STRICTLY CONVEX WULFF SHAPES
AND $C^1$ CONVEX INTEGRANDS

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Abstract. Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function and let $W_\gamma$ be the Wulff shape associated with $\gamma$. We show that Wulff shape $W_\gamma$ is strictly convex if and only if convex integrand of $W_\gamma$ is of class $C^1$. We also show that if the boundary of $W_\gamma$ is a $C^1$ submanifold, then $\gamma$ must be the convex integrand of $W_\gamma$.

1. Introduction

Let $n$ be a positive integer. Given a continuous function $\gamma : S^n \to \mathbb{R}_+$ where $S^n \subset \mathbb{R}^{n+1}$ is the unit sphere and $\mathbb{R}_+$ is the set consisting of positive real numbers, the Wulff shape associated with $\gamma$, denoted by $W_\gamma$, is the following intersection

$$W_\gamma = \bigcap_{\theta \in S^n} \Gamma_{\gamma, \theta}.$$ 

Here, $\Gamma_{\gamma, \theta}$ is the following half-space:

$$\Gamma_{\gamma, \theta} = \{ x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta) \}.$$

By definition, the Wulff shape $W_\gamma$ is a convex body such that the origin of $\mathbb{R}^{n+1}$ is an interior point of $W_\gamma$. The notion of Wulff shape was first introduced by G. Wulff in [9]. Let $Id : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \{1\}$ be the map defined by $Id(x) = (x, 1)$. Denote the point $(0, \ldots, 0, 1) \in \mathbb{R}^{n+2}$ by $N$. The set $S^{n+1} - H(-N)$ is denoted by

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{wulff_shape.png}
\caption{A Wulff shape $W_\gamma$.}
\end{figure}

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Let $\alpha_N : S_{N,+}^{n+1} \to \mathbb{R}^{n+1} \times \{1\}$ be the central projection relative to $N$, namely, $\alpha_N$ is defined as follows for any $P = (P_1, \ldots, P_{n+1}, P_{n+2}) \in S_{N,+}^{n+1}$ (see Figure 2):

$$\alpha_N (P_1, \ldots, P_{n+1}, P_{n+2}) = \left( \frac{P_1}{P_{n+2}}, \ldots, \frac{P_{n+1}}{P_{n+2}}, 1 \right).$$

Next, we consider the mapping $\Psi_N : S_{N,+}^{n+1} - \{\pm N\} \to S_{N,+}^{n+1}$ (see Figure 3), defined by

$$\Psi_N(P) = \frac{1}{\sqrt{1 - (N \cdot P)^2}} (N - (N \cdot P)P).$$

The mapping $\Psi_N$ was introduced in [5], has the following intriguing properties:

1. For any $P \in S_{N,+}^{n+1} - \{\pm N\}$, the equality $P \cdot \Psi_N(P) = 0$ holds,
2. for any $P \in S_{N,+}^{n+1} - \{\pm N\}$, the property $\Psi_N(P) \in \mathbb{R}N + \mathbb{R}P$ holds,
3. for any $P \in S_{N,+}^{n+1} - \{\pm N\}$, the property $N \cdot \Psi_N(P) > 0$ holds,
4. the restriction $\Psi_N |_{S_{N,+}^{n+1} - \{N\}} : S_{N,+}^{n+1} - \{N\} \to S_{N,+}^{n+1} - \{N\}$ is a $C^\infty$ diffeomorphism.

For any point $P \in S_{N,+}^{n+1}$, let $H(P)$ be the closed hemisphere centered at $P$, namely,

$$H(P) = \{Q \in S_{N,+}^{n+1} | P \cdot Q \geq 0\},$$
where the dot in the center stands for the scalar product of two vectors $P, Q \in \mathbb{R}^{n+2}$.

For any non-empty subset $\tilde{W} \subset S^{n+1}$, the spherical polar set of $\tilde{W}$, denoted by $\tilde{W}^\circ$, is defined as follows:

$$\tilde{W}^\circ = \bigcap_{P \in \tilde{W}} H(P).$$

for details on spherical polar set, see for instance [1, 6]

\textbf{Proposition 1 ([6]).} Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Let $\text{graph}(\gamma) = \{(\theta, \gamma(\theta)) \in \mathbb{R}^{n+1} - \{0\} \mid \theta \in S^n\}$, where $(\theta, \gamma(\theta))$ is the polar plot expression for a point of $\mathbb{R}^{n+1} - \{0\}$. Then, $\mathcal{W}_\gamma$ is characterized as follows:

$$\mathcal{W}_\gamma = \text{Id}^{-1} \circ \alpha_N \left(\left(\Psi_N \circ \alpha_N^{-1} \circ \text{Id} \left(\text{graph}(\gamma)\right)\right)\right).$$

\textbf{Proposition 2 ([6]).} For any Wulff shape $\mathcal{W}_\gamma$, the following set, too, is a Wulff shape:

$$\text{Id}^{-1} \circ \alpha_N \left(\left(\alpha_N^{-1} \circ \text{Id} \left(\mathcal{W}_\gamma\right)\right)\right).$$

\textbf{Definition 1 ([6]).} Let $\mathcal{W}_\gamma$ be a Wulff shape. The Wulff shape given in Proposition 2 is called the dual Wulff shape of $\mathcal{W}_\gamma$.

