Singularities of projections of surfaces in $\mathbb{R}^4$

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Abstract: We study the geometry of surfaces in $\mathbb{R}^4$ associated to contact with hyperplanes, planes and lines. In particular, we show the existence of multi-local robust feature of surface. These are smooth curves representing the various types of multi-local singularities.

1 Introduction

In this work we study the generic geometry of surfaces in $\mathbb{R}^4$ associated to its contact with hyperplanes, planes and lines. This contact is captured by the local and multi-local singularities of the height function and the orthogonal projection to 2,3-spaces. There are several results about the subject, see for exemple [1, 2, 4, 6]. We show the existence of multi-local robust features of the surface.

Given $M$ be a regular surface in Euclidean space $\mathbb{R}^4$. Given a point $p \in M$ consider the unit circle in $T_p M$ parametrized by $\theta \in [0,2\pi]$. The set of the curvature vectors $\eta(\theta)$ of the normal sections of $M$ by the hyperplane $\langle \theta \rangle \oplus N_p M$ form an ellipse in the normal plane $N_p M$ to $M$ at $p$, called the curvature ellipse ([4]).

The curvature ellipse is the image of the unit circle in $T_p M$ by a map formed by a pair of quadratic forms $(Q_1,Q_2)$. This pair of quadratic forms is the 2-jet of the 1-flat map $F : \mathbb{R}^2,0 \to \mathbb{R}^2,0$ whose graph, in orthogonal co-ordinates, is locally the surface $M$.

The flat geometry of surfaces is affine invariant. A different approach to the geometry of surfaces in $\mathbb{R}^4$ is given in [1]. This is via the pencil of the binary forms determined by the pair $(Q_1,Q_2)$. Each point on the surface determines a pair of quadratics: $(Q_1,Q_2) = (ax^2 + 2bxy + cy^2, lx^2 + 2mxy + ny^2)$.

If the forms $Q_1$ and $Q_2$ are independent, then we have the invariant

$$\delta(p) = (an - cl)^2 - 4(am - bl)(bn - cm).$$

A point $p$ is said to be elliptic/parabolic/hyperbolic if $\delta(p) < 0/ = 0/ > 0$. The set of points in $M$ where $\delta = 0$ is called the parabolic set of $M$ and is denoted by $\Delta$.

If $Q_1$ and $Q_2$ are dependent in a point $p$, the point is called inflection point.

The geometrical characterization of points on $M$ using singularity theory is first carried out in [6] via the family height function.

Definition 1.1 The family of height functions is defined by

$$h : M \times S^3 \to \mathbb{R}$$

$$\langle p,v \rangle \mapsto h(p,v) = \langle p,v \rangle$$

where $S^3$ denotes the unit sphere in $\mathbb{R}^4$. 
The height function $h_v$ ($v$ fix) is singular at $p$ if and only if $v \in N_pM$. It is shown in [6] that elliptic points are non-degenerate critical points of $h_v$ for any $v \in N_pM$. At a hyperbolic point, there are exactly two directions in $N_pM$, labeled \textit{binormal directions}, such that $p$ is a degenerate critical point of the corresponding height functions. The two binormal directions coincide at a parabolic point. The set of the $A_3$ singularities of the height function is a smooth curve in $M$ ($A_3$-curve) and the singularity $A_4$ of the height function occurs isolated on the $A_3$-curve [6].

The direction of the kernel of the Hessian of the height functions along a binormal direction is an \textit{asymptotic direction} associated to the given binormal direction ([6]). If $p$ is not an inflection point, there are $2/1/0$ asymptotic directions at $p$ depending on $p$ being a hyperbolic/parabolic/elliptic point.

### Definition 1.2

The family of \textit{orthogonal projections} is given by

$$P : M \times S^3 \rightarrow TS^3$$

$$(p, v) \mapsto (v, p - \langle p, v \rangle v).$$

For $v$ fixed, the projection can be viewed locally at a point $p \in M$ as a map germ $P_v : \mathbb{R}^2, 0 \rightarrow \mathbb{R}^3, 0$. If we allow smooth changes of coordinates in the source and target (i.e. consider the action of the Mather group $A$) then the generic $A$-singularities of $P_v$ are those that have $A$-codimension less than or equal to $3$ (which is the dimension of $S^3$). These are listed in [7].

The projection $P_v$ is singular at $p$ if and only if $v \in T_pM$. The singularity is a cross-cap unless $v$ is an asymptotic direction at $p$. The codimension 2 singularities occur generically on curves on the surface and the codimension 3 ones at special points on these curves. The $H_2$-curve coincides with the $\Delta$-set ([1]). The $B_2$-curve of $P_v$, with $v$ asymptotic, is also the $A_3$-set of the height function along the binormal direction associated to $v$ ([1]). This curve meets the $\Delta$-set tangentially at isolated points ([2]) and intersects the $S_2$-curve transversally at a $C_3$-singularity. At inflection points the $\Delta$-set has a Morse singularity and the configuration of the $B_2$ and $S_2$-curves there is given in [1].

