STUDIA MATHEMATICA

Online First version

On Boolean selfdecomposable distributions

by

Takahiro Hasebe (Sapporo), Kei Noba (Tokyo),
Noriyoshi Sakuma (Nagoya) and Yuki Ueda (Asahikawa)

Abstract. This paper introduces the class of selfdecomposable distributions concerning Boolean convolution. A general regularity property of Boolean selfdecomposable distributions is established; in particular, the number of atoms is at most 2 and the singular continuous part is 0. We then analyze how shifting probability measures changes Boolean selfdecomposability. Several examples are presented to supplement the above results. Finally, we prove that the standard normal distribution $N(0,1)$ is Boolean selfdecomposable but the shifted one $N(m,1)$ is not for sufficiently large $|m|$.

1. Introduction. In non-commutative probability theory, random variables are defined as elements in some $*$-algebra. A remarkable aspect of this theory is that various notions of independence exist for those random variables. From a certain viewpoint, notions of independence are classified into five kinds: tensor, free, Boolean, monotone and anti-monotone independence (see [M03]). Further, each notion of independence has the associated convolution of (Borel) probability measures on $\mathbb{R}$ that is defined to be the distribution of the sum of two independent (selfadjoint) random variables having prescribed distributions.

Limit theorems have been central subjects in both commutative (classical) and non-commutative probability theories. For instance, Khinchin introduced the class $\mathcal{L}$ of limit laws of certain independent triangular arrays. More precisely, a probability measure μ on $\mathbb{R}$ belongs to $\mathcal{L}$ if there exist a sequence $\{X_n\}_{n\geq 1}$ of independent $\mathbb{R}$-valued random variables and sequences $\{a_n\}_{n\geq 0} \subset \mathbb{R}$ and $\{b_n\}_{n\geq 1} \subset (0,\infty)$ of deterministic numbers such that

- the family $\{b_n X_k\}_{n\geq k\geq 1}$ forms an infinitesimal triangular array, i.e.

$$\lim_{n\to\infty} \sup_{1\leq k\leq n} P[|b_n X_k| \geq \varepsilon] = 0 \quad \text{for all } \varepsilon > 0,$$

2020 *Mathematics Subject Classification*: Primary 46L54; Secondary 60E07.
Key words and phrases: Boolean selfdecomposable distributions, Boolean convolution.
Received 27 January 2022; revised 5 October 2023.
Published online 24 January 2024.

- the law of the sequence of random variables

$$S_n = a_n + b_n(X_1 + \cdots + X_n), \qquad n \geq 1,$$

converges weakly to μ.

When $\{X_n\}_{n\geq 1}$ is furthermore identically distributed, the limit distribution μ, if it exists, is known to be a stable distribution. Thus the class $\mathcal{L}$ contains the class of stable distributions as a subset. In 1937, Lévy characterized $\mathcal{L}$ by the following property (see e.g. [GK54, Section 29, Theorem 1]): A probability measure μ on $\mathbb{R}$ is in $\mathcal{L}$ if and only if μ is *selfdecomposable*, i.e., for any $c \in (0,1)$, there exists a probability measure μ_c on $\mathbb{R}$ such that $\mu = D_c(\mu) * \mu_c$, where $D_c(\mu)$ is the push-forward of μ by the mapping $x \mapsto cx$, and $*$ denotes the classical convolution (see e.g. [Sat13]).

In non-commutative probability theory, an analogous limit theorem can be formulated for each notion of independence. Bercovici and Pata [BP00], Chistyakov and Götze [CG08] and Wang [W08] proved a parallelism between classical Khinchin limit theorem above and its non-commutative versions (for further details see Subsection 2.3 below). Correspondingly, notions of selfdecomposable distributions with respect to other convolutions are also defined (see Definition 3.1 below). In particular, the two classes of freely and monotonically selfdecomposable distributions were investigated in detail (see e.g. [BNT02, HST19, HT16] and [FHS20], respectively, where analogy and disanalogy with the classical class $\mathcal{L}$ are discussed). By contrast, little has been done on Boolean selfdecomposable distributions. Although our definition of the latter is natural for specialists, there has been no formal definition in the literature to the authors' best knowledge.

The purpose of this paper is to study the class of Boolean selfdecomposable distributions. Results will be presented along the following lines. In Section 2, we introduce some concepts and some preliminary results that are used in the remainder of the paper. In Section 3, we establish a general regularity result on Boolean selfdecomposable distributions. Specifically, we demonstrate that they have at most two atoms and do not have singular continuous part. We then investigate how Boolean selfdecomposability changes under shifts. It turns out that it is typically lost when the distribution is shifted by a sufficiently large positive or negative number. Furthermore, we observe some distributional properties of Boolean selfdecomposable distributions through several examples.

In Section 4, we study Boolean selfdecomposability for normal distributions. The results in that section are motivated by the fact that every normal distribution is both classically and freely selfdecomposable (see e.g. [Sat13] and [HST19], respectively). Our conclusion is that the standard normal distribution $N(0,1)$ is Boolean selfdecomposable too; however, the shifted one $N(m,1)$ is not Boolean selfdecomposable when $|m|$ is sufficiently large, as a

consequence of the aforementioned general result in Section 3. Simulations suggest that $|m| \simeq 3.09$ is the approximate threshold.

2. Preliminaries. In the first part of this section, we introduce analytic tools for understanding Boolean and free additive convolutions. Then we summarize a transfer principle for limit theorems for convolutions, which induces a bijection between different kinds of infinitely divisible distributions.

2.1. Boolean convolution and analytic tools. The Boolean convolution of probability measures on $\mathbb{R}$ was defined in [SW97]. We set $\mathcal{P}(\mathbb{R})$ to be the set of all Borel probability measures on $\mathbb{R}$, $\mathbb{C}^+$ the complex upper half-plane and $\mathbb{C}^-$ the complex lower half-plane. For $\mu \in \mathcal{P}(\mathbb{R})$, we define the *Cauchy transform* $G_\mu \colon \mathbb{C}^+ \to \mathbb{C}^-$ and the *F-transform* $F_\mu \colon \mathbb{C}^+ \to \mathbb{C}^+$ as follows:

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t}\,\mu(\mathrm{d}t) \quad \text{and} \quad F_\mu(z) := \frac{1}{G_\mu(z)}, \qquad z \in \mathbb{C}^+.$$

The *self-energy function* K_μ *of* μ is defined by

$$K_\mu(z) := z - F_\mu(z), \qquad z \in \mathbb{C}^+.$$

Since $\mathsf{Im}[F_\mu(z)] \geq \mathsf{Im}[z]$ for $z \in \mathbb{C}^+$ (see [BV93, Corollary 5.3]), the function K_μ is an analytic map from $\mathbb{C}^+$ to $\mathbb{C}^- \cup \mathbb{R}$. The self-energy functions are characterized in the following way (see [SW97, Proposition 3.1] for further details).

PROPOSITION 2.1. *Let K be an analytic function on $\mathbb{C}^+$. The following assertions are equivalent:*

(1) *There exists $\mu \in \mathcal{P}(\mathbb{R})$ such that $K = K_\mu$.*
(2) *There exist $b \in \mathbb{R}$ and a finite positive measure τ on $\mathbb{R}$, such that K has the form*

$$K(z) = b + \int_{\mathbb{R}} \frac{1+xz}{z-x}\,\tau(\mathrm{d}t), \qquad z \in \mathbb{C}^+. \tag{2.1}$$

If either of (1) *or* (2) *holds, then the pair (b,τ) is uniquely determined by μ (from formulas* (2.2)–(2.3) *below).*

We call such a pair (b,τ) the *Boolean generating pair* for μ. By Proposition 2.1, there is a one-to-one correspondence between $\mathcal{P}(\mathbb{R})$ and the set of all Boolean generating pairs. Hence, we denote by $\mu_{\uplus}^{(b,\tau)}$ the probability measure μ with Boolean generating pair (b,τ). The Boolean generating pair

for $\mu_{\uplus}^{(b,\tau)}$ can be computed from the following formulas:

$$K_{\mu_{\uplus}^{(b,\tau)}}(i) = b - i\tau(\mathbb{R}), \tag{2.2}$$

$$\int_{\alpha}^{\beta}(1+t^2)\,\tau(\mathrm{d}t) = -\frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{\alpha}^{\beta}\mathsf{Im}[K_{\mu_{\uplus}^{(b,\tau)}}(x+i\varepsilon)]\,\mathrm{d}x \tag{2.3}$$

for all $\alpha, \beta \in \mathbb{R}$ that are continuity points of τ. The latter formula is referred to as the *Stieltjes inversion formula*, due to

$$\mathsf{Im}[K_{\mu_{\uplus}^{(b,\tau)}}(x+i\varepsilon)] = -\int_{\mathbb{R}}\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\cdot(1+t^2)\,\tau(\mathrm{d}t).$$

For $\mu, \nu \in \mathcal{P}(\mathbb{R})$, their *Boolean convolution* $\mu \uplus \nu$ is the probability measure on $\mathbb{R}$ satisfying

$$K_{\mu\uplus\nu}(z) = K_\mu(z) + K_\nu(z), \qquad z \in \mathbb{C}^+. \tag{2.4}$$

The *η-transform* of $\mu \in \mathcal{P}(\mathbb{R})$ is defined by

$$\eta_\mu(z) := 1 - zF_\mu\left(\frac{1}{z}\right), \qquad z \in \mathbb{C}^-.$$

It is obvious that $\eta_\mu(z) = zK_\mu(1/z)$, and therefore $\eta_{\mu\uplus\nu}(z) = \eta_\mu(z) + \eta_\nu(z)$ for all $z \in \mathbb{C}^-$.

