On algebraic K-theory of quasi-coherent modules over spectral schemes

Mariko OHARA (Tohoku University)*

1. The main results

This is a summery of the consequence in [8]. More precise arguments or definitions, one can see [8].

1.1. The back ground

The algebraic K-theory is one of the important invariants of algebraic varieties. When we consider the Waldhousen construction of algebraic K-theory, we can find that the concept of a model category naturally appears and essentially plays an important role, so that we reach the concept of ∞-category developed by Lurie. Since the underlying ∞-categories of model categories are equivalent if the model categories are Quillen equivalent, we can regard the ∞-category as a homotopy invariant of the model structures. Also, the $\mathcal{G}$-structured ∞-topoi, the concept of the spectral schemes and their quasi-coherent sheaves are the one of the main topic of algebraic topology which is developed by Lurie.

1.2. Settings

Let $\mathcal{G}^{Sp}_{Zar}$ be the spectral Zariski geometry and $X \in \mathbf{Sch}(\mathcal{G}^{Sp}_{Zar})$ a spectral $\mathcal{G}^{Sp}_{Zar}$-scheme.

We study K-theory of spectral schemes by using quasi-coherent sheaves. we define K-theory of X by

$$K(X) = \Omega|S_\bullet(QCoh(X)^{lf})|,$$

where $QCoh(X)^{lf}$ is the ∞-category of the locally free sheaves defined in [6].

Let CAlg^{cn} be an ∞-category of connective $\mathbb{E}_\infty$-rings and CAlg^{e} a ∞-subcategory of CAlg^{cn} which consists of connective coherent $\mathbb{E}_\infty$-rings and the morphisms with the following condition: $R_1 \to R_2$ is a morphism of connective $\mathbb{E}_\infty$-rings which induces an exact functor $\mathrm{Mod}^{proj}_{R_1} \to \mathrm{Mod}^{proj}_{R_2}$. Note that any Zariski open immersion $R \to R[x^{-1}]$ for $x \in \pi_0 R$ induces an exact functor. Then we obtain the Zariski (resp. Nisnevich) ∞-topos $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^e)$ denoted by CAlg^{eZar} (resp. CAlg^{eNis}).

Let R be a connective $\mathbb{E}_\infty$-ring. We define a connective spectrum R^b to be the image of R under the morphism $\mathrm{Mod}_R \to \mathrm{Mod}^b_R$, where Mod^b_R is the ∞-subcategory of Mod_R bounded by the t-structure. We call such a spectrum R^b *bounded* since it has only finitely non-zero homotopy groups. Note that $\mathrm{Mod}^{proj}_{R^b} \simeq (\mathrm{Mod}^{proj}_R)^b$.

Let CAlg^b_{reg} be a ∞-subcategory of CAlg^{cn} which consists of bounded coherent regular $\mathbb{E}_\infty$-rings and the morphisms with the following condition: $R^b_1 \to R^b_2$ is a morphism of connective $\mathbb{E}_\infty$-rings such that the restriction of π_0 makes $\pi_0 R^b_2$ a finitely generated $\pi_0 R^b_1$-module. Those morphisms induces exact functors. Then we obtain the Zariski (resp. Nisnevich) ∞-topos $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^b_{reg})$ denoted by $\mathrm{CAlg}^{bZar}_{reg}$ (resp. $\mathrm{CAlg}^{bNis}_{reg}$).

We define a functor

$$K : \mathrm{CAlg}^{eZar} \to \widehat{\mathcal{S}} \ (resp. \mathrm{CAlg}^{eNis} \to \widehat{\mathcal{S}})$$

which carries an $\mathbb{E}_\infty$-ring A to the K-theory $K(\mathrm{Spec}^{\,g} A)$ defined above.

Let us denote the sheafification of K by $\widetilde{K}$.

*e-mail: sa9m08@math.tohoku.ac.jp

1.3. The main theorem

We regard the K-theory as a functor K on the affine spectral schemes and prove that the group completion $\Omega B(BGL)$ represents the sheafification of K with respect to Zariski (resp. Nisnevich). Here, we define BGL to be a classifying space of a colimit of affine spectral scheme GL_n. We also prove $K(R^b) \simeq K(\pi_0 R^b)$ for the bounded connective spectrum R^b.

