Remarks on the strong maximum principle involving p-Laplacian

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Let Ω be a bounded domain of $\mathbf{R}^{\mathbf{N}}$ $(N \geq 1)$. By Δ_p we denote the p-laplace operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \tag{0.1}$$

In this article, we shall study the strong maximum principle on the following quasilinear operator

$$-\Delta_p + a(x)Q(\cdot). \tag{0.2}$$

Here $1 < p < \infty$, $N \geq 1$, $a \in L^1(\Omega)$ and $Q(\cdot)$ is a nonlinear term satisfying the following properties:

$[\mathbf{Q_0}]$: $Q(t)$ is a strictly increasing and continuous function such that $Q(0) = 0$ and $t \cdot Q(t) > 0$ on $\mathbf{R} \setminus \{0\}$.

Moreover we assume in Theorem 1

$[\mathbf{Q_1}]$:

$$\limsup_{|t|\to 0} \frac{|Q(t)|}{|t|^{p-1}} < \infty. \tag{0.3}$$

Now let us recall some relating known results on the strong maximum principle assuming that $Q(t) = |t|^{p-2}t$ for simplicity. The classical strong maximum principle for a Laplacian asserts that if u is smooth, $u \geq 0$ and $-\Delta u \geq 0$ in a domain (a connected open set) $\Omega \subset \mathbf{R}^N$, then either $u \equiv 0$ or $u > 0$ in Ω. The same conclusion holds when $-\Delta u$ is replaced by $-\Delta + a(x)$ with $a \in L^s(\Omega), s > N/2$. Later these results were extended to the quasilinear operators $-\Delta_p u + a(x)u^{p-1}$ with $1 < p < \infty$, $a \in L^s(\Omega), s > N/p$. These are consequences of a weak Harnack's inequality. See [........] and [...] for $p = 2$ and [...] for $p > 1$. Another formulation of the same fact says that if $u(x) = 0$ for some point $x \in \Omega$, then $u \equiv 0$ in Ω.

However a similar conclusion does not hold when $a \notin L^s$, for any $s > N/p$.

Example 1. *Let B_1 be a unit ball in $\mathbf{R}^N$ with a center being* 0 *and*

$$\begin{cases} u = |x|^\alpha, \alpha > (p - N + 1)/(p-1), \\ a(x) = c(p,\alpha)|x|^{-p}, \\ c(p,\alpha) = \alpha^{p-1}(\alpha p - \alpha - p + N - 1). \end{cases} \tag{0.4}$$

Then we see $0 \leq a \notin L^{N/p}(B_1)$ and $-\Delta_p u + a(x)u^{p-1} = 0$ in B_1. Clearly $u(0) = 0$ but $u \not\equiv 0$ in B_1.

If u vanishes on a larger set, one may conclude that $u \equiv 0$ under some weaker condition on a. When $p = 2$, such a result was obtained by Bénilan-Brezis [] in the case where $a \in L^1(\Omega)$ and $\operatorname{supp} u$ is a compact subset of Ω. This maximum principle has been further extended by Ancona []. Later a more direct proof was given by Brezis - Ponce [] in the split of PDE's.

In the present paper we further study the case where $p \in (1,\infty)$ adopting a nonlinearlity $Q(t)$ in stead of $|t|^{p-2}t$. Now we describe our main result:

Theorem 1. *Let $N \geq 1$, $1 < p < \infty$ and $p^* = \max(0, p-1)$. Let Ω be a bounded domain of $\mathbf{R}^N$. Let u be a measurable function on Ω, $u \geq 0$ a.e. in Ω such that $u \in L^1(\Omega)$, $Q(u) \in L^1(\Omega)$, $|\nabla u| \in L^{p^*}_{loc}(\Omega)$ and $\Delta_p u$ is a Radon measure on Ω. Then we have the followings:*

1. *There exists $\tilde{u} : \Omega \mapsto \mathbf{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in Ω.*
2. *Let $a \in L^1(\Omega), a \geq 0$ a.e. in Ω. If*

$$-\Delta_p u + a(x)Q(u) \geq 0 \text{ in } \Omega, \tag{0.5}$$

in the following sense

$$\int_E \Delta_p u \, dx \leq \int_E aQ(u) \, dx \quad \text{for every Borel set } E \subset \Omega, \tag{0.6}$$

and if $\tilde{u} = 0$ on a set of positive p-capacity in Ω, then $u = 0$ a.e. in Ω.

