figure 2):

$$\alpha_N (P_1, \ldots, P_{n+1}, P_{n+2}) = \left(\frac{P_1}{P_{n+2}}, \ldots, \frac{P_{n+1}}{P_{n+2}}, 1\right).$$

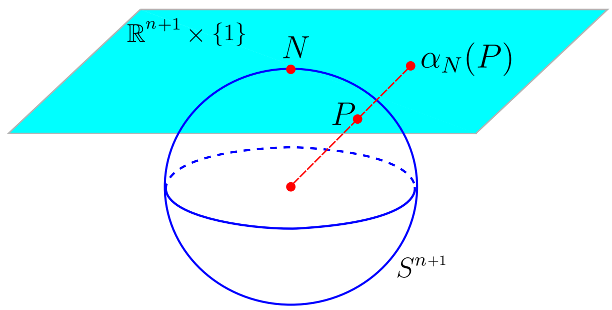

FIGURE 2. The central projection α_N.

Next, we consider the mapping $\Psi_N : S^{n+1} - \{\pm N\} \to S^{n+1}_{N,+}$ (see Figure 3), defined by

$$\Psi_N(P) = \frac{1}{\sqrt{1 - (N \cdot P)^2}}(N - (N \cdot P)P).$$

The mapping Ψ_N was introduced in [5], has the following intriguing properties:

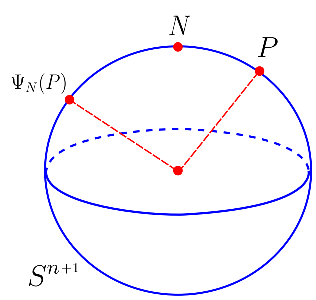

FIGURE 3. $P \cdot \Psi_N(P) = 0$.

(1) For any $P \in S^{n+1} - \{\pm N\}$, the equality $P \cdot \Psi_N(P) = 0$ holds,
(2) for any $P \in S^{n+1} - \{\pm N\}$, the property $\Psi_N(P) \in \mathbb{R}N + \mathbb{R}P$ holds,
(3) for any $P \in S^{n+1} - \{\pm N\}$, the property $N \cdot \Psi_N(P) > 0$ holds,
(4) the restriction $\Psi_N|_{S^{n+1}_{N,+} - \{N\}} : S^{n+1}_{N,+} - \{N\} \to S^{n+1}_{N,+} - \{N\}$ is a C^∞ diffeomorphism.

For any point $P \in S^{n+1}$, let $H(P)$ be the closed hemisphere centered at P, namely,

$$H(P) = \{Q \in S^{n+1} | P \cdot Q \geq 0\},$$

where the dot in the center stands for the scalar product of two vectors $P, Q \in \mathbb{R}^{n+2}$. For any non-empty subset $\widetilde{W} \subset S^{n+1}$, the *spherical polar set of* $\widetilde{W}$, denoted by $\widetilde{W}^\circ$, is defined as follows:

$$\widetilde{W}^\circ = \bigcap_{P \in \widetilde{W}} H(P).$$

for details on spherical polar set, see for instance [1, 6]

Proposition 1 ([6]). *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Let graph$(\gamma) = \{(\theta, \gamma(\theta)) \in \mathbb{R}^{n+1} - \{0\} \mid \theta \in S^n\}$, where $(\theta, \gamma(\theta))$ is the polar plot expression for a point of $\mathbb{R}^{n+1} - \{0\}$. Then, $\mathcal{W}_\gamma$ is characterized as follows:*

$$\mathcal{W}_\gamma = Id^{-1} \circ \alpha_N \left(\left(\Psi_N \circ \alpha_N^{-1} \circ Id\left(\mathrm{graph}(\gamma)\right)\right)^\circ \right).$$

Proposition 2 ([6]). *For any Wulff shape $\mathcal{W}_\gamma$, the following set, too, is a Wulff shape:*

$$Id^{-1} \circ \alpha_N \left(\left(\alpha_N^{-1} \circ Id\left(\mathcal{W}_\gamma\right)\right)^\circ \right).$$

Definition 1 ([6]). Let $\mathcal{W}_\gamma$ be a Wulff shape. The Wulff shape given in Proposition 2 is called the *dual Wulff shape* of $\mathcal{W}_\gamma$.

A Wulff shape $\mathcal{W}_\gamma$ said to be *self-dual Wulff shape* if the equality $W_\gamma = Id^{-1} \circ \alpha_N \left(\left(\alpha_N^{-1} \circ Id\left(\mathcal{W}_\gamma\right)\right)^\circ \right)$ holds, for details on self-dual Wulff shapes, see for instance [4].

The mapping inv : $\mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$, defined as follows, is called the *inversion* with respect to the origin of $\mathbb{R}^{n+1}$.

$$\mathrm{inv}(\theta, r) = \left(-\theta, \frac{1}{r} \right).$$

Let Γ_γ be the boundary of the convex hull of inv(graph(γ)). If the equality $\Gamma_\gamma =$ inv(graph(γ)) is satisfied, then γ is called a *convex integrand*. The notion of convex integrand was firstly introduced by J. Taylor in [8].

