TOPOLOGICAL CLASSIFICATION OF MAP GERMS USING REEB GRAPHS

ERICA BOIZAN BATISTA

1. Introduction

The classification problem of singular points of C^∞ map germs is one of the most important problems in Singularity theory. The classical classification is done via $\mathcal{A}$-equivalence, where we take C^∞-diffeomorphism germs in the source and the target. However, this is a difficult problem and it presents a lot of rigidity. Then it seems natural to investigate the classification of map germs up to weaker equivalence relations. Here we consider topological equivalence or C^0-$\mathcal{A}$-equivalence, where the changes of coordinates are homeomorphisms instead of C^∞-diffeomorphisms.

This work is devoted to the topological classification of C^∞ map germs from $\mathbb{R}^3$ to $\mathbb{R}^2$ which are finitely determined. The topological structure of a finitely determined map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is determined by the so-called link of f (cf. [6]). The link of f is obtained by taking a small enough representative $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ and the intersection of its image with a small enough sphere S^1_δ centered at the origin in $\mathbb{R}^2$. When f has isolated zeros (i.e., $f^{-1}(0) = \{0\}$), the link is a stable map $\gamma : S^2 \to S^1$ and f is topologically equivalent to the cone of γ. As a consequence, two finitely determined map germs $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ are topologically equivalent if their associated links are topologically equivalent.

2. Finite determinacy and the link of a map germ

Two C^∞ map germs $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ are *$\mathcal{A}$-equivalent* if there exist C^∞-diffeomorphism germs $\psi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ and $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f = \phi \circ g \circ \psi$. If ϕ, ψ are homeomorphisms instead of C^∞-diffeomorphisms, then we say that f and g are *topologically equivalent* (or *C^0-$\mathcal{A}$-equivalent*).

For simplicity, we will write just diffeomorphism instead of C^∞-diffeomorphism.

We say that $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is k-determined if for any map germ g with the same k-jet, we have that g is $\mathcal{A}$-equivalent to f. We say that f is finitely determined if it is k-determined for some k.

Let $f : U \to \mathbb{R}^2$ be a C^∞ map, where $U \subset \mathbb{R}^3$ is an open subset. We denote by $S(f) = \{p \in U \mid Jf(p) \text{ does not have rank } 2\}$ the *singular set* of f, where $Jf(p)$ is the Jacobian matrix of f. We also denote the *discriminant set* of f by $\Delta(f) = f(S(f))$.

Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then there exists a representative $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ such that

i) $S(f) \cap f^{-1}(0) = \{0\}$,
ii) the restriction $f|U - \{0\}$ has only definite and indefinite simple fold singularities.

If f is finitely determined, then its discriminant $\Delta(f)$ is a plane curve with an isolated singularity at the origin. The number of half branches of $\Delta(f)$ will play a crucial role in

Key words and phrases. topological equivalence, classification, link, Reeb graph.

the analysis of the Reeb graph associated to link of f and consequently, in the topological classification of f.

Denote by $J^r(n,p)$ the r-jet space from $(\mathbb{R}^n,0)$ to $(\mathbb{R}^p,0)$. For positive integers r and s with $s \geq r$, let $\pi^s_r : J^s(n,p) \to J^r(n,p)$ be the canonical projection defined by $\pi^s_r(j^s f(0)) = j^r f(0)$. For a positive number $\epsilon > 0$ we set

$$D^n_\epsilon = \{x \in \mathbb{R}^n \mid \|x\|^2 \leq \epsilon\},\ B^n_\epsilon = \{x \in \mathbb{R}^n \mid \|x\|^2 < \epsilon\} \text{ and } S^{n-1}_\epsilon = \{x \in \mathbb{R}^n \mid \|x\|^2 = \epsilon\}.$$

We denote D^n, B^n and S^{n-1} the standard disk, ball and sphere of radius 1, respectively.
T. Fukuda has proved the following cone structure theorem in his papers [5, 6]:

Theorem 2.1. *For any semialgebraic subset W of $J^r(n,p)$, there exist an integer s ($s \geq r$) depending only on n,p and r, and there exists a closed semialgebraic subset Σ_W of $(\pi^s_r)^{-1}(W)$ having codimension ≥ 1 such that for any C^∞ map $f : \mathbb{R}^n \to \mathbb{R}^p$ with $j^s f(0)$ belonging to $(\pi^s_r)^{-1}(W) \setminus \Sigma_W$ we have the following properties:*

(A) The case $f^{-1}(0) = \{0\}$: *there is $\epsilon_0 > 0$ such that for any number ϵ with $0 < \epsilon \leq \epsilon_0$ we have:*

(A-i) *the set $\tilde{S}^{n-1}_\epsilon = f^{-1}(S^{p-1}_\epsilon)$ is a C^∞ submanifold without boundary which is diffeomorphic to the standard unit sphere S^{n-1}.*

