Asymptotic behavior of solutions of renewal equations in epidemiology

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1 Introduction

Epidemiological models have played a crucial role to qualitatively understand the behavior of disease transmission. One of the pioneering works is carried out by Anderson, May [1], in which an SIRS (Susceptible-Infected-Recovered-Susceptible) model is introduced by a system of ordinary differential equations. By means of the model describing the experiments of disease dynamics on mice population, the authors have succeeded to fit the data of mice infected with diseases caused by a bacterium and virus. Later, identifying the basic reproduction number, global dynamics of the model has also been obtained; a disease-free equilibrium is globally asymptotically stable if the basic reproduction number is less than or equals to 1 and an endemic equilibrium is globally asymptotically stable if the basic reproduction number is greater than 1. Based on their modelling methods, epidemic models are proposed in a more general framework of differential equations such as multi-dimensional and delayed models.

Recently, in terms of Volterra type integral equations, structured population models have widely been formulated to investigate the effect of differently aged individuals for infectious disease process (see, e.g., Diekmann et al. [2]). One of the structured models is formulated by *age of infection*, denoting the time that has elapsed since the infection has started. First, we denote by a the age of infection and define $\mathcal{F}(a)$ for a probability to be infected until his or her infection-age becomes a. The function $\mathcal{F}$ is assumed to be decreasing with $\mathcal{F}(0) = 1$. Let us denote by $b(t)$ the incidence rate (the number of newly infected individuals) at time t. Then the number of infected individuals at time t is given as follows.

$$I(t) := \int_0^\infty b(t-a)\mathcal{F}(a)da.$$

Second, let us denote by $\beta(a)$ the age-specific transmission coefficient of infected individuals. By the three functions $\mathcal{F}$, b and β, we obtain the following renewal equation:

$$b(t) = S(t)\int_0^\infty \beta(a)b(t-a)\mathcal{F}(a)da$$

with the variable $S(t)$ denoting the number of susceptible individuals at time t (see also Figure 1.1). Let us define $\mathcal{F}(a) = e^{-(\mu+\eta+\gamma)a}$. Here $\mu > 0$, $\eta > 0$ and $\gamma > 0$ denote the natural mortality rate, the disease-induced death rate and the recovery rate, respectively.

Adding the variable $R(t)$, the number of recovered individuals at time t, we investigate the asymptotic

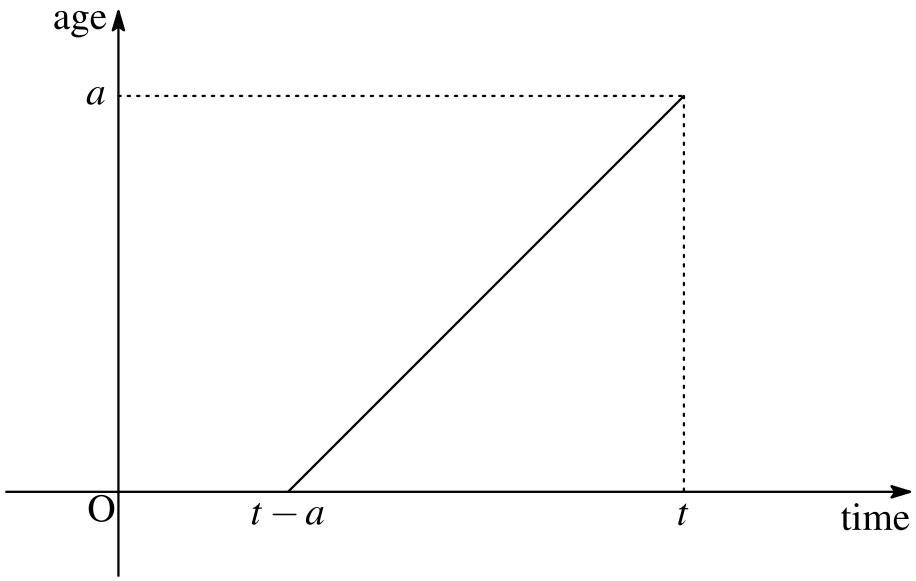

Figure 1.1: Infected individuals with infection age a at time t experience the incidence at time $t-a$.

behavior of the following model:

$$\begin{cases} \dfrac{dS(t)}{dt} = B - S(t)\displaystyle\int_0^\infty \beta(a)b(t-a)e^{-(\mu+\eta+\gamma)a}da - \mu S(t) + \delta R(t), \\ b(t) = S(t)\displaystyle\int_0^\infty \beta(a)b(t-a)e^{-(\mu+\eta+\gamma)a}da, \\ \dfrac{dR(t)}{dt} = \gamma\displaystyle\int_0^\infty b(t-a)e^{-(\mu+\eta+\gamma)a}da - (\mu+\delta)R(t) \end{cases} \tag{1.1}$$

with the initial conditions $(S(0), b(\theta), R(0)) = (s, \phi(\theta), r)$ for $\theta \in \mathbb{R}_-$. We assume that $(s, \phi, r) \in \mathbb{R}_+ \times L^1_\rho(\mathbb{R}_-; \mathbb{R}_+) \times \mathbb{R}_+$, the space consists of all equivalence classes of measurable functions $\phi : \mathbb{R}_- \longrightarrow \mathbb{R}_+$ such that $\|\phi\|_{L^1_\rho} = \int_0^\infty e^{-\rho a}|\phi(-a)|da < \infty$.