2. Main Results

Theorem 1 ([2]). *Let $W \subset \mathbb{R}^{n+1}$ be a convex body containing the origin of $\mathbb{R}^{n+1}$ as an interior point of W. Then, W is strictly convex if and only if its convex integrand γ_W is of class C^1.*

Theorem 2 ([3]). *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function and let $\mathcal{W}_\gamma$ be the Wulff shape associated with γ. Suppose that the boundary of $\mathcal{W}_\gamma$ is a C^1 submanifold. Then, γ must be the convex integrand of $\mathcal{W}_\gamma$.*

3. Applications of Theorem 1

Since the boundary of the convex hull of a C^1 closed submanifold is a C^1 closed submanifold (for instance, see [7, 10]), as a corollary of Theorem 1, we have the following:

Corollary 1 ([2]). *Let $\gamma : S^n \to \mathbb{R}_+$ be a function of class C^1. Then, $\mathcal{W}_\gamma$ is strictly convex.*

In particular, we have the following:

Corollary 2 ([6], Theorem 1.3)**.** *Let $\gamma : S^n \to \mathbb{R}_+$ be a function of class C^1. Then, $\mathcal{W}_\gamma$ is never a polytope.*

On the other hand, the converse of Corollary 1 does not hold in general (see Figure 4).

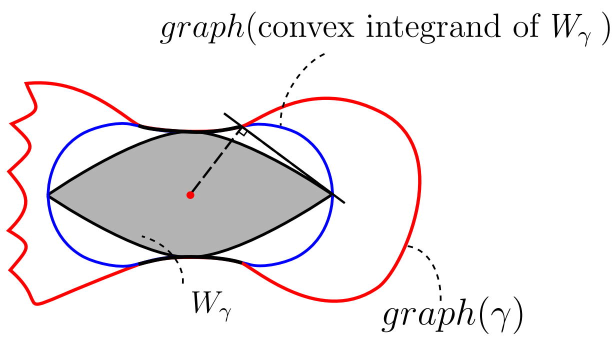

FIGURE 4. A strictly convex Wulff shape $\mathcal{W}_\gamma$ having non smooth support function γ.

Combining Theorem 1 and Proposition 1 yields the following:

Corollary 3 ([2])**.** *A Wulff shape in $\mathbb{R}^{n+1}$ is strictly convex if and only if the boundary of its dual Wulff shape is C^1 diffeomorphic to S^n.*

In particular, we have the following:

Corollary 4 ([2])**.** *A Wulff shape in $\mathbb{R}^{n+1}$ is strictly convex and its boundary is C^1 diffeomorphic to S^n if and only if its dual Wulff shape is strictly convex and the boundary of it is C^1 diffeomorphic to S^n.*

It is interesting to compare Corollary 4 and the following proposition:

Proposition 3 ([6])**.** *A Wulff shape in $\mathbb{R}^{n+1}$ is a polytope if and only if its dual Wulff shape is a polytope.*

Finally, we give an application of Theorem 1 from the view point of pedal.

Definition 2 ([2])**.** Let p (resp., $F : S^n \to \mathbb{R}^{n+1}$) be a point of $\mathbb{R}^{n+1}$ (resp., a C^1 embedding). Then, the *pedal of $F(S^n)$ relative to p* is the mapping $G : S^n \to \mathbb{R}^{n+1}$ which maps $\theta \in S^n$ to the nearest point in the tangent hyperplane to $F(S^n)$ at $F(\theta)$ from the given point p.

Let W be a Wulff shape in $\mathbb{R}^{n+1}$. Suppose that ∂W is C^1 diffeomorphic to S^n. Then, ∂W may be regarded as the graph of a certain C^1 embedding $F : S^n \to \mathbb{R}^{n+1}$, and γ_W is exactly the pedal of ∂W relative to the origin. Theorem 1 gives a sufficient condition for the pedal of ∂W relative to the origin to be smooth:

Corollary 5 ([2])**.** *Suppose that a Wulff shape W in $\mathbb{R}^{n+1}$ is strictly convex and its boundary is C^1 diffeomorphic to S^n. Then, the pedal of ∂W relative to the any interior point of W is of class C^1.*

References

[1] H. Han and T. Nishimura, *The spherical dual transform is an isometry for spherical Wulff shapes*, preprint (available from arXiv:1504.02845 [math.MG]).

[2] H. Han and T. Nishimura, *Strictly convex Wulff shapes and C^1 convex integrands* , preprint (available from arXiv:1507.05162 [math.MG]).

[3] H. Han and T. Nishimura, *Uniqueness of the surface energy density for a Wulff shape with C^1 boundary* , preprint (available from arXiv:1509.02786 [math.MG]).

[4] H. Han and T. Nishimura, *Self-dual Wulff shapes and spherical convex bodies of constant width $\pi/2$* , preprint (available from arXiv:1511.04165 [math.MG]).

[5] T. Nishimura, *Normal forms for singularities of pedal curves produced by non-singular dual curve germs in S^n*, Geom Dedicata **133**(2008), 59–66.

[6] T. Nishimura and Y. Sakemi, *Topological aspect of Wulff shapes*, J. Math. Soc. Japan, **66** (2014), 89–109.

[7] S. A. Robertson and M. C. Romero-Fuster, *The convex hull of a hypersurface*, Proc. London Math. Soc., **50**(1985), 370–384.

[8] J. E. Taylor, *Crystalline variational problems*, Bull. Amer. Math. Soc., **84**(1978), 568–588.

[9] G. Wulff, *Zur frage der geschwindindigkeit des wachstrums und der auflösung der krystallflachen*, Z. Kristallographine und Mineralogie, **34**(1901), 449–530.

[10] V. M. Zakalyukin, *Singularities of convex hulls of smooth maniifolds*, Functional Anal. Appl., **11**(1977), 225–227(1978).

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