(A-ii) *The restricted map $f|\tilde{S}^{n-1}_\epsilon : \tilde{S}^{n-1}_\epsilon \to S^{p-1}_\epsilon$ is topologically stable (C^∞ stable if (n,p) is a nice pair in Mather's sense).*

(A-iii) *If $\tilde{D}^{n-1}_\epsilon = f^{-1}(D^{p-1}_\epsilon)$, then the restricted map $f|\tilde{D}^{n-1}_\epsilon : \tilde{D}^{n-1}_\epsilon \to D^p_\epsilon$ is topologically equivalent to the cone of $f|\tilde{S}^{n-1}_\epsilon$.*

(B) The case $f^{-1}(0) \neq \{0\}$: *there exist a positive number ϵ_0 and a strictly increasing C^∞ function $\delta : [0,\epsilon_0] \to [0,\infty)$ with $\delta(0) = 0$ such that for every ϵ and δ with $0 < \epsilon \leq \epsilon_0$ and $0 < \delta \leq \delta(\epsilon)$ we have:*

(B-i) *$f^{-1}(0) \cap S^{n-1}_\epsilon$ is an $(n-p-1)$-dimensional manifold and it is diffeomorphic to $f^{-1}(0) \cap S^{n-1}_{\epsilon_0}$.*

(B-ii) *$D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)$ is a C^∞ manifold, in general with boundary and it is diffeomorphic to $D^n_{\epsilon_0} \cap f^{-1}(S^{p-1}_{\delta(\epsilon_0)})$.*

(B-iii) *the restriction $f|D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta) : D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta) \to S^{p-1}_\delta$ is a topologically stable map (C^∞ stable if (n,p) is a nice pair in Mather's sense) and its topological class is independent of ϵ and δ.*

Assuming that f is r-determined for some r and taking $W = \{j^r f(0)\}$, we can apply Theorem 2.1 to obtain a representative of f satisfying (A) or (B), depending on if $f^{-1}(0) = \{0\}$ or $f^{-1}(0) \neq \{0\}$. Note that when $n \leq p$ we always have $f^{-1}(0) = \{0\}$ but when $n > p$ we may have the two possibilities.

Definition 2.2. Let $f : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ be a finitely determined map germ such that $f^{-1}(0) = \{0\}$. We say that the stable map $f|\tilde{S}^2_\epsilon : \tilde{S}^2_\epsilon \to S^1_\epsilon$ is the *link* of f, where f is a representative that satisfies the Fukuda's conditions (A) of Theorem 2.1 adapted for case $n = 3$ and $p = 2$.

Corollary 2.3. *Two finitely determined map germs $f,g : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ with $f^{-1}(0) = \{0\} = g^{-1}(0)$ are topologically equivalent if their associated links are topologically equivalent.*

3. The generalized Reeb graph

The Reeb graph was introduced by Reeb in [7] and it is well known that it is a complete topological invariant for Morse functions from S^2 to $\mathbb{R}$ (see [1]).

Proposition 3.1. *Let $\gamma : S^2 \to S^1$ be a stable map. Then γ is not a regular map.*

Given a continuous map $f : X \to Y$ between topological spaces, we consider the following equivalence relation on X: $x \sim y \Leftrightarrow f(x) = f(y)$ and x and y are in the same connected component of $f^{-1}(f(x))$.

Proposition 3.2. *Let $\gamma : S^2 \to S^1$ be a stable map. Then the quotient space $S^2/\sim$ admits the structure of a connected graph in the following way:*

(1) *the vertices are the connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a critical value;*
(2) *each edge is formed by points that correspond to connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a regular value.*

Each vertex of the graph can be of three topological types, depending on if the connected component has a maximum/minimum critical point, a saddle point or just regular points.

Let $v_1, \ldots, v_r \in S^1$ be the critical values of γ. We choose a base point $v_0 \in S^1$ and an orientation. We can reorder the critical values such that $v_0 \leq v_1 < \ldots < v_r$ and we label each vertex with the index $i \in \{1, \ldots, r\}$, if it corresponds to the critical value v_i.

Definition 3.3. The graph given by $S^2/\sim$ together with the the labels of the vertices, as previously defined, is said to be the *generalized Reeb graph* associated to $\gamma : S^2 \to S^1$.

For simplicity, from now on we will just call Reeb graph to the generalized Reeb graph, unless otherwise specified.