A probability measure μ is said to be *Boolean infinitely divisible* if for each $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{P}(\mathbb{R})$ such that $\mu = \mu_n \uplus \cdots \uplus \mu_n$ (n times). For each $\mu \in \mathcal{P}(\mathbb{R})$ with Boolean generating pair (b, τ), set μ_n to be the probability measure with Boolean generating pair $(b/n, \tau/n)$ for each $n \in \mathbb{N}$. Then μ is the n-fold Boolean convolution of μ_n. Therefore every probability measure on $\mathbb{R}$ is Boolean infinitely divisible.

For a probability measure μ on $\mathbb{R}$ with Boolean generating pair (b, τ), let us set

$$\begin{cases} a = \tau(\{0\}), \\ \nu(\mathrm{d}t) = \dfrac{1+t^2}{t^2}\cdot\mathbf{1}_{\mathbb{R}\setminus\{0\}}(t)\tau(\mathrm{d}t), \\ \gamma = b + \displaystyle\int_{\mathbb{R}} t\left(\mathbf{1}_{[-1,1]}(t) - \frac{1}{1+t^2}\right)\nu(\mathrm{d}t). \end{cases} \tag{2.5}$$

The triplet (a, ν, γ) thus defined satisfies

(T) $a \geq 0$, $\gamma \in \mathbb{R}$ and ν is a positive measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge x^2)\,\nu(\mathrm{d}x) < \infty$.

Moreover, the set of triplets (a, ν, γ) satisfying (T) is in bijection with the set of pairs (b, τ) of a real number b and a finite positive measure τ on $\mathbb{R}$. In

terms of this bijection, formula (2.1) has the following equivalent form:

$$\eta_\mu(z) = zK_\mu\left(\frac{1}{z}\right) = \gamma z + az^2 + \int_{\mathbb{R}}\left(\frac{1}{1-zt} - 1 - zt\mathbf{1}_{[-1,1]}(t)\right)\nu(\mathrm{d}t). \tag{2.6}$$

The real number $a \geq 0$ is called the *Boolean Gaussian component* for μ, and the measure ν is called the *Boolean Lévy measure* for μ. The triplet (a, ν, γ) is called the *Boolean Lévy triplet*.

Let $\mu \in \mathcal{P}(\mathbb{R})$ and ν be its Boolean Lévy measure. Let ν^{ac} be the Lebesgue absolutely continuous part of ν. We introduce functions $k_\mu, \ell_\mu \colon \mathbb{R} \setminus \{0\} \to [0, \infty)$ by

$$k_\mu(x) = |x|\frac{\mathrm{d}\nu^{\mathrm{ac}}}{\mathrm{d}x}(x) \quad \text{and} \quad \ell_\mu(x) = |x|k_\mu(x), \qquad x \in \mathbb{R} \setminus \{0\}. \tag{2.7}$$

Those functions are key ingredients in our main results. We observe that $\nu^{\mathrm{ac}}(\mathrm{d}x)$ can be expressed in the form $x^{-2}\ell_\mu(x)\,\mathrm{d}x = |x|^{-1}k_\mu(x)\,\mathrm{d}x$.

Remark 2.2. The pair $(k_\mu(x),\ell_\mu(x))$ is determined uniquely in the sense of Remark 15.12(ii) in the book by Sato [Sat13].

2.2. Free convolution and analytic tools. The free convolution of general probability measures on $\mathbb{R}$ was defined in [BV93]. According to [BV93, Proposition 5.4], for any $\mu \in \mathcal{P}(\mathbb{R})$ and any $\lambda > 0$ there exist $\alpha, \beta, M > 0$ such that F_μ is univalent on the set $\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ : \mathsf{Im}[z] > \beta, |\mathsf{Re}[z]| < \alpha\,\mathsf{Im}[z]\}$ and $\Gamma_{\lambda,M} \subset F_\mu(\Gamma_{\alpha,\beta})$. This implies that the right inverse function F_μ^{-1} of F_μ exists on $\Gamma_{\lambda,M}$. The *Voiculescu transform* φ_μ is defined by

$$\varphi_\mu(z) := F_\mu^{-1}(z) - z, \qquad z \in \Gamma_{\lambda,M}.$$

For $\mu, \nu \in \mathcal{P}(\mathbb{R})$, their *free convolution* $\mu \boxplus \nu$ is the probability measure satisfying

$$\varphi_{\mu\boxplus\nu}(z) = \varphi_\mu(z) + \varphi_\nu(z) \tag{2.8}$$

for z in the common domain in which the three transforms are defined.

The *R-transform* of $\mu \in \mathcal{P}(\mathbb{R})$ is defined by

$$R_\mu(z) := z\varphi_\mu\left(\frac{1}{z}\right), \qquad z \in \mathbb{C}^- \text{ with } \frac{1}{z} \in \Gamma_{\lambda,M}.$$

It is obvious that $R_{\mu\boxplus\nu}(z) = R_\mu(z) + R_\nu(z)$ for all z in the common domain in which the three transforms are defined.

A probability measure μ on $\mathbb{R}$ is said to be *freely infinitely divisible* (denoted by $\mu \in I(\boxplus)$) if for each $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{P}(\mathbb{R})$ such that $\mu = \mu_n \boxplus \cdots \boxplus \mu_n$ (n times). Several criteria for free infinite divisibility were given by using harmonic analysis, complex analysis, and combinatorics (see e.g. [BBLS11, AH13]). In particular, the following characterization of $I(\boxplus)$ is well known (see [BV93, Theorem 5.10]).

PROPOSITION 2.3. *For $\mu \in \mathcal{P}(\mathbb{R})$, the following conditions are equivalent.*

(1) $\mu \in I(\boxplus)$.
(2) *The Voiculescu transform φ_μ has an analytic extension (denoted by the same symbol φ_μ) defined on $\mathbb{C}^+$ with values in $\mathbb{C}^- \cup \mathbb{R}$.*
(3) *There exist $b \in \mathbb{R}$ and a finite positive measure τ on $\mathbb{R}$ such that*

$$\varphi_\mu(z) = b + \int_{\mathbb{R}} \frac{1+tz}{z-t}\,\tau(\mathrm{d}t), \qquad z \in \mathbb{C}^+. \tag{2.9}$$

Note that such a pair (b,τ) is uniquely determined by μ for the same reason as (2.2)–(2.3). Conversely, given $b \in \mathbb{R}$ and a finite positive measure τ on $\mathbb{R}$, there exists $\mu \in I(\boxplus)$ such that (2.9) holds.

The above pair (b,τ) is called the *free generating pair* for μ, and we denote by $\mu_{\boxplus}^{(b,\tau)}$ the freely infinitely divisible distribution with free generating pair (b,τ).

For $\mu \in I(\boxplus)$ with free generating pair (b,τ), formula (2.9) is equivalent to

$$R_\mu(z) = \gamma z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1-zt} - 1 - zt\mathbf{1}_{[-1,1]}(t) \right) \nu(\mathrm{d}t), \qquad z \in \mathbb{C}^-, \tag{2.10}$$

where the triplet (a,ν,γ) is given by (2.5); see [BNT02] for further details. The real number $a \geq 0$ is called the *free Gaussian component* for μ, and the measure ν is called the *free Lévy measure* for μ.

2.3. Boolean-to-free Bercovici–Pata bijection. In [BP99, Theorem 6.3], Bercovici and Pata found a remarkable equivalence between limit theorems for the convolutions $*$, $\boxplus$ and $\uplus$. It leads to a bijection between the corresponding three classes of infinitely divisible distributions. Bercovici and Pata's work concerned limit theorems for i.i.d. random variables. We present here a statement in a generalized setting of infinitesimal triangular arrays, combining [BP00, Theorem 1], [CG08, Theorem 2.1] and [W08, Theorem 5.3]. When μ_n, μ are finite positive measures on $\mathbb{R}$, we use the notation $\mu_n \xrightarrow{w} \mu$ to mean that μ_n converges weakly to μ.