All new born are assumed to be susceptible. $B > 0$ denotes the birth rate of susceptible individuals and $\delta \geq 0$ denotes the rate of immunity loss of recovered individuals. Let us also assume that $\beta \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$. For system (1.1), the basic reproduction number R_0, given as the dominant eigenvalue of a positive linear operator, is computed as

$$R_0 = B\int_0^\infty \beta(a)\mathcal{F}(a)da.$$

Let $F : \mathbb{R}_+ \times L^1_\rho(\mathbb{R}_-; \mathbb{R}_+) \times \mathbb{R}_+ \longrightarrow \mathbb{R}^3$ with

$$F(s, \phi, r) = \begin{pmatrix} B - \mu s - s\displaystyle\int_0^\infty \beta(a)\phi(-a)\mathcal{F}(a)da + \delta r \\ s\displaystyle\int_0^\infty \beta(a)\phi(-a)\mathcal{F}(a)da \\ \gamma\displaystyle\int_0^\infty \phi(-a)\mathcal{F}(a)da - (\mu+\delta)r \end{pmatrix}.$$

Then system (1.1) can be written as the following abstract form:

$$\begin{pmatrix} \frac{d}{dt}S(t) \\ b(t) \\ \frac{d}{dt}R(t) \end{pmatrix} = F(S(t), b_t, R(t)),$$

where $b_t : \mathbb{R}_- \longrightarrow \mathbb{R}_+$ is given by $b_t(\theta) = b(t+\theta)$. One can prove that system (1.1) has a unique solution defined on $(0, \infty)$ by the similar proof in [7, Theorem 3.1] and $S(t), b(t), R(t) \geq 0$ for all $t > 0$. One can see that system (1.1) always has a disease-free equilibrium in X_+. Furthermore, system (1.1) admits an endemic equilibrium in X_+ if and only if $R_0 > 1$. In this talk, we establish global staility of the endemic equilibrium and introduce a several open problems when the rate of immunity loss δ is positive.

2 Stability of the endemic equilibrium

We first consider the case where $\beta \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$ satisfies the following assumption:

Assumption 2.1. *For $h \in \mathbb{R}_+$, the function β satisfies*

(H1) $\beta(0) = 0$. (H2) $\beta(a) \equiv \beta > 0$ *for* $a > h$. (H3) β *is nondecreasing.*

We note that β is a function of bounded variation on $\mathbb{R}_+$ under the hypotheses (H1)-(H3).

Proposition 2.1. *Let us assume that $h > 0$. Then it holds that*

$$b(t) = S(t) \int_0^h \mathcal{F}(\tau) I(t-\tau) d\beta(\tau). \tag{2.1}$$

Here the integral in (2.1) *is a Riemann-Stieltjes integral.*

From Proposition 2.1, system (1.1) is expressed as a system of delay differential equations as follows.

$$\begin{cases} \dfrac{dS(t)}{dt} = B - S(t) \displaystyle\int_0^\infty \beta(a) b(t-a) e^{-(\mu+\eta+\gamma)a} da - \mu S(t) + \delta R(t), \\ \dfrac{dI(t)}{dt} = S(t) \displaystyle\int_0^h e^{-(\mu+\eta+\gamma)\tau} I(t-\tau) d\beta(\tau) - (\mu+\eta+\gamma) I(t), \\ \dfrac{dR(t)}{dt} = \gamma \displaystyle\int_0^\infty b(t-a) e^{-(\mu+\eta+\gamma)a} da - (\mu+\delta) R(t). \end{cases} \tag{2.2}$$

By means of Lyapunov functional approach, it is proven that the endemic equilibrium of (2.2) is globally asymptotically stable when δ is small [3]. On the other hand, by monotone iterative methods, it is proven that the endemic equilibrium of (2.2) is globally asymptotically stable when δ is large [5].

In contrast to the case $\delta = 0$ (see Magal et al. [4]), due to the cyclicity of the model structure, there are still few results on the stability of the equilibria of system (1.1) when $\delta > 0$. Computing the Frechét derivative of F evaluated at the endemic equilibrium, the following result is obtained [6].

Theorem 2.1. *Let us assume $R_0 > 1$. Then the endemic equilibrium is locally asymptotically stable.*

We further elaborate our stability analysis when η, γ are age-specific and discuss open problems remained.

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