Remark 0.1. *1. In the section 2 the definitions of quasicontinuity and p-capacity denoted by $C_p(E, \Omega)$ are given together with their fundamental properties.*

2. In (0.4), $u = |x|^\alpha$ satisfies $-\Delta_p u + a(x)u^{p-1} = 0$ in B_1. If $p > N$, then $C_p(\{0\}, B_1) > 0$ holds. But we note that $a \notin L^1(B_1)$.

Remark 0.2. *Let us set $Q(t) = |t|^{q-2}t$ for $q > 1$ which clearly satisfies $[\mathbf{Q_0}]$. Then the condition $[\mathbf{Q_1}]$ is satisfied if and only if $q \geq p$. In this case Example 2 below shows the necessity of the condition $[\mathbf{Q_1}]$.*

In order to study the necessity of the condition $[\mathbf{Q_1}]$ in Theorem, let us introduce another condition $[\mathbf{Q_2}]$.

$[\mathbf{Q_2}]$: There exists a $q \in (1, p)$ such that we have

$$\liminf_{|t| \to 0} \frac{|Q(t)|}{|t|^{q-1}} > 0 \tag{0.7}$$

Then we have the following.

Example 2. *We assume that $\Omega = B_1$, the conditions $[\mathbf{Q_0}]$ and $[\mathbf{Q_2}]$, and we fix a nonnegative integer $m \leq N-1$.*

Let $\mathcal{M}_0 = \{0\}$ and let $\mathcal{M}_m \subset \mathbf{R}^N$ for $m > 0$ be an m dimensional linear subspace defined by

$$\mathcal{M}_m = \{y = (y_1, y_2, \ldots, y_N) \in \mathbf{R}^N : y_{m+1} = y_{m+2} = \cdots y_N = 0\}, \tag{0.8}$$

and we put $K_m = \mathcal{M}_m \cap \overline{B_{1/2}}$. , Let us set

$$d_m(x) = dist(x, \mathcal{M}_m) \equiv \sqrt{\sum_{k=m+1}^{N} x_k^2}. \tag{0.9}$$

Then clearly $d_m \in C^\infty(\mathbf{R}^N \setminus \mathcal{M}_m)$ and $|\nabla d_m(x)| = 1$ in $\mathbf{R}^N \setminus \mathcal{M}_m$. Now we construct a null solution U for (0.2) in B_1 of the form

$$U(x) = d_m(x)^\alpha \tag{0.10}$$

as before. By a direct calculation we have

$$-\Delta_p U + a(x)Q(U) = 0 \quad \text{in } B_1, \tag{0.11}$$

where

$$a(x) = \frac{\Delta_p U}{Q(U)} = \frac{U^{q-1}}{Q(U)} \alpha^{p-1}(d_m \Delta d_m + (\alpha-1)(p-1)) d_m^{\alpha(p-q)-p}.$$

Here we note that

$$d_m(x)\Delta d_m(x) = N - m - 1. \tag{0.12}$$

By virtue of $[\mathbf{Q_2}]$ *we have for a sufficiently large* $\alpha > 0$

$$0 \leq a(x) \leq C d_m(x)^{\alpha(p-q)-p} \in L^1(B_1), \quad \textit{for some positive constant } C.$$

Clearly $U = 0$ *on* $K_m \subset \mathcal{M}_m$ *and* $U \not\equiv 0$.

Now we choose a nonnegative interger m *so that* $m > N - p$.