In [5], carried out a study using the family orthogonal projections in planes.

### Definition 1.3

The \textit{orthogonal projections to planes} is given by

$$\Pi : M \times G(2, 4) \rightarrow \mathbb{R}^2$$

$$(p, v) \mapsto \Pi(p, v) = (\langle p, a \rangle, \langle p, b \rangle),$$

where $G(2, 4)$ is the Grassmanian of 2-planes in $\mathbb{R}^4$ and, $a$ and $b$ are unit vectors linearly independent generating the plane $v \in G(2, 4)$.

The family projections in planes can be seen to 4-parameters and, moreover, fixed $v$ is locally in $p \in M$ a germ of a map $\Pi_v : \mathbb{R}^2, 0 \rightarrow \mathbb{R}^2, 0$ which give that contact between $M$ and the plane which is the orthogonal complement of plane $v$. The generic $A$-singularities of that germs are those in [8].

$\Pi_v$ is singular at $p \in M$, if only if, the intersection of the tangent plane to $M$ at $p$ with the orthogonal complement of the plane $v$, denoted by $v^\perp$, is different of zero, i.e., there is a tangent vector not null $u \in T_pM$ such that $u \in v^\perp$. The singularities of corank 2 correspond to normal planes to surface. For singularities of corank 1, at hyperbolic (resp.}


parabolic) points there are two (resp. one) degenerate planes whose projection is of type \(4_2\) (or worse) and at elliptic points there is only transversal singularities to \(\Sigma_1\). \((\Sigma_1\) is the set of singular points with corank 1). On the plane of degenerate projection \(v\) chosen a direction in this plane, the \(A_2\)-set of the height function coincides with the \(4_2\)-set of \(\Pi_v\).

2 The multi-local curves and their applications

We are seeking the loci of points in \(M\) where \(h_v, P_v\) has a multi-local singularity of \(A_e\)-codimension \(\leq 2\).

**Theorem 1** In the family the height function \(h_v\). The \(A_3\)-curve and the \(A_1A_2\)-curve are generically tangential at \(A_4\) with contact of order 2.

![Figure 1: multi-local curves in \(M\) away of the inflection points](image)

**Theorem 2** In family of the orthogonal projection \(P_v\). For a generic surface \(M\) embedded in \(\mathbb{R}^4\), the multi-local singularities of \(P_v\) of codimension 2 that are adjacent to local singularities occur only at codimension 3 local singularities of \(P_v\).

(i) At a \(B_3\)-singularity of \(P_v\), there is an \([A_2]\)-curve which meets the \(B_2\)-curve transversally;

(ii) At an \(S_3\)-singularity of \(P_v\), there are no codimension 2 multi-local singularities;

(iii) At a \(C_3\)-singularity of \(P_v\), there are no codimension 2 multi-local singularities;

(iv) At an \(H_3\)-singularity of \(P_v\), there is an \((A_0S_0)_2\)-curve which meets the \(\Delta\)-set tangentially;

(v) At a \(P_3(c)\)-singularity of \(P_v\), there are the curves \(A_0S_1, (A_0S_0)_2\) and \(A_0S_0|A_1^\perp\) which meets the \(\Delta\)-set tangentially.

All the tangential are generically of order 2, (see Figure 2). Where \([A_2]\), \((A_0S_0)_2\), \(A_0S_1\) and \(A_0S_0|A_1^\perp\) are bigerms the \(A_e\)-codimension 2 classified in [3].

**Proposition 2.1** At a point \(P_3(c)\), the surface is projectively equivalent to \((x, y, f_1(x, y), f_2(x, y))\), where

\[
j^4(f_1(x, y), f_2(x, y)) = (x^2 + xy^2 + \alpha y^4, xy + \beta y^3 + \phi_4).
\]
We denote tangents at $P_3(c)$ to the Legendrian lifts of the parabolic, $B_2$, $(A_0S_0)_2$, $A_0S_1$ and $A_0S_2|A_1^\pm$ curves by $l_P$, $l_b$, $l_{s_0}$, $l_{s_1}$, and $l_{s_2}$, respectively. We denote by $l_g$ the contact element to the point $P_3(c)$ (the vertical line in the contact plane at that point).

**Theorem 3** At a generic point $P_3(c)$, two cross-ratios can permit recover the projective invariants $\alpha$ and $\beta$ of surface.

**References**