Theorem 1.1. *Let BGL be a classifying space of a colimit of affine spectral scheme GL_n and $\Omega B(BGL)$ the group completion. Let $\mathrm{CAlg}^{e\mathcal{G}}$ be either Zariski ∞-topos CAlg^{eZar} or Nisnevich ∞-topos CAlg^{eNis}.*

(i) There is an equivalence of ∞-groupoids:

$$\mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{e\mathcal{G}})}(\mathrm{Spec}^{\,g} R,\, \Omega B(BGL)) \simeq \widetilde{K}(\mathrm{Spec}^{\,g} R).$$

(ii) $K(R^b) \simeq K(\pi_0 R)$ as a functor on $\mathrm{CAlg}^{bZar}_{reg}$ (resp. $\mathrm{CAlg}^{bNis}_{reg}$).

Let $K^f : \mathrm{CAlg}^{bZar}_{reg} \to \widehat{\mathcal{S}}$ (resp. $\mathrm{CAlg}^{bNis}_{reg} \to \widehat{\mathcal{S}}$) be a functor which carries an $\mathbb{E}_\infty$-ring A to the K-theory KA of Elmendorf-Kriz-Mandell-May. Let us denote the sheafification of K^f by $\widehat{K^f}$. Then:

(iii) $\widehat{K^f} R^b \simeq K(R^b)$, and K is a sheaf on $\mathrm{CAlg}^{bZar}_{reg}$ (resp. $\mathrm{CAlg}^{bNis}_{reg}$).

1.4. Remarks

The sheafification makes no defference between objectwize group completion a functor $R \mapsto \Omega B(BGL(R))$ and the group completion of sheaf $R \mapsto (\Omega BBGL)(R)$.

The left hand side of Theorem 1.1(i) is equivalent to $\widetilde{K^f} R$, so it gives a generalization of the consequence of Elmendorf-Kriz-Mandell-May [4, VI, Theorem 7.1] to the algebraic K-theory sheaf in certain ∞-topos.

In bounded case, by combining Theorem 1.1(ii) and (iii), a functors K^f is characterized by π_0-part of R^b.

2. Outline of the proof of Main results

In this section, we explain the outline of proof of Theorem 1.1(i). The key lemma is as follows.

Let R be an $\mathbb{E}_\infty$-ring and Mod^{nproj}_R an ∞-category of rank n projective R-modules in Mod^{cn}_R. We defined spectral affine group scheme GL_n. We regard $BGL_n(R)$ is a ∞-groupoid which consists of the single object R and equivalences as morphisms.

Lemma 2.1. *Let $X : \mathrm{CAlg}^{cn}_{\mathcal{G}} \to \widehat{\mathcal{S}}$ be a spectral sheaf and $X \to BGL_n$ a morphism in $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{cn}_{\mathcal{G}})$. Let $\mathrm{QCoh}(X)^{lf}_n$ be the ∞-category of locally free sheaves of rank n. Then there is an equivalence of ∞-groupoids;*

$$\mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{cn}_{\mathcal{G}})}(X,\, BGL_n) \simeq \mathrm{QCoh}(X)^{lf}_n.$$

Proof. Roughly speaking, for each point $x \in X$ and sufficiently small open $x \in \mathrm{Spec}^{\,g} A \to X$, we assign the stalk of the morphism $X \to BGL_n$ at x to the finitely generated free A-module $F(x)$. Then the sheaf condition gives a locally free sheaf F and a one-to-one correspondence.

By [6, Proposition 2.7.14], the sheafification of the functor X with respect to Zariski or Nisnevich topology does not change the ∞-category $\mathrm{QCoh}(X)$. The functor BGL_n

is a hypercomplete sheaf with respect to those topology by [7, Theorem 6.1]. Therefore, it is sufficient to say that there is a one-to-one correspondence in Zariski case.

Now, we have $Mod_R^{nproj} \simeq \mathrm{QCoh}(\mathrm{Spec}^{\,g} R)_n^{lf}$. Take $M \in \mathrm{Mod}_R^{nproj}$. Then $\pi_0 M$ is a locally free $\pi_0 R$-module of finite rank. We can choose elements $x_1, \cdots, x_m \in \pi_0 R$ such that they generate the unit ideal and each localization $(\pi_0 M)[x_i^{-1}]$ is a free module over $(\pi_0 R)[x_i^{-1}]$ of finite rank. It follows that $M[x_i^{-1}]$ is a free module over $R[x_i^{-1}]$. Thus, M is free locally with respect to the Zariski topology on $\mathrm{Spec}^{\,g} R$ [6, Remark 2.7.30]. The lemma follows from the fact that, for a spectral sheaf X, $\mathrm{QCoh}(X)_n^{lf}$ is glued up by affine cover in the ∞-topos $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}_{\mathcal{G}}^{cn})$. □