Then it follows from Lemma 1(2) that $C_p(K_m, B_1) > 0$ *provided that* $1 < p < N$. *If* $p > N$, *then we set* $m = 0$ *and* $K_0 = \mathcal{M}_0 = \{0\}$ *so that we have* $C_p(\{0\}, B_1) > 0$. *Lastly we assume* $1 < p = N$. *Again it follows from Lemma 1(2) that we have* $C_{N-\eta}(K_1, B_1) > 0$ *for a sufficiently small* $\eta > 0$. *Hence* $C_N(K_1, B_1) > 0$ *by a Hölder inequality.*

Lemma 1. *Let* p *satisfy* $1 < p < N$ *and let* E *be a compactum in* B_1.

1. *Assume that* $H_{N-p+\varepsilon}(E) > 0$ *for some* $\varepsilon > 0$. *Then* $C_p(E, B_1) > 0$.
2. *Assume that a nonnegative integer* m *satisfies* $N - p < m\, (\leq N - 1)$. *Then* $C_p(K_m, B_1) > 0$,*where* $K_m = \mathcal{M}_m \cap \overline{B_{1/2}}$.

Here by $H_d(E)$ *for* $0 \leq d \leq N$ *we denote a d-dimensional Hausdorff measure.*

Proof: The assertion 1 is a fundamental property of capacity. For instance see [; Proposition 3.1].

Since $\dim \mathcal{M}_m = m$, we see $H_{N-p+\varepsilon}(K_m) > 0$ for a sufficiently small $\varepsilon > 0$. Hence the assertion 2 is a direct consequence of the previous one. □

Then we have the following.

Proposition 0.1. *Let us set* $Q(t) = |t|^{q-2}t$ *for* $1 < q$. *Then, in the hypotheses of theorem 1 the condition* $[\mathbf{Q_1}]$ *is necessary.*

Proof: If $1 < q < p$, then Q satisfies $[\mathbf{Q_2}]$ and we already have the counter-examples. □

Definition 0.1. *Definitions of* $W^{1,p}_{loc}(\Omega)$ *and* $W^{1,p}(\Omega)$ *and* $p-capacity$.

$L^1_{loc}(\Omega)$*-functions are an important class of distributions,but we can usefully refine that class by studying functions whose distributeonal derivatives are also* $L^1_{loc}(\Omega)$*-function. This class is denoted by* $W^{1,1}_{loc}(\Omega)$. *Furthermore,just as* $L^p_{loc}(\Omega)$ *is related to* $L^1_{loc}(\Omega)$ *we can also define the class of functions* $W^{1,p}_{loc}(\Omega)$ *for each* $1 \leq p \leq \infty$. *Thus,*

$$W^{1,p}_{loc} = \{f : \Omega \to C : f \in L^p_{loc}(\Omega) \textit{ and } \partial_i f, \textit{ as a distribution in } D'(\Omega), \textit{ is an } L^p_{loc}(\Omega)\textit{-fuction for } i = 1, ..., n\}.$$

Definition 0.2. *The p-capacity of a compact set* $\Sigma \subset \Omega$ *is defined as*

$$C_p(\Sigma, \Omega) = inf\{\int_\Omega \mid \nabla\varphi \mid^p : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \textit{ in some neightborhood of } \Sigma\}.$$

Definition 0.3. *In this paper we shall consider the operators defined by*

$$L_p(u) = \Delta_p(u) = div(| \nabla u |^{p-2} \nabla u)$$

Definition 0.4. *We recall that a function v:* $\Omega \rightarrow \mathbf{R}$ *is quasicontinuous if there exists a sequence of open subsets* (ω_n) *of* Ω *such that* $v |_{\Omega \setminus \omega_n}$ *is continuous* $\forall n \geq 1$ *and* $cap\omega_n \rightarrow 0$ *as* $n \rightarrow \infty$*, where cap* ω_n *denotes the* $H^1 - capacity$ *of* ω_n*.*

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