THEOREM 2.4. *Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers and let $\{\mu_{n,k}\}_{1\leq k\leq k_n,\, n\geq 1}$ be a family (or an array) of probability measures on $\mathbb{R}$ such that $\{k_n\}_{n\geq 1}$ is a sequence of natural numbers which tends to infinity and*

$$\lim_{n\to\infty} \sup_{1\leq k\leq k_n} \mu_{n,k}(\mathbb{R}\setminus(-\varepsilon,\varepsilon)) = 0 \quad \text{for all } \varepsilon > 0. \tag{2.11}$$

We set

$$a_{n,k} = \int_{|x|<1} t\,\mu_{n,k}(\mathrm{d}t) \quad \text{and} \quad \mathring{\mu}_{n,k}(B) = \mu_{n,k}(B + a_{n,k})$$

for Borel subsets B of $\mathbb{R}$. Then the following conditions are equivalent:

(1) $\delta_{a_n} \uplus \mu_{n,1} \uplus \mu_{n,2} \uplus \cdots \uplus \mu_{n,k_n} \xrightarrow{w} \mu$ *for some* $\mu \in \mathcal{P}(\mathbb{R})$ *as* $n \to \infty$.
(2) $\delta_{a_n} \boxplus \mu_{n,1} \boxplus \mu_{n,2} \boxplus \cdots \boxplus \mu_{n,k_n} \xrightarrow{w} \mu'$ *for some* $\mu' \in \mathcal{P}(\mathbb{R})$ *as* $n \to \infty$.
(3) $\delta_{a_n} * \mu_{n,1} * \mu_{n,2} * \cdots * \mu_{n,k_n} \xrightarrow{w} \mu''$ *for some* $\mu'' \in \mathcal{P}(\mathbb{R})$ *as* $n \to \infty$.
(4) *There exist* $b \in \mathbb{R}$ *and a positive finite measure* τ *on* $\mathbb{R}$ *such that*

$$\lim_{n\to\infty}\left[a_n + \sum_{k=1}^{k_n}\left(a_{n,k} + \int_{\mathbb{R}} \frac{t}{t^2+1}\,\mathring{\mu}_{n,k}(\mathrm{d}t)\right)\right] = b,$$

$$\sum_{k=1}^{k_n} \frac{t^2}{t^2+1}\,\mathring{\mu}_{n,k}(\mathrm{d}t) \xrightarrow{w} \tau(\mathrm{d}t) \quad \text{as } n \to \infty.$$

Moreover, if one of those statements holds then $\mu = \mu_{\uplus}^{(b,\tau)}$, $\mu' = \mu_{\boxplus}^{(b,\tau)}$ *and* $\mu'' = \mu_{*}^{(b,\tau)}$, *where the last measure is the infinitely divisible distribution characterized by the Lévy–Khinchin representation*

$$\int_{\mathbb{R}} e^{zt}\,\mu_{*}^{(b,\tau)}(\mathrm{d}t) = \exp\left[bz + \int_{\mathbb{R}}\left(e^{zt} - 1 - \frac{zt}{1+t^2}\right)\frac{1+t^2}{t^2}\,\tau(\mathrm{d}t)\right], \qquad z \in i\mathbb{R}.$$

From this result, it is natural to identify the limit distributions μ, μ', μ''. For later use, we only formulate the map Λ_B between the two classes $\mathcal{P}(\mathbb{R})$ and $I(\boxplus)$ sending $\mu_{\uplus}^{(b,\tau)} \in \mathcal{P}(\mathbb{R})$ to $\mu_{\boxplus}^{(b,\tau)} \in I(\boxplus)$ for each $b \in \mathbb{R}$ and each finite positive measure τ on $\mathbb{R}$. This map Λ_B is obviously a bijection and is called the *Boolean-to-free Bercovici-Pata bijection*. It turns out that Λ_B has the following properties.

LEMMA 2.5. *Let* $\mu_1, \mu_2, \mu \in \mathcal{P}(\mathbb{R})$ *and* $c \in \mathbb{R}$. *Then:*

(1) $R_{\Lambda_B(\mu)} = \eta_\mu$ *on* $\mathbb{C}^-$.
(2) $\Lambda_B(\mu_1 \uplus \mu_2) = \Lambda_B(\mu_1) \boxplus \Lambda_B(\mu_2)$.
(3) $\Lambda_B(D_c(\mu)) = D_c(\Lambda_B(\mu))$.
(4) $\Lambda_B(\delta_c) = \delta_c$.
(5) Λ_B *is a homeomorphism with respect to weak convergence.*

Formulas (1)–(4) can be checked by straightforward algebraic calculations, while assertion (5) needs careful analysis. The reader is referred to [BN08, (2.20)] for formula (1) and to [BNT02] for the other assertions.

Theorem 2.4 implies the following equivalence of Lévy's limit theorems (the classical one is already mentioned in the Introduction). Although this will not be directly used in our paper, it will serve as a good motivation for studying Boolean selfdecomposable distributions.

COROLLARY 2.6. *Let* $\{a_n\}_{n\geq 1}$ *be a sequence of real numbers,* $\{b_n\}_{n\geq 1}$ *be a sequence of positive real numbers and* $\{\mu_n\}_{n\geq 1}$ *a sequence of probability*

measures on $\mathbb{R}$ *such that* (2.11) *is satisfied for the array* $\mu_{n,k} := D_{b_n}(\mu_k)$, $1 \le k \le n$, $1 \le n$. *Then the following conditions are equivalent:*

(1) $\delta_{a_n} \uplus D_{b_n}(\mu_1 \uplus \cdots \uplus \mu_n) \xrightarrow{w} \mu$ *for some* $\mu \in \mathcal{P}(\mathbb{R})$ *as* $n \to \infty$.
(2) $\delta_{a_n} \boxplus D_{b_n}(\mu_1 \boxplus \cdots \boxplus \mu_n) \xrightarrow{w} \mu'$ *for some* $\mu' \in \mathcal{P}(\mathbb{R})$ *as* $n \to \infty$.
(3) $\delta_{a_n} * D_{b_n}(\mu_1 * \cdots * \mu_n) \xrightarrow{w} \mu''$ *for some* $\mu'' \in \mathcal{P}(\mathbb{R})$ *as* $n \to \infty$.

The possible limit distributions μ, μ', μ'' in Corollary 2.6 belong to certain subclasses of freely, Boolean and classically infinitely divisible distributions, respectively. Those subclasses are all characterized by the respective notions of *selfdecomposability* defined later in Definition 3.1. This fact follows e.g. from the last part of Theorem 2.4, [GK54, Section 29, Theorem 1] and [BNT02, Theorem 4.8] (or [CG08, Theorem 2.10]) for $*$ and $\boxplus$. The case of $\uplus$ can be treated similarly to (and is actually easier than) the free case. See also the paragraph following Definition 3.1.

3. Boolean selfdecomposable distributions

3.1. The class of Boolean selfdecomposable distributions. The classical notion of selfdecomposability can be extended to general convolutions of probability measures in the following way.

Definition 3.1. Let $\circ$ be a binary operation on $\mathcal{P}(\mathbb{R})$. Then a measure $\mu \in \mathcal{P}(\mathbb{R})$ is said to be $\circ$-*selfdecomposable* (denoted $\mu \in L(\circ)$) if for any $c \in (0,1)$ there is $\mu_c \in \mathcal{P}(\mathbb{R})$ such that $\mu = D_c(\mu) \circ \mu_c$. In particular, if $\circ = \uplus$ (resp. $\circ = \boxplus$), then μ is said to be *Boolean selfdecomposable* (resp. *freely selfdecomposable*).

By Lemma 2.5(2)–(3) and [BNT02, Theorem 4.6, Proposition 4.7], we have

$$\Lambda_B(L(\uplus)) = L(\boxplus) \quad \text{and} \quad L(\uplus) = \Lambda_B^{-1}(L(\boxplus)). \tag{3.1}$$

It is known that the free selfdecomposability has the following characterization (see e.g. [HT16, Subsection 2.2]).

Lemma 3.2. *A probability measure* μ *on* $\mathbb{R}$ *belongs to the class* $L(\boxplus)$ *if and only if* μ *is a freely infinitely divisible distribution whose free Lévy measure* ν *is Lebesgue absolutely continuous and the function* $\mathbb{R} \setminus \{0\} \to [0,\infty)$, $x \mapsto |x|\frac{d\nu}{dx}(x)$, *has a version with respect to the Lebesgue measure that is unimodal with mode* 0, *i.e. non-decreasing on* $(-\infty, 0)$ *and non-increasing on* $(0,\infty)$.

By the definition of Λ_B, we obtain a characterization for the Boolean selfdecomposability exactly in the same way as Lemma 3.2, in which the free Lévy measure is replaced with the Boolean Lévy measure.

Proposition 3.3. *A probability measure* μ *on* $\mathbb{R}$ *belongs to the class* $L(\uplus)$ *if and only if its Boolean Lévy measure is Lebesgue absolutely continuous and*

the function k_μ in (2.7) *has a version with respect to the Lebesgue measure that is unimodal with mode* 0.

When we consider $\mu \in L(\uplus)$, for simplicity we always take k_μ itself to be unimodal with mode 0 unless specified otherwise.

Given a probability measure μ, a typical method for proving it to be Boolean selfdecomposable is Proposition 3.3. To check that the function k_μ is unimodal with mode 0, we can calculate k_μ from Proposition 3.4. To check that the Boolean Lévy measure is Lebesgue absolutely continuous, we provide a practical sufficient condition in Proposition 3.5.

PROPOSITION 3.4. *Suppose that $\mu \in \mathcal{P}(\mathbb{R})$. The function $k_\mu \colon \mathbb{R} \setminus \{0\} \to [0,\infty)$ defined by* (2.7) *is then given by*

$$k_\mu(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi |x|} \operatorname{Im}[F_\mu(x + i\varepsilon)] \quad \text{for a.e. } x \in \mathbb{R} \setminus \{0\} \tag{3.2}$$

and the Boolean Gaussian component of μ is given by $-\lim_{\varepsilon \to 0^+} i\varepsilon F_\mu(i\varepsilon)$.

