Effective nonvanishing of pluriadjoint line bundles

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1. Introduction

Let X be a smooth projective variety defined over $\mathbf{C}$ and L an ample line bundle over X. Then the pair (X, L) is called a **polarized manifold**.

In the classification theory of polarized manifolds, it is important to study a condition on the integer m for which $|K_X + mL|$ is free. Fujita's freeness conjecture predicts that $|K_X + mL|$ is free for any $m \geq \dim X + 1$. It is known that the above conjecture is true when $\dim X \leq 4$. In higher dimensional case, it is proved that $|K_X + mL|$ is free for every integer $m \geq \dim X(\dim X + 1)/2 + 1$ (see [1], [9]).

On the other hand when $K_X + L$ is nef, by the nonvanishing theorem due to V. Shokurov, we see that $|m(K_X + L)| \neq \emptyset$ holds for $m \gg 0$. Then it is important to find an integer m with $|m(K_X + L)| \neq \emptyset$. Concerning this, Y. Kawamata ([7]) proposed the following conjecture:

Conjecture 1.1. *Let X be a normal projective variety and let B be a $\mathbf{Q}$-effective divisor on X such that (X, B) is a KLT pair. Let D be a nef Cartier divisor on X such that $D - (K_X + B)$ is nef and big. Then $H^0(X, \mathcal{O}_X(D)) \neq 0$ holds.*

When X is smooth, $B = 0$ and $D := K_X + L$ is nef, this implies that $|K_X + L| \neq \emptyset$ holds for any polarized manifold (X, L) with $K_X + L$ nef. In [7], Kawamata solved the conjecture above when X is 2-dimensional and when X is a minimal 3-fold. A. Höring ([6, Theorem 1.5]) solved it when X is a normal projective 3-fold with at most $\mathbf{Q}$-factorial canonical singularities, $B = 0$, and $D - K_X$ is a nef and big Cartier divisor. These results are immediate consequences of the Hirzebruch-Riemann-Roch theorem and some classical results on surfaces and 3-folds. In higher dimensional case, it is rather difficult to calculate $\dim H^0(X, \mathcal{O}_X(D))$. Indeed, Conjecture 1.1 is still widely open for the case of $\dim X \geq 4$.

Concerning the effective nonvanishing of global sections of pluri-adjoint line bundles, Y. Fukuma proposed the following problem:

Problem 1.2([4, Problem 3.2]). *For any fixed positive integer n, find the smallest positive integer m_n depending only on n such that $H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$ for every $m \geq m_n$ and for every polarized manifold (X, L) of dimension n with $\kappa(K_X + L) \geq 0$.*

It is known that $m_1 = 1$, $m_2 = 1$ (cf. [4, Theorem 2.8]) and $m_3 = 1$ ([6]). Recently, Fukuma also treated the case of $\dim X = 4$ ([5]).

Our main result is the following:

Theorem 1.3. *Let (X, L) be a polarized manifold of dimension n with $K_X + L$ nef. Then $H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$ holds for every positive integer $m \geq n(n + 1)/2 + 2$.*

The above theorem gives a partial answer to Problem 1.2 in higher dimensional case. We give the proof in Section 3; our basic tool is singular hermitian metrics, which will be reviewed in the next section.

2. Preliminaries

We introduce the notions of singular hermitian metrics and multiplier ideal sheaves.

Definition 2.1. Let L be a holomorphic line bundle over a complex manifold X. A **singular hermitian metric** h on L is given by $h = h_0 \cdot e^{-\varphi}$, where h_0 is a C^∞-hermitian metric on

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L and $\varphi \in L^1_{\mathrm{loc}}(X)$. The **curvature current** Θ_h of h is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1}\partial\overline{\partial}\varphi,$$

where Θ_{h_0} denotes the curvature form of h_0, and $\partial\overline{\partial}\varphi$ is taken in the sense of currents.

EXAMPLE 2.2. Let L be a holomorphic line bundle over a complex manifold X. Suppose that there exists a positive integer m such that $\Gamma(X, \mathcal{O}_X(mL)) \neq 0$. Let $\sigma \in \Gamma(X, \mathcal{O}_X(mL))$ be a nontrivial section. Then

$$h := \frac{1}{|\sigma|^{2/m}} = \frac{h_0}{h_0^{\otimes m}(\sigma, \sigma)^{1/m}}$$

is a singular hermitian metric on L, where h_0 is an arbitrary C^∞-hermitian metric on L. By Poincaré-Lelong's formula, we have $\Theta_h = 2\pi/m(\sigma)$, where (σ) denotes the current of integration over the divisor of σ. In particular, we see that Θ_h is a positive current.

