Self-avoiding walk on random conductors

Yuki Chino*
Department of Mathematics
Hokkaido University

Joint work with Akira Sakai (Hokkaido University)†

Self-avoiding walk (SAW) is a statistical-mechanical model that the chemist P. J. Froly first introduced for studying the behavior of linear polymers [4, 5]. Now we have many rigorous results on SAW, especially in $d > 4$ due to the lace expansion [1, 7]. However in two or three dimensions, there still remain open problems [8]. In 1981, B. K. Chakrabarti and J. Kartész first introduced the random environment to SAW [2]. Our interest is to understand how the random environment affects the behavior of the observables concerning SAW around the critical point. In this talk, we will show the quenched critical point is almost surely a constant and estimate upper and lower bounds.

Model and the results

Let $\mathbb{B}^d$ denote the set of nearest-neighbor bonds in $\mathbb{Z}^d$, let $\Omega(x)$ be the set of nearest-neighbor self-avoiding paths on $\mathbb{Z}^d$ from x. The self-avoiding walk is the set of the trajectries of the walk that can not return the point once it visited. We call this property self-avoidance constraint. By this property, we can regard SAW paths as the statistical-mechanical model for linear polymers. Denoting the length of ω by $|\omega|$ (i.e., $|\omega| = n$ for $\omega = (\omega_0, \ldots, \omega_n)$) and the energy cost of a bond between consecutive monomers by $h \in \mathbb{R}$, we define the susceptibility as

$$\chi_h = \sum_{\omega \in \Omega(x)} e^{-h|\omega|},$$

which is independent of the location of the reference point $x \in \mathbb{Z}^d$. Two other key observables are the two-point function and the number of SAWs

*chino@math.sci.hokudai.ac.jp
†sakai@math.sci.hokudai.ac.jp

of length n:

$$G_h(x) = \sum_{\omega \in \Omega(o,x)} e^{-h|\omega|}, \qquad c(n) = \sum_{\omega \in \Omega(x)} \mathbf{1}_{\{|\omega|=n\}},$$

where o is the origin of $\mathbb{Z}^d$, $\mathbf{1}_{\{\cdots\}}$ is the indicator function, and $\Omega(o,x)$ is the set of nearest-neighbor self-avoiding paths on $\mathbb{Z}^d$ from o to x. Obviously,

$$\chi_h = \sum_{x \in \mathbb{Z}^d} G_h(x) = \sum_{n=0}^{\infty} e^{-hn} c(n).$$

Due to subadditivity of $c(n)$, we can show that $\chi_h < \infty$ if and only if $h > \log \mu$, where μ is the connective constant for SAW [7]:

$$\mu = \lim_{n \to \infty} c(n)^{1/n} = \inf_n c(n)^{1/n}.$$

Therefore, $h = \log \mu$ is the critical point of the susceptibility. Many rigorous results on the behavior of these observables around the critical point $h = \log \mu$ have been proven. However, there still remain many challenging open problems in two and three dimensions. See [8] and the references therein.

Let $\boldsymbol{X} = \{X_b\}_{b \in \mathbb{B}^d}$ be a collection of i.i.d. bounded random variables whose law and expectation are denoted by $\mathbb{P}_{\boldsymbol{X}}$ and $\mathbb{E}_{\boldsymbol{X}}$, respectively. Similarly to the homogeneous case, we define the quenched susceptibility at $x \in \mathbb{Z}^d$:

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{\omega \in \Omega(x)} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})},$$

where b_j is the j-th bond of ω. Because of the inhomogeneity of $\boldsymbol{X}$, the quenched susceptibility is not translation invariant and does depend on the location of the reference point x. We also define the random media version counterpart of the number of SAWs $c(n)$ in random environment:

$$\hat{c}_{\beta,\boldsymbol{X}}(x;n) = \sum_{\omega \in \Omega(x)} e^{-\beta \sum_{j=1}^{|\omega|} X_{b_j}} \mathbf{1}_{\{|\omega|=n\}}.$$