Proof. Both formulas are basic. Let ν be the Boolean Lévy measure for μ and (b, τ) be the Boolean generating pair for μ. By Proposition 2.1 and relation (2.5), the F-transform of μ has the form

$$\begin{aligned} F_\mu(z) &= z - b - \int_{\mathbb{R}} \left(\frac{1+t^2}{z-t} + t \right) \tau(\mathrm{d}t) \\ &= z - b - \frac{a}{z} + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) |t|^2 \, \nu(\mathrm{d}t). \end{aligned} \tag{3.3}$$

Since the Lebesgue absolutely continuous part of $|t|^2 \nu(\mathrm{d}t)$ is $|t| k_\mu(t)\, \mathrm{d}t$, the desired formula for k_μ is a consequence of the Stieltjes inversion formula (see e.g. [HO07, Corollary 1.103] or [Sch12, Theorem F.6]). The formula for the Boolean Gaussian component follows from the Dominated Convergence Theorem. ∎

PROPOSITION 3.5. *Let $\mu \in \mathcal{P}(\mathbb{R})$. Suppose that there exists an at most countable subset C of $\mathbb{R}$ such that*

(1) $0 \in C$,
(2) $\lim_{\varepsilon \to 0^+} \operatorname{Im}[F_\mu(x + i\varepsilon)] \in [0,\infty)$ *exists for all* $x \in \mathbb{R} \setminus C$,
(3) $\lim_{\varepsilon \to 0^+} \varepsilon F_\mu(x + i\varepsilon) = 0$ *for all* $x \in C \setminus \{0\}$.

Then the Boolean Lévy measure of μ is Lebesgue absolutely continuous.

Proof. Let ν be the Boolean Lévy measure of μ and $\rho(\mathrm{d}t) := |t|^2 \nu(\mathrm{d}t)$. Note that $\rho(\{0\}) = 0$. It suffices to show that ρ is Lebesgue absolutely continuous. Let $\rho = \rho_{\mathrm{sing}} + \rho_{\mathrm{ac}}$ be the decomposition of ρ into the singular part and the absolutely continuous part with respect to the Lebesgue measure.

From formula (3.3) and [Sch12, Theorem F.6], ρ_{sing} is supported on the set

$$S_\rho^{\mathrm{sing}} := \left\{ x \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \mathsf{Im}[F_\mu(x+i\varepsilon)] = \infty \right\}$$

in the sense that $\rho_{\mathrm{sing}}(\mathbb{R} \setminus S_\rho^{\mathrm{sing}}) = 0$. By assumption (2), the set S_ρ^{sing} is contained in C and hence is at most countable. This implies that ρ_{sing} is a discrete measure. Then assumption (3) implies that ρ does not have an atom in C, so that $\rho_{\mathrm{sing}} = 0$. ∎

Remark 3.6. It is clear that $L(\uplus) \not\subseteq L(\boxplus)$ since the Bernoulli distribution $\frac{1}{2}(\delta_1 + \delta_{-1})$ is in $L(\uplus)$, but not in $L(\boxplus)$.

The following example shows that $L(\boxplus) \not\subseteq L(\uplus)$. Let $\mathbf{f}_{1/2}$ be the positive free 1/2-stable distribution defined by

$$\varphi_{\mathbf{f}_{1/2}}(z) = \sqrt{-z}, \qquad z \in \mathbb{C}^+,$$

where the square root above is defined as the principal branch (see [BP99, Appendix] for details). Stability implies selfdecomposability, so $\mathbf{f}_{1/2} \in L(\boxplus)$. To see $\mathbf{f}_{1/2} \notin L(\uplus)$, inverting $F^{-1}_{\mathbf{f}_{1/2}}(z) = z + \sqrt{-z}$ yields

$$F_{\mathbf{f}_{1/2}}(z) = \frac{2z - 1 - \sqrt{1-4z}}{2}, \qquad z \in \mathbb{C}^+.$$

By Proposition 3.5, the Boolean Lévy measure is Lebesgue absolutely continuous. By Proposition 3.4 the function $k_{\mathbf{f}_{1/2}}$ can be calculated as

$$k_{\mathbf{f}_{1/2}}(x) = \frac{\sqrt{4x-1}}{2\pi x} \cdot \mathbf{1}_{[1/4,\infty)}(x).$$

Since $k_{\mathbf{f}_{1/2}}$ is not non-increasing on $(0,\infty)$, the desired conclusion $\mathbf{f}_{1/2} \notin L(\uplus)$ follows.

Example 3.7. The probability measure having the Boolean Lévy triplet $(a, 0, \gamma)$ with $a > 0$ is called a *Boolean Gaussian distribution* and will be denoted $B(\gamma, a)$. It has mean γ, variance a and is of the form

(3.4)
$$\left(\frac{1}{2} - \frac{\gamma}{2\sqrt{\gamma^2+4a^2}}\right)\delta_{(\gamma-\sqrt{\gamma^2+4a^2})/2} + \left(\frac{1}{2} + \frac{\gamma}{2\sqrt{\gamma^2+4a^2}}\right)\delta_{(\gamma+\sqrt{\gamma^2+4a^2})/2}.$$

Because the Boolean Lévy measure is zero, $B(\gamma, a)$ is Boolean selfdecomposable. Note that $B(a, \gamma)$ can be written in the simpler form

$$\frac{1}{\beta - \alpha}(\beta\delta_\beta - \alpha\delta_\alpha)$$

with parameters $\alpha = (\gamma - \sqrt{\gamma^2+4a^2})/2 < 0$ and $\beta = (\gamma + \sqrt{\gamma^2+4a^2})/2 > 0$. In fact, the map $(\gamma, a) \mapsto (\alpha, \beta)$ gives a bijection from $\mathbb{R} \times (0,\infty)$ to $(-\infty, 0) \times (0,\infty)$.

In particular, one can see that $p\delta_1 + (1-p)\delta_{-1}$ for $p \in (0,1)$ is a Boolean Gaussian distribution if and only if $p = 1/2$.

EXAMPLE 3.8. The probability measure $\mathbf{b}_{\alpha,\rho}$ characterized by

$$\eta_{\mathbf{b}_{\alpha,\rho}}(z) = -(e^{i\rho\pi}z)^{\alpha}, \quad z \in \mathbb{C}^-,$$

is called a *Boolean* (*strictly*) *stable distribution*, where the parameter (α,ρ) belongs to the set

$$\mathfrak{A} = \{(\alpha,\rho) : \alpha \in (0,1],\ \rho \in [0,1]\} \cup \{(\alpha,\rho) : \alpha \in (1,2],\ \rho \in [1-\alpha^{-1}, \alpha^{-1}]\}.$$

The Boolean stable distributions satisfy the relation $\mathbf{b}_{\alpha,\rho} = D_c(\mathbf{b}_{\alpha,\rho}) \uplus D_{(1-c^\alpha)^{1/\alpha}}(\mathbf{b}_{\alpha,\rho})$ for any $c \in (0,1)$, and hence are Boolean selfdecomposable. Proposition 3.4 implies the formula

$$k_{\mathbf{b}_{\alpha,\rho}}(x) = \frac{\sin(\alpha\rho\pi)}{\pi} x^{-\alpha}\mathbf{1}_{(0,\infty)}(x) + \frac{\sin(\alpha(1-\rho)\pi)}{\pi}|x|^{-\alpha}\mathbf{1}_{(-\infty,0)}(x).$$

The special case $\mathbf{b}_{2,1/2}$ coincides with the Bernoulli distribution $B(0,1)$. For further information on $\mathbf{b}_{\alpha,\rho}$, see e.g. [SW97, HS15].

3.2. Regularity for Boolean selfdecomposable distributions. We establish a general regularity result on Boolean selfdecomposable distributions. Note that we exclude the well understood Boolean Gaussian distributions (see Example 3.7) from the statement in order to have a non-empty support for the function k_μ.

THEOREM 3.9. *Let $\mu \in L(\uplus)$ with the function k_μ not identically equal to zero. Let $\alpha := \inf\{x \in \mathbb{R} : k_\mu(x) \neq 0\} \in [-\infty, 0]$ and $\beta := \sup\{x \in \mathbb{R} : k_\mu(x) \neq 0\} \in [0,\infty]$. Then*

(1) *$\mu|_{(\alpha,\beta)}$ is absolutely continuous with respect to the Lebesgue measure.*
(2) *$\mu|_{(-\infty,\alpha]}$ is the zero measure or a delta measure if $\alpha > -\infty$. The same holds for $\mu|_{[\beta,\infty)}$ if $\beta < \infty$.*
(3) *$\mu(\{0\}) = 0$.*

In particular, μ has no singular continuous part with respect to the Lebesgue measure and has at most two atoms.

Proof. Let us take the Pick–Nevanlinna representation of F_μ in the form

$$F_\mu(z) = z - b - \frac{a}{z} + \int_\alpha^\beta \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)|t|k_\mu(t)\,\mathrm{d}t, \tag{3.5}$$

where $a \geq 0$ and $b \in \mathbb{R}$; see (3.3).

