Singularities of projections of surfaces in $\mathbb{R}^4$

Jorge Luiz Deolindo Silva

Abstract: We study the geometry of surfaces in $\mathbb{R}^4$ associated to contact with hyperplanes, planes and lines. In particular, we show the existence of multi-local robust feature of surface. These are smooth curves representing the various types of multi-local singularities.

1 Introduction

In this work we study the generic geometry of surfaces in $\mathbb{R}^4$ associated to its contact with hyperplanes, planes and lines. This contact is captured by the local and multi-local singularities of the height function and the orthogonal projection to $2, 3$-spaces. There are several results about the subject, see for exemple [1, 2, 4, 6]. We show the existence of multi-local robust features of the surface.

Given M be a regular surface in Euclidean space $\mathbb{R}^4$. Given a point $p \in M$ consider the unit circle in T_pM parametrized by $\theta \in [0, 2\pi]$. The set of the curvature vectors $\eta(\theta)$ of the normal sections of M by the hyperplane $\langle\theta\rangle \oplus N_pM$ form an ellipse in the normal plane N_pM to M at p, called the *curvature ellipse* ([4]).

The curvature ellipse is the image of the unit circle in T_pM by a map formed by a pair of quadratic forms (Q_1, Q_2). This pair of quadratic forms is the 2-jet of the 1-flat map $F : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ whose graph, in orthogonal co-ordinates, is locally the surface M.

The flat geometry of surfaces is affine invariant. A different approach to the geometry of surfaces in $\mathbb{R}^4$ is given in [1]. This is via the pencil of the binary forms determined by the pair (Q_1, Q_2). Each point on the surface determines a pair of quadratics: $(Q_1, Q_2) = (ax^2 + 2bxy + cy^2, lx^2 + 2mxy + ny^2)$.

If the forms Q_1 and Q_2 are independent, then we have the invariant

$$\delta(p) = (an - cl)^2 - 4(am - bl)(bn - cm).$$

A point p is said to be *elliptic/parabolic/hyperbolic* if $\delta(p) < 0/ = 0/ > 0$. The set of points in M where $\delta = 0$ is called the *parabolic set* of M and is denoted by Δ.

If Q_1 and Q_2 are dependent in a point p, the point is called *inflection point.*

The geometrical characterization of points on M using singularity theory is first carried out in [6] via the family height function.

Definition 1.1 *The family of* ***height functions*** *is defined by*

$$\begin{array}{rcl} h : M \times S^3 & \to & \mathbb{R} \\ (p, v) & \mapsto & h(p, v) = \langle p, v\rangle \end{array}$$

where S^3 denotes the unit sphere in $\mathbb{R}^4$.

The height function h_v (v fix) is singular at p if and only if $v \in N_pM$. It is shown in [6] that elliptic points are non-degenerate critical points of h_v for any $v \in N_pM$. At a hyperbolic point, there are exactly two directions in N_pM, labeled *binormal directions*, such that p is a degenerate critical point of the corresponding height functions. The two binormal directions coincide at a parabolic point. The set of the A_3 singularities of the height function is a smooth curve in M (A_3-curve) and the singularity A_4 of the height function occurs isolated on the A_3-curve [6].

The direction of the kernel of the Hessian of the height functions along a binormal direction is an *asymptotic direction* associated to the given binormal direction ([6]). If p is not an inflection point, there are 2/1/0 asymptotic directions at p depending on p being a hyperbolic/parabolic/elliptic point.

Definition 1.2 *The family of* ***orthogonal projections*** *is given by*

$$\begin{array}{rcl} P : M \times S^3 & \rightarrow & TS^3 \\ (p, v) & \mapsto & (v, p - \langle p, v\rangle v). \end{array}$$

For v fixed, the projection can be viewed locally at a point $p \in M$ as a map germ $P_v : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$. If we allow smooth changes of coordinates in the source and target (i.e. consider the action of the Mather group $\mathcal{A}$) then the generic $\mathcal{A}$-singularities of P_v are those that have $\mathcal{A}$-codimension less than or iqual to 3 (which is the dimension of S^3). These are listed in [7].

The projection P_v is singular at p if and only if $v \in T_pM$. The singularity is a cross-cap unless v is an asymptotic direction at p. The codimension 2 singularities occur generically on curves on the surface and the codimension 3 ones at special points on these curves. The H_2-curve coincides with the Δ-set ([1]). The B_2-curve of P_v, with v asymptotic, is also the A_3-set of the height function along the binormal direction associated to v ([1]). This curve meets the Δ-set tangentially at isolated points ([2]) and intersects the S_2-curve transversally at a C_3-singularity. At inflection points the Δ-set has a Morse singularity and the configuration of the B_2 and S_2-curves there is given in [1].

In [5], carried out a study using the family orthogonal projections in planes.

Definition 1.3 *The* ***orthogonal projections to planes*** *is given by*

$$\begin{array}{rcl} \Pi : M \times G(2,4) & \rightarrow & \mathbb{R}^2 \\ (p, v) & \mapsto & \Pi(p, v) = (\langle p, a\rangle, \langle p, b\rangle), \end{array}$$

where $G(2,4)$ is the Grassmanian of 2-planes in $\mathbb{R}^4$ and, a and b are unit vectors linearly independents generating the plane $v \in G(2,4)$.

The family projections in planes can be seen to 4-parameters and, moreover, fixed v is locally in $p \in M$ a germ of a map $\Pi_v : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ which give that contact between M and the plane which is the orthogonal complement of plane v. The generic $\mathcal{A}$-singularities of that germs are those in [8].