Therefore, we have

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{n=0}^{\infty} e^{-hn} \hat{c}_{\beta,\boldsymbol{X}}(x;n).$$

Since $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ is monotonic in h, we can define the quenched critical point:

$$\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x) = \inf\{h \in \mathbb{R} : \hat{\chi}_{h,\beta,\boldsymbol{X}}(x) < \infty\}.$$

We denote $c(n)$ be the number of the homogeneous SAWs. By virtue of the self-avoidance constraint on ω and the i.i.d. property of $\boldsymbol{X}$, we can directly compute the annealed susceptibility $\mathbb{E}_{\boldsymbol{X}}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]$

$$\mathbb{E}_{\boldsymbol{X}}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] = \sum_{n=0}^{\infty} e^{-(h-\log\lambda_\beta)n} c(n),$$

where $\lambda_\beta = \mathbb{E}_{\boldsymbol{X}}[e^{-\beta X_b}]$. Then the annealed critical point must be defined:

$$h_\beta^{\mathsf{a}} = \log\mu + \log\lambda_\beta,$$

so that $\mathbb{E}_{\boldsymbol{X}}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] < \infty$ if and only if $h > h_\beta^{\mathsf{a}}$. By Jensen's inequality,

$$h_\beta^{\mathsf{a}} \geq \log\mu - \beta\mathbb{E}_{\boldsymbol{X}}[X_b].$$

The following theorem is the main result of this talk.

Theorem 1. *Let $d \geq 1$ and $\beta \geq 0$. The quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ is $\mathbb{P}_{\boldsymbol{X}}$-almost surely a constant that does not depend on the location of the reference point $x \in \mathbb{Z}^d$. Moreover, by abbreviating $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ as $\hat{h}^{\mathsf{q}}_\beta$, we have*

$$\log\mu - \beta\mathbb{E}_{\boldsymbol{X}}[X_b] \leq \hat{h}^{\mathsf{q}}_\beta \leq h^{\mathsf{a}}_\beta, \qquad \mathbb{P}_{\boldsymbol{X}}\text{-almost surely.}$$

For $d = 1$, in particular, the lower bound is an equality.

The key elements for the proof are the following:

- To prove that the quenched critical point is $\mathbb{P}_{\boldsymbol{X}}$-a.s. a constant we show translation invariance and ergodicity by following similar analysis to that in H. Lacoin [6].
- The upper bound (generally called the annealed bound) is trivial. On the other hand, the lower bound is derived from the second moment estimate by using the Paley-Zygmund inequality.

In our main theorem, for $d = 1$, the lower bound is an equality. We want to discuss what happens in $d \geq 2$:

- In two dimension, H. Lacoin improve the upper bound by using fractional moment method [6].
- In high dimensions, can we estimate the upper and lower bound more sharply??

References

[1] D. Brydges and T. Spencer. Self-avoiding walk in 5 or more dimensions. *Commun. Math. Phys.* **97** (1985): 125–148.

[2] B. K. Chakrabarti and J. Kartész. The statistics of self-avoiding walk on a disordered lattice. *Z. Phys. B Cond. Mat.* **44** (1981): 221–223

[3] Y. Chino and A. Sakai. The quenched critical point for self-avoiding walk on random conductors. arXiv:1508.01262

[4] P. J. Flory. The configuration of a real polymer chain. *J. Chem. Phys.* **17** (1949): 303–310.

[5] P. J. Flory. *Principles of Polymer Chemistry* (Cornell Univ. Press, 1953).

[6] H. Lacoin. Non-coincidence of quenched and annealed connective constants on the supercritical planar percolation cluster. *Probab. Theory Relat. Fields* **159** (2014): 777–808.

[7] N. Madras and G. Slade. *The Self-Avoiding Walk* (Birkhäuser, 2013).

[8] G. Slade. The self-avoiding walk: A brief survey. *Surveys in Stochastic Processes* (J. Blath et al. eds., European Mathematical Society, 2011): 181–199.