Proposition 3.4. *Let $\gamma : S^2 \to S^1$ be a stable map. Then the Reeb graph of γ is a tree.*

Let $\gamma, \delta : S^2 \to S^1$ be two stable maps. Let Γ_γ and Γ_δ be their respective Reeb graphs. Consider the induced quotient maps $\bar{\gamma} : \Gamma_\gamma \to S^1_\gamma$ and $\bar{\delta} : \Gamma_\delta \to S^1_\delta$, where S^1_γ, S^1_δ denote S^1 with the graph structure whose vertices are the critical values of γ, δ respectively.

Definition 3.5. An *isomorphism* between two graphs Γ_1 and Γ_2 is a bijection f from $V(\Gamma_1)$ to $V(\Gamma_2)$ such that two vertices v and w are adjacent in Γ_1 if and only if $f(v)$ and $f(w)$ are adjacent in Γ_2, where $V(\Gamma_i) = \{\text{vertices of } \Gamma_i\}$.

Definition 3.6. We say that Γ_γ *is equivalent to* Γ_δ and we denote it by $\Gamma_\gamma \sim \Gamma_\delta$, if there exist graph isomorphisms $j : \Gamma_\gamma \to \Gamma_\delta$ and $l : S^1_\gamma \to S^1_\delta$, such that the following diagram is commutative:

$$\begin{array}{ccc} V_\gamma & \xrightarrow{\bar{\gamma}|V_\gamma} & \Delta_\gamma \\ {\scriptstyle j|V_\gamma}\downarrow & & \downarrow{\scriptstyle l|\Delta_\gamma} \\ V_\delta & \xrightarrow{\bar{\delta}|V_\delta} & \Delta_\delta \end{array}$$

where $V_\gamma = \{\text{vertices of } \Gamma_\gamma\}$, $V_\delta = \{\text{vertices of } \Gamma_\delta\}$ and Δ_γ and Δ_δ are their respective discriminant sets.

Theorem 3.7. *Let $\gamma, \delta : S^2 \to S^1$ be two stable maps. If γ and δ are topologically equivalent then their respective Reeb graphs are equivalent.*

The above theorem allows us to extend the definition of Reeb graph for C^0-stable maps between topological spheres.

Theorem 3.8. *Let $\gamma, \delta : S^2 \to S^1$ be two stable maps such that $\Gamma_\gamma \sim \Gamma_\delta$. Then γ is $\mathcal{A}$-equivalent to δ.*

Corollary 3.9. *Let $\gamma, \delta : S^2 \to S^1$ be two stable maps. Then the following statements are equivalent:*

(1) *γ, δ are $\mathcal{A}$-equivalent,*
(2) *γ, δ are topologically equivalent,*
(3) *$\Gamma_\gamma \sim \Gamma_\delta$.*

Theorem 3.10. *Let $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs such that $f^{-1}(0) = \{0\} = g^{-1}(0)$. If f and g are topologically equivalent then the Reeb graphs of their links are equivalent.*

Again, Theorem 3.10 together with Corollary 2.3 and Theorem 3.8 show that the Reeb graph is a complete topological invariant for map germs from with isolated zeros.

Corollary 3.11. *Let $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be finitely determined map germs such that $f^{-1}(0) = \{0\} = g^{-1}(0)$. Then the following statements are equivalent:*

(1) *f, g are topologically equivalent,*
(2) *the Reeb graphs of the links of f, g are equivalent,*
(3) *the links of f, g are topologically equivalent.*

References

[1] V.I. Arnold, *Topological classification of Morse functions and generalisations of Hilbert's 16-th problem*, Math. Phys. Anal. Geom. Vol. 10 (2007) 227–236.

[2] E.B. Batista, J.C.F. Costa, J.J. Nuño-Ballesteros, *The Reeb graph of a map germ from $\mathbb{R}^3$ to $\mathbb{R}^2$ with isolated zeros*, to appear in Proc. Edinb. Math. Soc. (2).

[3] E.B. Batista, J.C.F. Costa, J.J. Nuño-Ballesteros, *The Reeb graph of a map germ from $\mathbb{R}^3$ to $\mathbb{R}^2$ with non isolated zeros*, preprint (2015).

[4] J.C.F. Costa, J.J. Nuño-Ballesteros, *Topological $\mathcal{K}$-classification of finitely determined map germs*, Geom. Dedicata 166 (2013) 147–162.

[5] T. Fukuda, *Local Topological Properties of Differentiable Mappings I*, Invent. Math. 65 no. 2 (1981/82) 227–250.

[6] T. Fukuda, *Local Topological Properties of Differentiable Mappings II*, Tokyo J. Math. 8, no.2 (1985) 501–520.

[7] G. Reeb, *Sur les points singuliers d'une forme de Pfaff completement intégrable ou d'une fonction numérique*, C. R. Acad. Sci. Paris 222 (1946) 847–849.

Research Institute of Environment and Information Sciences, Yokohama National University, Yokohama 240-8501, Japan

E-mail address: ericabbatista@gmail.com