DEFINITION 2.3. Let L be a line bundle over a complex manifold X and h a singular hermitian metric on L. We shall write h as $h = h_0 \cdot e^{-\varphi}$, where h_0 is a C^∞-hermitian metric on L and $\varphi \in L^1_{\mathrm{loc}}(X)$. Then we define the **multiplier ideal sheaf** $\mathcal{I}(h)$ of (L, h) by

$$\Gamma(U, \mathcal{I}(h)) := \{f \in \Gamma(U, \mathcal{O}_X) \mid |f|^2 \cdot e^{-\varphi} \in L^1_{\mathrm{loc}}(U)\},$$

where U runs over the open subsets of X.

The following vanishing theorem due to A. Nadel ([8]) plays a crucial role in the proof of Theorem 1.3 (cf. Remark 2.4.1).

THEOREM 2.4. *Let L be a line bundle over a compact Kähler manifold (X, ω), and h a singular hermitian metric on L. Suppose that the curvature current Θ_h of h is strictly positive, i.e., there exists a constant $\varepsilon > 0$ such that $\Theta_h - \varepsilon\omega$ is a positive $(1,1)$-current. Then $\mathcal{I}(h)$ is a coherent sheaf on X, and*

$$H^q(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h)) = 0$$

holds for every $q \geq 1$.

REMARK 2.4.1. We shall explain how to establish the effective nonvanishing of global sections of (multi-)adjoint line bundles by using the above theorem. Suppose that there exists a singular hermitian metric h on a line bundle L such that

1. Θ_h is strictly positive;
2. $\mathcal{O}_X/\mathcal{I}(h)$ has isolated support at a point x in X.

Then by Theorem 2.4, we have $H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h)) = 0$. This implies that the map:

$$H^0(X, \mathcal{O}_X(K_X + L)) \longrightarrow H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{O}_X/\mathcal{I}(h))$$

is surjective. Therefore, since the support of $\mathcal{O}_X/\mathcal{I}(h)$ is isolated at x, we can take a global section $\sigma \in H^0(X, \mathcal{O}_X(K_X + L))$ with $\sigma(x) \neq 0$. In particular, we conclude $H^0(X, \mathcal{O}_X(K_X + L)) \neq 0$.

3. Sketch of the proof of Theorem 1.3

We shall prove Theorem 1.3. Let $\Phi_m : X \dashrightarrow \mathbf{P}H^0(X, \mathcal{O}_X(m(K_X + L)))^*$ denote the rational map associated with $|m(K_X + L)|$. By the base point free theorem and by taking an integer $m \gg 1$, we obtain a surjective morphism $f := \Phi_m : X \longrightarrow Y$, where Y denotes the image of X. We may assume that $\kappa(X, K_X + L) = \dim Y$ and $\kappa(F, K_F + L|_F) = 0$ for the general fiber F of f. Taking a suitable modification, we may also assume that Y is smooth. Now we define the reflexive sheaf B on Y by $B := f_*\mathcal{O}_X(K_{X/Y} + L)^{**}$. Since $K_F + L|_F$ is trivial, B is an invertible sheaf on Y. Moreover we have the following:

LEMMA 3.1. *B is big, and $K_Y + B$ is nef and big.*

PROOF. Let h_L be a C^∞-hermitian metric on L with strictly positive curvature. Then we define the singular hermitian metric h_B on B by

$$h_B(\sigma, \sigma) := \int_{X/Y} h_L \cdot \sigma \wedge \overline{\sigma}, \tag{3.1}$$

where $\sigma \in \Gamma(Y, B)$ is a global section of B. Then by [3, Theorem 0.1], we see that h_B has strictly positive curvature current. This implies that B is big. On the other hand, by the construction of B, it follows immediately that $K_Y + B$ is big. (For the nefness of $K_Y + B$, see [2, Lemma 4.3].) □

So it suffices to show the following:

LEMMA 3.2. *$H^0(Y, \mathcal{O}_Y(m(K_Y + B))) \neq 0$ holds for every integer $m \geq d(d+1)/2 + 2$, where $d := \dim Y$.*

SKETCH OF THE PROOF OF LEMMA 3.2. We use the technique adopted by Angehrn and Siu ([1]) and Tsuji ([9]) in their study of Fujita's freeness conjecture. We set $\mu_0 := N^d$ and fix a point y_0 on Y. First, by a dimension counting argument, we have the following:

LEMMA 3.3. *$H^0(Y, \mathcal{O}_Y(m(K_Y + B)) \otimes \mathfrak{m}_{y_0}^{\lceil \sqrt[d]{\mu_0}(1-\varepsilon)m \rceil}) \neq 0$ holds for every $0 < \varepsilon < 1$ and every $m \gg 0$.*