Proposition 2.2. *Let BGL be a classifying space of colimit of affine spectral schemes GL_n and $\Omega B(BGL)$ the group completion. Let $\mathrm{CAlg}^{e\mathcal{G}}$ be either the Zariski ∞-topos CAlg^{eZar} or the Nisnevich ∞-topos CAlg^{eNis}. There is an equivalence of ∞-groupoids:*

$$\mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{e\mathcal{G}})}(\mathrm{Spec}^{\,g} R,\, \Omega B(BGL)) = \widetilde{K}(\mathrm{Spec}^{\,g} R).$$

Proof. By [7, Lemma 3.21], we have an equivalence:

$$\underset{n}{colim}\, \mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{e\mathcal{G}})}(\mathrm{Spec}^{\,g} R,\, BGL_n) \simeq \mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{e\mathcal{G}})}(\mathrm{Spec}^{\,g} R,\, BGL).$$

By Lemma 2.1, we have an equivalence

$$\Omega B(BGL(R)) \simeq \Omega B((\mathrm{Mod}_R^{proj})^{\simeq}),$$

where $(-)^{\simeq}$ denotes the maximal Kan complex. We can take the category of semi-finite R-modules. Since all w-cofibrations in Mod_R^{proj} is split and the homotopy category of Mod_R^{proj} is additive, by applying Waldhausen's additive K-theory to $\Omega B((\mathrm{Mod}_R^{proj})^{\simeq})$, we obtain that $\Omega B((\mathrm{Mod}_R^{proj})^{\simeq})$ is equivalent to the algebraic K-theory $K(\mathrm{Mod}_R^{proj}) = K(\mathrm{Spec}^{\,g} R)$ which is obtained by $S_\bullet$ construction.

On the other hand, we have an equivalence induced by Yoneda embedding;

$$\mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{e\mathcal{G}})}(\mathrm{Spec}^{\,g} R,\, \Omega B(BGL)) \simeq (\Omega BBGL)(R).$$

Consider the following commutative diagram of the ∞-categories:

$$\begin{array}{ccc}
\mathcal{P}(\mathrm{CAlg}^{e\mathcal{G}}) & \xrightarrow{\Omega B} & \mathcal{P}(\mathrm{CAlg}^{e\mathcal{G}}) \\
\big\downarrow \widetilde{(-)} & & \big\downarrow \widetilde{(-)} \\
\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathcal{S}hv(\mathrm{CAlg}^{e\mathcal{G}})) & \xrightarrow{\Omega B} & \mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathcal{S}hv(\mathrm{CAlg}^{e\mathcal{G}})).
\end{array}$$

We have an equivalence $(\Omega BBGL)(R) \simeq \Omega B(BGL(R))$ if the objectwise group completion functor $R \mapsto \Omega B(BGL(R))$ is a spectral sheaf.

□

The space $\Omega B(BGL(R))$ has the same construction of the K-theory KR defined in [4] as a space by [4, Theorem 7.1].

References

[1] C. Barwick and T. Lawson, *Regularity of structured ring spectra and localization in K-theory*, Preprint, available at arxiv:1402.6038v2 (2014).

[2] A. Blumberg, D. Gepner, and G. Tabuada, *A universal characterization of higher algebraic K-theory*, To appear in Geometry and Topology, available at arxiv:1001.2282v4 (2014).

[3] J. Blumberg and A. Mandell, *The localization sequence for the algebraic K-theory of topological K-theory*, Acta Mathematica **200** (2008), 155–179.

[4] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole. MR1417719 (97h:55006)

[5] J. Lurie, *Higher algebra*, Preprint, available at www.math.harvard.edu/lurie (2011).

[6] ———, *Quasi-Coherent Sheaves and Tannaka Duality Theorems*, Preprint, available at www.math.harvard.edu/lurie (2011).

[7] ———, *Spectral Schemes*, Preprint, available at www.math.harvard.edu/lurie (2011).

[8] M. Ohara, *Algebraic K-theory of quasi-coherent modules over spectral schemes and the representation in affine case*, Preprint, available at arxiv.org/abs/1411.2306 (2014).