(2) It suffices to prove the statement for $(-\infty,\alpha]$ by symmetry. We assume $\alpha > -\infty$. Formula (3.5) enables F_μ to be extended analytically to $\mathbb{C}^+ \cup (-\infty,\alpha)$. The extended function, still denoted F_μ, takes real values on $(-\infty,\alpha)$; hence, the value of the function $\operatorname{Im}[G_\mu(z)] = -\operatorname{Im}[F_\mu(z)]/|F_\mu(z)|^2$

is zero on $(-\infty,\alpha)\setminus\{\text{zeros of } F_\mu|_{(-\infty,\alpha)}\}$. By the Stieltjes inversion formula, we conclude that

$$\mu\left((-\infty,\alpha)\setminus\{\text{zeros of } F_\mu|_{(-\infty,\alpha)}\}\right)=0.$$

Because $F_\mu'>1$ on $(-\infty,\alpha)$, F_μ has at most one zero on $(-\infty,\alpha)$.

If F_μ has no zeros on $(-\infty,\alpha)$, then $\mu|_{(-\infty,\alpha]}$ is the zero measure or a delta measure at α.

If F_μ has a zero x_0 in $(-\infty,\alpha)$, then

$$\mu(\{x_0\})=\lim_{z\to x_0}\frac{z-x_0}{F_\mu(z)}=\frac{1}{F_\mu'(x_0)}>0.$$

In this case we need to check that $\mu(\{\alpha\})=0$. Because F_μ is strictly increasing and $F_\mu(x_0)=0$, we have $\lim_{x\to\alpha-0}F_\mu(x)\in(0,\infty]$ and hence $G_\mu(\alpha):=\lim_{x\to\alpha-0}G_\mu(x)\in[0,\infty)$. Directly or by Lindelöf's theorem [BCDM20, Theorem 1.5.7], it can be verified that $G_\mu(\alpha)=\lim_{\varepsilon\to0^+}G_\mu(\alpha+i\varepsilon)$. Therefore, $\mu(\{\alpha\})=\lim_{\varepsilon\to0^+}i\varepsilon G_\mu(\alpha+i\varepsilon)=0$.

(3) Because k_μ is unimodal and is not identically zero, we have either $\alpha<0$ or $\beta>0$. Without loss of generality we may work in the latter case. Then $\delta:=\inf_{x\in(0,\beta/2)}k_\mu(x)>0$. Because $\mu(\{0\})=\lim_{\varepsilon\to0^+}i\varepsilon/F_\mu(i\varepsilon)$, the desired assertion $\mu(\{0\})=0$ follows from $\lim_{\varepsilon\to0^+}\mathsf{Im}[F_\mu(i\varepsilon)]/\varepsilon=\infty$. The latter is the case because

$$\begin{aligned}\frac{\mathsf{Im}[F_\mu(i\varepsilon)]}{\varepsilon}&=1+\frac{a}{\varepsilon^2}+\int_\alpha^\beta\frac{|t|k_\mu(t)}{t^2+\varepsilon^2}\,\mathrm{d}t\\&\geq\delta\int_0^{\beta/2}\frac{t}{t^2+\varepsilon^2}\,\mathrm{d}t\to\delta\int_0^{\beta/2}\frac{1}{t}\,\mathrm{d}t=\infty\quad\text{as }\varepsilon\to0^+.\end{aligned}$$

(1) Let $\mu=\mu_{\mathrm{sing}}+\mu_{\mathrm{ac}}$ be the decomposition of μ into the singular part and absolutely continuous part with respect to the Lebesgue measure. From (3), it suffices to prove $\mu_{\mathrm{sing}}((\alpha,\beta)\setminus\{0\})=0$. As in the proof of Proposition 3.5, $\mu_{\mathrm{sing}}(\mathbb{R}\setminus S)=0$, where

$$S:=\Big\{x\in\mathbb{R}:\lim_{\varepsilon\to0^+}\mathsf{Im}[-G_\mu(x+i\varepsilon)]=\infty\Big\}.$$

Let us assume that $\beta>0$; otherwise we need not work on the interval $(0,\beta)$. Because k_μ is unimodal with mode 0, for every $0<\beta'<\beta$ the number $\delta':=\inf_{x\in(0,\beta']}k_\mu(x)$ is positive. For every $\varepsilon>0$ and $x\in(0,\beta')$ we have

$$\begin{aligned}\mathsf{Im}[F_\mu(x+i\varepsilon)]&=\varepsilon+\frac{a}{\varepsilon}+\int_\alpha^\beta\frac{\varepsilon|t|k_\mu(t)}{(t-x)^2+\varepsilon^2}\,\mathrm{d}t\\&\geq\delta' x\int_x^{\beta'}\frac{\varepsilon}{(t-x)^2+\varepsilon^2}\,\mathrm{d}t=\delta' x\arctan\frac{\beta'-x}{\varepsilon},\end{aligned}$$

which tends to $\delta' x\pi/2$ as ε tends to 0. This implies that

$$\limsup_{\varepsilon\to 0^+} \mathsf{Im}[-G_\mu(x+i\varepsilon)] = \limsup_{\varepsilon\to 0^+} \frac{\mathsf{Im}[F_\mu(x+i\varepsilon)]}{|F_\mu(x+i\varepsilon)|^2}$$
$$\leq \limsup_{\varepsilon\to 0^+} \frac{1}{\mathsf{Im}[F_\mu(x+i\varepsilon)]} \leq \frac{2}{\delta' x\pi} < \infty$$

for all $x \in (0, \beta')$. We can thus conclude that $(0,\beta) \subseteq \mathbb{R}\setminus S$, i.e. $\mu_{\mathrm{sing}}((0,\beta))$ $= 0$. Similarly or by symmetry, we get $\mu_{\mathrm{sing}}((\alpha,0)) = 0$ if $\alpha < 0$. ∎

Remark 3.10. The delta measures and Boolean Gaussian distributions are obviously singular distributions. Other Boolean selfdecomposable distributions may also have a non-zero singular part; see e.g. Example 3.17(3, 4). By contrast, all classical selfdecomposable distributions and free distributions except for the delta measures are Lebesgue absolutely continuous (see [Sat13, Theorem 27.13] and [HT16, Remark 2], respectively).

Remark 3.11. In the setting of Theorem 3.9, the points α and β may or may not be atoms of μ. To see this let μ be the Boolean selfdecomposable probability measure defined by

$$F_\mu(z) = z - b + \int_0^1 \frac{1}{t-z}|t|\frac{(1-t)^p}{|t|}\,\mathrm{d}t, \qquad z \in \mathbb{C}^+,$$

where $p > 0$ is a parameter and $b \in \mathbb{R}$ is defined so that $F_\mu(1) := 1 - b - \int_0^1 (1-t)^{p-1}\,\mathrm{d}t = 0$. One sees that

$$\frac{F_\mu(z)}{z-1} = \frac{F_\mu(z)-F_\mu(1)}{z-1} = 1 + \int_0^1 \frac{(1-t)^{p-1}}{z-t}\,\mathrm{d}t$$
$$\to q := 1 + \int_0^1 (1-t)^{p-2}\,\mathrm{d}t \in (1,\infty]$$

as $z \to 1$ satisfying $\mathsf{Re}[z] \geq 1$. This implies that $\mu(\{1\}) = 1/q$, which is positive if $p > 1$ and zero if $0 < p \leq 1$.

Example 3.12 (Mixture of a Cauchy distribution and δ_0). Let us set

$$\kappa_p = \frac{p}{\pi(1+x^2)}\mathbf{1}_{\mathbb{R}}(x)\,\mathrm{d}x + (1-p)\delta_0$$

for $p \in [0,1]$. Its reciprocal Cauchy transform is given by

$$F_{\kappa_p}(z) = \frac{1}{\frac{p}{z+i} + \frac{1-p}{z}} = \frac{z(z+i)}{z+(1-p)i}, \qquad z \in \mathbb{C}^+.$$

By Proposition 3.5, the Boolean Lévy measure is Lebesgue absolutely con-

tinuous. By Proposition 3.4, the function k_{κ_p} is given by

$$k_{\kappa_p}(x) = \frac{1}{\pi} \cdot \frac{p|x|}{x^2 + (1-p)^2} \cdot \mathbf{1}_{\mathbb{R}\setminus\{0\}}(x).$$

As long as $p \in (0,1)$, this is not unimodal with mode 0 and hence κ_p is not Boolean selfdecomposable (cf. Theorem 3.9(3)).

According to Theorem 3.9, every Boolean selfdecomposable distribution μ with $k_\mu \neq 0$ has at most two atoms. Here we completely determine the two-point measures which are Boolean selfdecomposable.

PROPOSITION 3.13. *A two-point probability measure on $\mathbb{R}$ belongs to $L(\uplus)$ if and only if it is a Boolean Gaussian distribution.*

Proof. The Boolean Gaussian distribution $B(\gamma, a)$ is a Boolean selfdecomposable two-point probability measure for all $\gamma \in \mathbb{R}$ and $a > 0$; see (3.4).