Fix $0 < \varepsilon < 1$ and a positive integer m_0 as in the above lemma, and take a nontrivial global section:

$$\sigma_0 \in H^0(Y, \mathcal{O}_Y(m_0(K_Y + B)) \otimes \mathfrak{m}_{y_0}^{\lceil \sqrt[d]{\mu_0}(1-\varepsilon)m_0 \rceil}).$$

We define the singular hermitian metric h_0 on $K_Y + B$ by $h_0 = |\sigma_0|^{-2/m_0}$. Let α_0 be the positive number defined by $\alpha_0 := \inf\{\alpha > 0 \mid \mathcal{I}(h_0^\alpha)_{y_0} \neq \mathcal{O}_{Y,y_0}\}$. Then by the fact that $(\sum_{i=1}^n |z_i|^2)^{-n}$ is not locally integrable around the origin of $\mathbf{C}^n$, we get $\alpha_0 \leq (d/\sqrt[d]{\mu_0})(1-\varepsilon)^{-1}$. Fix an arbitrary positive number $\lambda \ll 1/d$. Since $\mu_0 \geq 1$ holds, by taking ε sufficiently small, we may assume that $\alpha_0 \leq d + \lambda$ holds.

Let V_1 be the analytic set whose structure sheaf is $\mathcal{O}_Y/\mathcal{I}(h_0^{\alpha_0})$, and Y_1 an irreducible component of V_1 which passes through y_0. Here, for simplicity, we suppose that $\dim Y_1 = 0$. Then we have the following:

LEMMA 3.4. *$H^0(Y, \mathcal{O}_Y(m(K_Y + B))) \neq 0$ holds for every $m \geq d + 2$.*

PROOF. Fix an integer $m \geq \alpha_0$. Then by $\alpha_0 \leq d + \lambda$, we have $m \geq d + 1$.

Since $K_Y + B$ is big, by Kodaira's lemma, we have an effective $\mathbf{Q}$-divisor G on Y such that $K_Y + B - G$ is ample. We may assume that the support of G does not contain y_0. Let $0 < \delta \ll 1$ be a rational number, and we set $A := (m - \alpha_0)(K_Y + B) - \delta G$. Note that A is ample, because $K_Y + B$ is nef. Fix a C^∞-hermitian metric h_A on A with strictly positive curvature. Let $G = \sum e_i E_i$ be the irreducible decomposition of G and $\sigma_i \in \Gamma(Y, E_i)$ a global section with $(\sigma_i) = E_i$. Then we define the singular hermitian metric h on $\mathcal{O}_Y(m(K_Y + B))$ by

$$h := \frac{h_0^{\alpha_0} \cdot h_A}{\prod_i |\sigma_i|^{2\delta e_i}}.$$

Since $h \cdot h_B$ has strictly positive curvature current, by virtue of Theorem 2.4 (cf. Remark 2.4.1), we see that the restriction map:

$$H^0(Y, \mathcal{O}_Y((m+1)(K_Y + B))) \longrightarrow H^0(Y, \mathcal{O}_Y((m+1)(K_Y + B)) \otimes \mathcal{O}_Y/\mathcal{I}(h \cdot h_B)) \tag{3.2}$$

is surjective. Now we may assume that y_0 is not on the singular locus of h_B, and hence $\mathcal{O}_Y/\mathcal{I}(h \cdot h_B)$ has isolated support at y_0. Therefore by the surjectivity of (3.2), there exists a global section $\tau \in H^0(Y, \mathcal{O}_Y((m+1)(K_Y + B)))$ with $\tau(y_0) \neq 0$. We have thus proved the lemma. □

When $\dim Y_1 > 0$, we need to cut down the support of $\mathcal{O}_Y/\mathcal{I}(h \cdot h_B)$ in order to construct a singular hermitian metric as in Remark 2.4.1; by Angehrn-Siu's argument, we obtain the following lemma (see [1], [2, Section 3] for details).

LEMMA 3.5. *Let m be an integer with $m \geq d(d+1)/2+1$. Then there exists a singular hermitian metric h_{y_0} on $\mathcal{O}_Y(m(K_Y+B))$ such that h_{y_0} has strictly positive curvature current, and $\mathcal{O}_Y/\mathcal{I}(h_{y_0})$ has isolated support at y_0.*

Then by an similar argument to that in the proof of Lemma 3.4, we see that there exists a global section $\tau \in H^0(Y, \mathcal{O}_Y(m(K_Y+B))$ with $\tau(y_0) \neq 0$ for every $m \geq d(d+1)/2+2$. □

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