A Wulff shape $\mathcal{W}_\gamma$ said to be self-dual Wulff shape if the equality $W_\gamma = \text{Id}^{-1} \circ \alpha_N \left(\left(\alpha_N^{-1} \circ \text{Id} \left(\mathcal{W}_\gamma\right)\right)\right)$ holds, for details on self-dual Wulff shapes, see for instance [4].

The mapping $\text{inv} : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$, defined as follows, is called the \text{inversion} with respect to the origin of $\mathbb{R}^{n+1}$.

$$\text{inv}(\theta, r) = \left(-\theta, \frac{1}{r}\right).$$

Let $\Gamma_\gamma$ be the boundary of the convex hull of $\text{inv}(\text{graph}(\gamma))$. If the equality $\Gamma_\gamma = \text{inv}(\text{graph}(\gamma))$ is satisfied, then $\gamma$ is called a \text{convex integrand}. The notion of convex integrand was firstly introduced by J. Taylor in [8].

2. \textbf{Main Results}

\textbf{Theorem 1 ([2])}. Let $W \subset \mathbb{R}^{n+1}$ be a convex body containing the origin of $\mathbb{R}^{n+1}$ as an interior point of $W$. Then, $W$ is strictly convex if and only if its convex integrand $\gamma_W$ is of class $C^1$.

\textbf{Theorem 2 ([3])}. Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function and let $\mathcal{W}_\gamma$ be the Wulff shape associated with $\gamma$. Suppose that the boundary of $\mathcal{W}_\gamma$ is a $C^1$ submanifold. Then, $\gamma$ must be the convex integrand of $\mathcal{W}_\gamma$.

3. \textbf{Applications of Theorem 1}

Since the boundary of the convex hull of a $C^1$ closed submanifold is a $C^1$ closed submanifold (for instance, see [7, 10]), as a corollary of Theorem 1, we have the following:

\textbf{Corollary 1 ([2])}. Let $\gamma : S^n \to \mathbb{R}_+$ be a function of class $C^1$. Then, $\mathcal{W}_\gamma$ is strictly convex.

In particular, we have the following:
Corollary 2 ([6], Theorem 1.3). Let \( \gamma : S^n \to \mathbb{R}_+ \) be a function of class \( C^1 \). Then, \( W_\gamma \) is never a polytope.

On the other hand, the converse of Corollary 1 does not hold in general (see Figure 4).

\[ \text{graph(\text{convex integrand of } W_\gamma)} \]

\[ \text{graph(\gamma)} \]

**Figure 4.** A strictly convex Wulff shape \( W_\gamma \) having non smooth support function \( \gamma \).

Combining Theorem 1 and Proposition 1 yields the following:

**Corollary 3** ([2]). A Wulff shape in \( \mathbb{R}^{n+1} \) is strictly convex if and only if the boundary of its dual Wulff shape is \( C^1 \) diffeomorphic to \( S^n \).

In particular, we have the following:

**Corollary 4** ([2]). A Wulff shape in \( \mathbb{R}^{n+1} \) is strictly convex and its boundary is \( C^1 \) diffeomorphic to \( S^n \) if and only if its dual Wulff shape is strictly convex and the boundary of it is \( C^1 \) diffeomorphic to \( S^n \).

It is interesting to compare Corollary 4 and the following proposition:

**Proposition 3** ([6]). A Wulff shape in \( \mathbb{R}^{n+1} \) is a polytope if and only if its dual Wulff shape is a polytope.

Finally, we give an application of Theorem 1 from the view point of pedal.

**Definition 2** ([2]). Let \( p \) (resp., \( F : S^n \to \mathbb{R}^{n+1} \)) be a point of \( \mathbb{R}^{n+1} \) (resp., a \( C^1 \) embedding). Then, the pedal of \( F(S^n) \) relative to \( p \) is the mapping \( G : S^n \to \mathbb{R}^{n+1} \) which maps \( \theta \in S^n \) to the nearest point in the tangent hyperplane to \( F(S^n) \) at \( F(\theta) \) from the given point \( p \).

Let \( W \) be a Wulff shape in \( \mathbb{R}^{n+1} \). Suppose that \( \partial W \) is \( C^1 \) diffeomorphic to \( S^n \). Then, \( \partial W \) may be regarded as the graph of a certain \( C^1 \) embedding \( F : S^n \to \mathbb{R}^{n+1} \), and \( \gamma_\omega \) is exactly the pedal of \( \partial W \) relative to the origin. Theorem 1 gives a sufficient condition for the pedal of \( \partial W \) relative to the origin to be smooth:

**Corollary 5** ([2]). Suppose that a Wulff shape \( W \) in \( \mathbb{R}^{n+1} \) is strictly convex and its boundary is \( C^1 \) diffeomorphic to \( S^n \). Then, the pedal of \( \partial W \) relative to the any interior point of \( W \) is of class \( C^1 \).
References


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