Conversely, we assume that a two-point probability measure μ belongs to $L(\uplus)$. By Proposition 3.4 we get $k_\mu(x) = 0$ for a.e. $x \in \mathbb{R} \setminus \{0\}$ because F_μ is of the form $(z^2 + pz + q)/(z - r)$, $p, q, r \in \mathbb{R}$. Therefore μ has a Boolean Lévy triplet $(a, 0, \gamma)$ for some $\gamma \in \mathbb{R}$ and some $a \geq 0$. The Boolean Gaussian component a is non-zero; otherwise μ would be a Dirac measure. Consequently, we have $\mu = B(\gamma, a)$ for $\gamma \in \mathbb{R}$ and $a > 0$. ∎

3.3. Boolean selfdecomposability of shifted probability measures. For any $a \in \mathbb{R}$, "the Boolean shift" $\mu \mapsto \mu \uplus \delta_a$ preserves the class $L(\uplus)$. On the other hand, the usual shift $\mu \mapsto \mu * \delta_a$ does not preserve $L(\uplus)$, which can be observed from Proposition 3.13. This phenomenon is investigated in detail below. The function ℓ_λ defined in (2.7) plays a key role.

LEMMA 3.14. *Suppose that $\lambda \in \mathcal{P}(\mathbb{R})$ satisfies the condition*

(C) *the Boolean Gaussian component is zero and the Boolean Lévy measure is Lebesgue absolutely continuous.*

Then for any $m \in \mathbb{R}$ the measure $\lambda * \delta_m$ also satisfies condition (C) *and*

$$\ell_{\lambda * \delta_m}(t) = \ell_\lambda(t - m) \quad \text{for a.e. } t \in \mathbb{R}. \tag{3.6}$$

Proof. According to (2.7), formula (3.3) can be expressed in the form

$$F_\lambda(z) = z - b - \frac{a}{z} + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \ell_\lambda(t)\, \mathrm{d}t, \tag{3.7}$$

where a is the Boolean Gaussian component, which is now 0. This yields

$$\begin{aligned} F_{\lambda * \delta_m}(z) &= F_\lambda(z - m) \\ &= z - b_m + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \ell_\lambda(t - m)\, \mathrm{d}t \end{aligned} \tag{3.8}$$

for some $b_m \in \mathbb{R}$. Therefore, the Boolean Gaussian component of $\lambda * \delta_m$ is zero and its Boolean Lévy measure equals $t^{-2}\ell_\lambda(t-m)\mathbf{1}_{\mathbb{R}\setminus\{0\}}(t)\,\mathrm{d}t$, as desired. ∎

For a clear statement, we define $\mathcal{Q}$ to be the set of probability measures λ on $\mathbb{R}$ such that

(H1) the Boolean Gaussian component of λ is zero,
(H2) the Boolean Lévy measure of λ has the form $t^{-2}\ell_\lambda(t)\mathbf{1}_{\mathbb{R}\setminus\{0\}}(t)\,\mathrm{d}t$, where $\ell_\lambda\colon \mathbb{R}\to[0,\infty)$ is right continuous on $\mathbb{R}\setminus\{m_\lambda\}$ for some $m_\lambda\in\mathbb{R}$.

THEOREM 3.15.

(1) *If $\lambda \in \mathcal{P}(\mathbb{R})\setminus\mathcal{Q}$ then $\lambda*\delta_m \notin L(\uplus)$ for any $m\in\mathbb{R}\setminus\{0\}$.*
(2) *If $\lambda\in\mathcal{Q}$ and there exist $a,b\in\mathbb{R}\setminus\{m_\lambda\}$ with $a<b$ and $\ell_\lambda(a)<\ell_\lambda(b)$ then $\lambda*\delta_m\notin L(\uplus)$ for any $m>\frac{b\ell_\lambda(a)-a\ell_\lambda(b)}{\ell_\lambda(b)-\ell_\lambda(a)}$.*
(3) *If $\lambda\in\mathcal{Q}$ and there exist $c,d\in\mathbb{R}\setminus\{m_\lambda\}$ with $c<d$ and $\ell_\lambda(c)>\ell_\lambda(d)$ then $\lambda*\delta_m\notin L(\uplus)$ for any $m<\frac{c\ell_\lambda(d)-d\ell_\lambda(c)}{\ell_\lambda(c)-\ell_\lambda(d)}$.*
(4) *If $\lambda\in\mathcal{Q}$ and ℓ_λ is constant on $\mathbb{R}\setminus\{m_\lambda\}$ then $\lambda*\delta_m\in L(\uplus)$ for all $m\in\mathbb{R}$.*

Proof. (1) Suppose first that λ does not satisfy (H1), i.e. it has a positive Boolean Gaussian component a. Similarly to (3.8), we obtain

$$F_{\lambda*\delta_m}(z) = z - b_m + \frac{a}{m-z} + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)(t-m)^2\,\mathrm{d}\nu(t-m) \tag{3.9}$$

for some $b_m\in\mathbb{R}$, where ν is the Boolean Lévy measure of λ. This implies that for $m\neq 0$, the Boolean Lévy measure of $\lambda*\delta_m$ is given by $am^{-2}\delta_m + t^{-2}(t-m)^2\mathrm{d}\nu(t-m)$. Because this is not Lebesgue absolutely continuous, $\lambda*\delta_m$ is not Boolean selfdecomposable.

To complete the proof of (1), it suffices to prove that if λ satisfies (H1) and $\lambda*\delta_{m_0}\in L(\uplus)$ for some $m_0\in\mathbb{R}$ then λ satisfies (H2). According to Lemma 3.14, λ has a Lebesgue absolutely continuous Boolean Lévy measure and

$$\begin{aligned}\ell_\lambda(t) &= \ell_{(\lambda*\delta_{m_0})*\delta_{-m_0}}(t) = \ell_{\lambda*\delta_{m_0}}(t+m_0)\\ &= |t+m_0|k_{\lambda*\delta_{m_0}}(t+m_0) \quad \text{for a.e. } t\in\mathbb{R}\setminus\{0\}.\end{aligned} \tag{3.10}$$

Because $k_{\lambda*\delta_{m_0}}$ is unimodal with mode 0, it has a version that is right continuous on $\mathbb{R}\setminus\{0\}$. Therefore, ℓ_λ has a version that is right continuous on $\mathbb{R}\setminus\{-m_0\}$, i.e. (H2) holds.

(2) For all $m>M := \frac{b\ell_\lambda(a)-a\ell_\lambda(b)}{\ell_\lambda(b)-\ell_\lambda(a)}$, observe that $0<m+a<m+b$ and, by Lemma 3.14,

$$k_{\lambda*\delta_m}(m+a) - k_{\lambda*\delta_m}(m+b) = \frac{\ell_\lambda(a)}{m+a} - \frac{\ell_\lambda(b)}{m+b} < 0. \tag{3.11}$$

Because ℓ_λ is right continuous at a and b, inequality (3.11) still holds when a, b are replaced with $a+\varepsilon, b+\varepsilon$ for sufficiently small $\varepsilon > 0$; hence, for any $m > M$, $k_{\lambda * \delta_m}$ does not have a version that is non-increasing on $(0, \infty)$, and consequently $\lambda * \delta_m$ is not in $L(\uplus)$. The proof of (3) is similar.

(4) In this case, the distribution $\lambda * \delta_m$ is a delta measure (when $\ell_\lambda = 0$) or a Cauchy distribution (when ℓ_λ is a positive constant function) for any $m \in \mathbb{R}$, which is Boolean selfdecomposable. ∎

REMARK 3.16. In case (2) it might happen that $\lambda * \delta_m$ is in $L(\uplus)$ for all sufficiently small m. To obtain such an example, the function ℓ_λ needs to be non-decreasing on $\mathbb{R}$; otherwise case (3) would apply. Let λ be a probability measure with Boolean Lévy triplet $(0, k_\lambda(t)\,\mathrm{d}t, 0)$, where

$$k_\lambda(t) = \frac{1}{|t|}\left[\arctan t + \frac{\pi}{2}\right], \quad t \in \mathbb{R} \setminus \{0\}.$$

It can be checked by calculus that the function

$$k_{\lambda * \delta_m}(t) := \frac{1}{|t|}\left[\arctan(t-m) + \frac{\pi}{2}\right], \quad t \in \mathbb{R} \setminus \{0\},$$

is unimodal with mode 0 if and only if $m \le \pi/2$, and hence $\lambda * \delta_m \in L(\uplus)$ for any such m.

3.4. Several examples of Boolean selfdecomposable distributions. We observe distributional properties of Boolean selfdecomposable distributions through several examples.

EXAMPLE 3.17. We give several examples in the class $L(\uplus)$.