Π_v is singular at $p \in M$, if only if, the intersection of the tangent plane to M at p with the orthogonal complement of the plane v, denoted by $v^\perp$, is different of zero, i. e., there is a tangent vector not null $u \in T_pM$ such that $u \in v^\perp$. The singularities of corank 2 correspond to normal planes to surface. For singularities of corank 1, at hyperbolic (resp.

parabolic) points there are two (resp. one) degenerate planes whose projection is of type 4_2 (or worse) and at elliptic points there is only transversal singularities to Σ^1. (Σ^1 is the set of singular points with corank 1). On the plane of degenerate projection v chosen a direction in this plane, the A_2-set of the height function coincides with the 4_2-set of Π_v.

2 The multi-local curves and their applications

We are seeking the loci of points in M where h_v, P_v has a multi-local singularity of $\mathcal{A}_e$-codimension ≤ 2.

Theorem 1 *In the family the height function h_v. The A_3-curve and the A_1A_2-curve are generically tangential at A_4 with contact of order 2.*

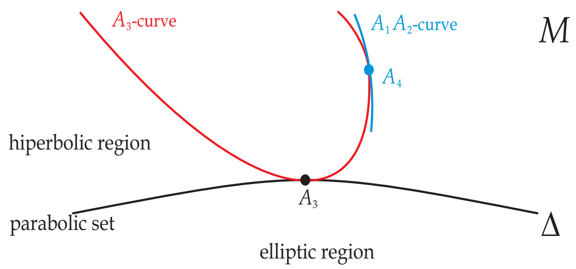

Figure 1: multi-local curves in M away of the inflection points

Theorem 2 *In family of the orthogonal projection P_v. For a generic surface M embedded in $\mathbb{R}^4$, the multi-local singularities of P_v of codimension 2 that are adjacent to local singularities occur only at codimension 3 local singularities of P_v.*

(i) At a B_3-singularity of P_v, there is an $[A_2]$-curve which meets the B_2-curve transversally;

(ii) At an S_3-singularity of P_v, there are no codimension 2 multi-local singularities;

(iii) At a C_3-singularity of P_v, there are no codimension 2 multi-local singularities;

(iv) At an H_3-singularity of P_v, there is an $(A_0S_0)_2$-curve which meets the Δ-set tangentially;

(v) At a $P_3(c)$-singularity of P_v, there are the curves A_0S_1, $(A_0S_0)_2$ and $A_0S_0|A_1^{\pm}$ which meets the Δ-set tangentially.

All the tangential are generically of order 2, (see Figure 2). Where $[A_2]$, $(A_0S_0)_2$, A_0S_1 and $A_0S_0|A_1^{\pm}$ are bigerms the $\mathcal{A}_e$-codimension 2 classified in [3].

Proposition 2.1 *At a point $P_3(c)$, the surface is projectively equivalent to $(x, y, f_1(x, y), f_2(x, y))$, where*

$$j^4(f_1(x, y), f_2(x, y)) = (x^2 + xy^2 + \alpha y^4, xy + \beta y^3 + \phi_4).$$

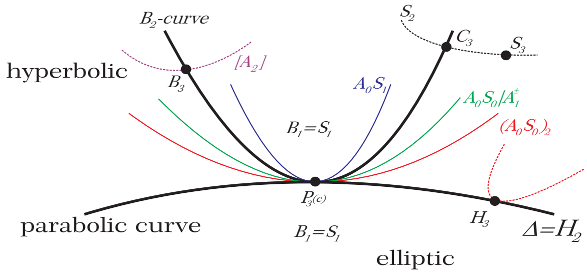

Figure 2: multi-local curves in M away of the inflection points

We denote tangents at $P_3(c)$ to the Legendrian lifts of the parabolic, B_2, $(A_0S_0)_2$, A_0S_1 and $A_0S_0|A_1^{\pm}$ curves by $l_P, l_b, l_{s_0}, l_{s_1}$, and $l_{s_{02}}$, respectively. We denote by l_g the contact element to the point $P_3(c)$ (the vertical line in the contact plane at that point).

Theorem 3 *At a generic point $P_3(c)$, two cross-ratios can permit recover the projective invariants α and β of surface.*

References

[1] J. W. Bruce and A. C. Nogueira, Surfaces in $\mathbb{R}^4$ and duality. *Quart. J. Math. Oxford Ser.* Ser. (2), 49 (1998), 433–443.

[2] J. W. Bruce and F. Tari, Families of surfaces in $\mathbb{R}^4$. *Proc. Edinb. Math. Soc.* 45 (2002), 181–203.

[3] C. A. Hoobs, N. P. Kirk, On the classification and bifurcation of multi-germs of maps from surfaces to 3-space, Math. Scand. 89 (2001), 57–96

[4] J. A. Little, On the singularities of submanifolds of heigher dimensional Euclidean space. *Annli Mat. Pura et Appl.* (4A) 83 (1969), 261–336.

[5] A. C. Nogueira, Superfícies em $\mathbb{R}^4$ e dualidade. Tese de Doutorado, ICMC, São Carlos-USP, 1998.

[6] D. K. H. Mochida, M. C. Romero-Fuster and M. A. S. Ruas, The geometry of surfaces in 4-space from a contact viewpoint. *Geometria Dedicata* 54 (1995), 323–332.

[7] D. M. Q. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$. *Proc. London Math. Soc.* 50 (1985), 333–369.

[8] J. H. Rieger, Families os maps from the plane to the plane.*J. London Math. Soc.* (1987), 351–369.