(1) The *free Poisson* (or *Marchenko–Pastur*) *distribution* $\mathbf{MP}_\lambda$ is a probability measure defined by

$$\max\{1-\lambda, 0\}\delta_0 + \frac{1}{2\pi x}\sqrt{(\lambda_+ - x)(x - \lambda_-)}\cdot \mathbf{1}_{(\lambda_-, \lambda_+)}(x)\,\mathrm{d}x, \quad \lambda > 0,$$

where $\lambda_\pm = (1 \pm \sqrt{\lambda})^2$. Its F-transform is given by

$$F_{\mathbf{MP}_\lambda}(z) = \frac{z + 1 - \lambda + \sqrt{(z-\lambda_-)(z-\lambda_+)}}{2}, \quad z \in \mathbb{C}^+,$$

where the square root $\sqrt{w}$ is defined continuously on the angles $\arg w \in (0, 2\pi)$. By Propositions 3.5 and 3.4, the Boolean Lévy measure is Lebesgue absolutely continuous and the function $k_{\mathbf{MP}_\lambda}$ is given by

$$k_{\mathbf{MP}_\lambda}(x) = \frac{\sqrt{(x-\lambda_-)(\lambda_+ - x)}}{2\pi x}\cdot \mathbf{1}_{(\lambda_-, \lambda_+)}(x),$$

and therefore $\mathbf{MP}_\lambda \in L(\uplus)$ if and only if $\lambda = 1$. Consequently, $L(\uplus)$ is not closed under the free convolution since $\mathbf{MP}_1 \boxplus \mathbf{MP}_1 = \mathbf{MP}_2 \notin L(\uplus)$, for example.

(2) Let $S(m,\sigma^2)$ be *Wigner's semicircle law*, that is, its probability density function is defined by

$$\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-(x-m)^2}\cdot\mathbf{1}_{[m-2\sigma,m+2\sigma]}(x),\qquad m\in\mathbb{R},\ \sigma>0.$$

Its F-transform is known to be

$$F_{S(m,\sigma^2)}(z)=\frac{z-m+\sqrt{(z-m)^2-4\sigma^2}}{2},\qquad z\in\mathbb{C}^+.$$

By Proposition 3.5 the Boolean Lévy measure is Lebesgue absolutely continuous and by Proposition 3.4 we have

$$k_{S(m,\sigma^2)}(x)=\frac{\sqrt{4\sigma^2-(x-m)^2}}{2\pi|x|}\cdot\mathbf{1}_{[m-2\sigma,m+2\sigma]\setminus\{0\}}(x),$$

and therefore $S(m,\sigma^2)\in L(\uplus)$ if and only if $m-2\sigma\le 0\le m+2\sigma$.

(3) For any $t>1$, the measure $B(0,1)^{\boxplus t}$ is given by

$$\max\left\{\frac{2-t}{2},0\right\}(\delta_t+\delta_{-t})+\frac{1}{2\pi}\cdot\frac{t\sqrt{4(t-1)-x^2}}{t^2-x^2}\cdot\mathbf{1}_{[-2\sqrt{t-1},2\sqrt{t-1}]}(x)\,\mathrm{d}x.$$

It is called the *Kesten distribution*. The F-transform is given by

$$F_{B(0,1)^{\boxplus t}}(z)=\frac{(t-2)z+t\sqrt{z^2-4(t-1)}}{2(t-1)},\qquad z\in\mathbb{C}^+.$$

By Propositions 3.5 and 3.4, the Boolean Lévy measure is Lebesgue absolutely continuous and

$$k_{B(0,1)^{\boxplus t}}(x)=\frac{t\sqrt{4(t-1)-x^2}}{2\pi(t-1)|x|}\cdot\mathbf{1}_{[-2\sqrt{t-1},2\sqrt{t-1}]\setminus\{0\}}(x).$$

Because $k_{B(0,1)^{\boxplus t}}$ is unimodal with mode 0 for all $t>1$, we conclude that $B(0,1)^{\boxplus t}\in L(\uplus)$ for all $t>1$.

(4) Let $\mu(p,r)$ be the 2-*parameter Fuss–Catalan distribution*. Recall that $\mu(p,r)\in I(\boxplus)$ if and only if either $0<r\le\min\{p/2,p-1\}$ or $1\le p=r\le 2$ (see [MSU20, Theorem 4.1]). Then we can define $\widetilde{\mu}(p,r):=\Lambda_B^{-1}(\mu(p,r))$ for such pairs (p,r). Since $\mu(p,r)\in L(\boxplus)$ if and only if $1\le p=r\le 2$ (see [MSU20, Theorem 4.3]), we have $\widetilde{\mu}(p,p)\in L(\uplus)$ for $1\le p\le 2$ by (3.1). In particular, $\widetilde{\mu}(2,2)=B(2,1)$ is a Boolean Gaussian distribution. By Lemma 2.5(1) and [MSU20, Proposition 3.6], for $1\le p<2$ the Boolean Lévy triplet of $\widetilde{\mu}(p,p)$ is given by $(0,|x|^{-1}k_{\mu(p,p)}(x)\,\mathrm{d}x,p)$, where

$$k_{\mu(p,p)}(x)=-\frac{\sin(p\pi)}{\pi}\left(\frac{1+x}{-x}\right)^p\cdot\mathbf{1}_{(-1,0)}(x).$$

From computations similar to [MSU20, Proposition 3.6], we obtain

$$\eta_{\widetilde{\mu}(p,p)}(z)=(1+z)^p-1,\qquad z\in\mathbb{C}^-.$$

Then the Cauchy transform is as follows:

$$G_{\widetilde{\mu}(p,p)}(z) = \frac{1}{z[2-(1+z^{-1})p]}, \quad z \in \mathbb{C}^+.$$

By the Stieltjes inversion formula, the measure $\widetilde{\mu}(p,p)$ for $1 \le p < 2$ has the following explicit form:

$$\frac{1}{p(2-2^{1-1/p})}\delta_{a_p} + \frac{\sin(p\pi)\left|\frac{1+x}{x}\right|^p}{\pi x(4-4\cos(p\pi)\left|\frac{1+x}{x}\right|^p + \left|\frac{1+x}{x}\right|^{2p})} \cdot \mathbf{1}_{(-1,0)}(x)\,\mathrm{d}x,$$

where $a_p = (2^{1/p}-1)^{-1}$. According to this example, the discrete parts of Boolean selfdecomposable distributions come from not only the Boolean Gaussian component but also its Boolean Lévy measure.

4. Boolean selfdecomposability for normal distributions. In this section, we investigate Boolean selfdecomposability for normal distributions $N(m,v)$. The question is unimodality of $k_{N(m,v)}$ with mode 0. It turns out that $k_{N(m,v)}$ has an explicit formula. To compute it, we make use of the following differential equation (see [K98, (8.1.6)]), which can be derived by integration by parts.

LEMMA 4.1. *The following differential equation holds:*

$$G'_{N(0,1)}(z) = 1 - zG_{N(0,1)}(z), \quad z \in \mathbb{C}^+.$$

Using the above lemma, we can explicitly compute the functions $G_{N(0,1)}$ and $k_{N(0,1)}$.

PROPOSITION 4.2. *Let $h(z) = \int_0^z e^{t^2}\,\mathrm{d}t$, $z \in \mathbb{C}$. Then*

$$G_{N(0,1)}(z) = e^{-\frac{1}{2}z^2}\left[-i\sqrt{\frac{\pi}{2}} + \sqrt{2}\,h\left(\frac{z}{\sqrt{2}}\right)\right], \quad z \in \mathbb{C}^+, \tag{4.1}$$

and the Boolean Lévy triplet for $N(0,1)$ is $(0, k_{N(0,1)}(x)\,\mathrm{d}x, 0)$, where

$$k_{N(0,1)}(x) = \frac{1}{\pi}\sqrt{\frac{2}{\pi}} \cdot \frac{1}{|x|e^{-x^2/2}\left[1+\frac{4}{\pi}h\left(\frac{x}{\sqrt{2}}\right)^2\right]} \cdot \mathbf{1}_{\mathbb{R}\setminus\{0\}}(x). \tag{4.2}$$

Proof. The standard variation of constants method for the differential equation in Lemma 4.1 yields the solution

$$G_{N(0,1)}(z) = e^{-z^2/2}\Big[C + \int_0^z e^{t^2/2}\,\mathrm{d}t\Big], \quad z \in \mathbb{C}^+,$$

for some $C \in \mathbb{C}$. Because $N(0,1)$ is symmetric about 0, $\mathsf{Re}[G_{N(0,1)}(iy)] = 0$ for all $y > 0$ and so $\mathsf{Re}[C] = 0$. In order that the density of $N(0,1)$ is obtained as the limit $(-1/\pi)\lim_{\varepsilon\to 0^+} G_{N(0,1)}(x+i\varepsilon)$, we must have $\mathsf{Im}[C] = -\sqrt{\pi/2}$, proving the desired formula (4.1). Because $F_{N(0,1)}$ has a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$, the Stieltjes inversion (2.3) implies that the Boolean Gaussian

component is zero and the Boolean Lévy measure is Lebesgue absolutely continuous. By Proposition 3.4 we have (4.2). By (2.2) and $\mathsf{Re}[F_{N(0,1)}(i)] = 0$ we obtain $b = 0$. Finally, because $\tau(\mathrm{d}x) = \frac{|x|k_{N(0,1)}(x)}{1+x^2}\,\mathrm{d}x$ is symmetric about zero, (2.5) yields $\gamma = 0$. ∎

Analyzing the function $k_{N(0,1)}$ we are able to demonstrate the following.

THEOREM 4.3. *For all $v > 0$, we have $N(0, v) \in L(\uplus)$.*

Proof. Since $D_{\sqrt{v}}(N(0,1)) = N(0,v)$, it suffices to show $N(0,1) \in L(\uplus)$. To do so, we prove unimodality for $k_{N(0,1)}$ with mode 0 using Proposition 4.2. Because the function $k_{N(0,1)}$ is symmetric with respect to 0, it is enough to prove that $k_{N(0,1)}(x)$ is non-increasing on $(0,\infty)$. For this it suffices to show that $\ell_{N(0,1)}$ is non-increasing on $(0,\infty)$, i.e. the function

$$f(x) = e^{-x^2/2}\left[1 + \frac{4}{\pi}h\left(\frac{x}{\sqrt{2}}\right)^2\right] \tag{4.3}$$

is non-decreasing on $(0,\infty)$. For concise calculations, let us instead work with the rescaled function $g(x) = f(\sqrt{2}\,x)$.

To begin we compute

$$\begin{aligned} g'(x) &= -2xe^{-x^2}\left[1 + \frac{4}{\pi}h(x)^2\right] + \frac{8}{\pi}h(x) \\ &= -\frac{8xe^{-x^2}}{\pi}\left[h(x)^2 - \frac{e^{x^2}}{x}h(x) + \frac{\pi}{4}\right]. \end{aligned}$$

The desired inequality, $g' \geq 0$ on $(0,\infty)$, is therefore equivalent to

$$\frac{1}{2}\left[\frac{e^{x^2}}{x} - \sqrt{\left(\frac{e^{x^2}}{x}\right)^2 - \pi}\right] \leq h(x) \leq \frac{1}{2}\left[\frac{e^{x^2}}{x} + \sqrt{\left(\frac{e^{x^2}}{x}\right)^2 - \pi}\right]. \tag{4.4}$$

Note here that $(e^{x^2}/x)^2 \geq 2e > \pi$ for all $x > 0$.

For the upper bound of (4.4), we use the supplementary inequality

$$h(x) < \frac{e^{x^2} - 1}{x}, \qquad x > 0. \tag{4.5}$$

This can be easily verified with calculus: $H(x) := (e^{x^2} - 1)/x - h(x)$ satisfies $H' > 0$ on $(0,\infty)$ and $H(+0) = 0$. Thanks to (4.5), for the upper bound of (4.4) it suffices to show that

$$\frac{1}{2}\left[\frac{e^{x^2}}{x} + \sqrt{\left(\frac{e^{x^2}}{x}\right)^2 - \pi}\right] \geq \frac{e^{x^2} - 1}{x},$$

or equivalently

$$\sqrt{\left(\frac{e^{x^2}}{x}\right)^2 - \pi} \geq \frac{e^{x^2} - 2}{x}.$$

The latter is obvious if $e^{x^2}-2<0$. In the case $e^{x^2}-2\geq 0$ the desired inequality is equivalent to

$$\left(\frac{e^{x^2}}{x}\right)^2-\pi\geq\frac{e^{2x^2}-4e^{x^2}+4}{x^2},$$

or equivalently

$$4e^{x^2}-\pi x^2-4\geq 0.$$

By calculus, this is the case. Thus we are done.

The lower bound of (4.4) can also be proved by calculus. Let

$$J(x):=2h(x)-\left[\frac{e^{x^2}}{x}-\sqrt{\left(\frac{e^{x^2}}{x}\right)^2-\pi}\right].$$

It is elementary to see that $J(+0)=0$. It then suffices to show that $J'>0$. To begin, we compute

$$\begin{aligned}J'(x)&=2e^{x^2}-\left[-\frac{e^{x^2}}{x^2}+2e^{x^2}-\frac{-2e^{2x^2}/x^3+4e^{2x^2}/x}{2\sqrt{(e^{x^2}/x)^2-\pi}}\right]\\&=\frac{e^{2x^2}}{x^3}\left[xe^{-x^2}+\frac{-1+2x^2}{\sqrt{(e^{x^2}/x)^2-\pi}}\right].\end{aligned}$$

Hence J' is obviously positive on $[1/\sqrt{2},\infty)$. Suppose that $J'(x)=0$ held for some $0<x<1/\sqrt{2}$. This would imply

$$x^2e^{-2x^2}\left[\left(\frac{e^{x^2}}{x}\right)^2-\pi\right]=(2x^2-1)^2,$$

or equivalently

$$\pi e^{-2x^2}+4x^2-4=0.$$

However, by elementary calculus the left hand side of the last equation must be negative, a contradiction. ∎

Next, we show the failure of Boolean selfdecomposablity for normal distributions.

THEOREM 4.4. *There exists $M>0$ such that $|m/\sqrt{v}|>M$ implies that $N(m,v)\notin L(\uplus)$.*

Proof. It suffices to find $M>0$ such that $N(m,1)\notin L(\uplus)$ for $|m|>M$ thanks to the fact that dilation preserves $L(\uplus)$. Recall

$$\ell_{N(0,1)}(x)=|x|k_{N(0,1)}(x)=\frac{c}{f(x)},$$

where $c=\pi^{-1}\sqrt{2/\pi}$ and f is defined in (4.3). Because $N(0,1)$ has no Boolean Gaussian component by Proposition 4.2 and the function $\ell_{N(0,1)}$

is not constant on $(0,\infty)$ and is symmetric on $\mathbb{R}$, the desired conclusion follows by Theorem 3.15. ∎

REMARK 4.5 (Estimate of M). According to Theorem 3.15, $N(m,1)$ is not in $L(\uplus)$ if $|m| > M_0$, where

$$M_0 = \inf\left\{\frac{b\ell_{N(0,1)}(a) - a\ell_{N(0,1)}(b)}{\ell_{N(0,1)}(b) - \ell_{N(0,1)}(a)} : \begin{array}{l} -\infty < a < b < \infty, \\ \ell_{N(0,1)}(a) < \ell_{N(0,1)}(b) \end{array}\right\}. \tag{4.6}$$

With some inspection helped by the identity

$$\begin{aligned}\frac{b\ell_{N(0,1)}(a) - a\ell_{N(0,1)}(b)}{\ell_{N(0,1)}(b) - \ell_{N(0,1)}(a)} &= \frac{(b-a)\ell_{N(0,1)}(a)}{\ell_{N(0,1)}(b) - \ell_{N(0,1)}(a)} - a \\ &= \frac{(b-a)\ell_{N(0,1)}(b)}{\ell_{N(0,1)}(b) - \ell_{N(0,1)}(a)} - b,\end{aligned}$$

the infimum (4.6) turns out to be achieved in the limit $|b-a| \to 0$ and only when a, b are both negative, so that we obtain

$$M_0 = \inf\{p(a) : -\infty < a < 0\}, \quad \text{where} \quad p(a) := \frac{\ell_{N(0,1)}(a)}{\ell'_{N(0,1)}(a)} - a.$$

In order to find the infimum of the function p, observe that

$$p'(a) = \frac{-\ell_{N(0,1)}(a)\ell''_{N(0,1)}(a)}{\ell'_{N(0,1)}(a)^2}.$$

According to simulations, it is likely that $\ell''_{N(0,1)}$ in $(-\infty, 0)$ has a unique zero (denoted a_0 below) and hence p takes its minimum at a_0; see Figure 1.

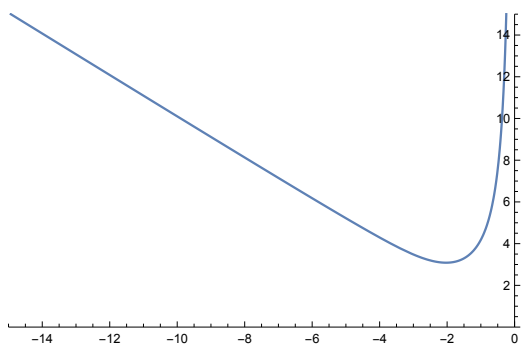

Fig. 1. $p(a)$

The values a_0 and $M_0 = p(a_0)$ are approximately -2.03 and 3.09, respectively. Moreover, simulations also suggest that M_0 is the precise threshold, i.e. $N(m,1)$ is Boolean selfdecomposable if and only if $|m| \le M_0$; see Figures 2, 3. Those simulations are performed on Mathematica Version 12.1.1, Wolfram Research, Inc., Champaign, IL.

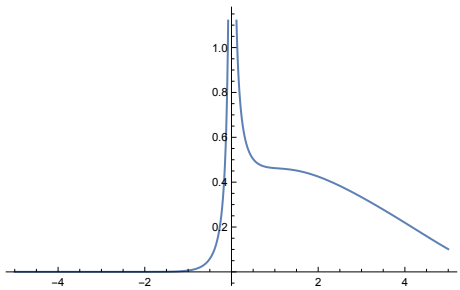

Fig. 2. $k_{N(3.05,1)}(a)$

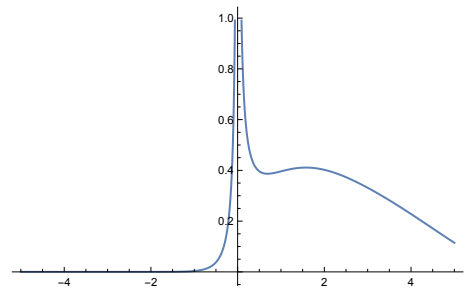

Fig. 3. $k_{N(3.2,1)}(a)$

Acknowledgements. The authors express sincere thanks to the referee, who gave useful comments to improve the manuscript.

This research is supported by JSPS Open Partnership Joint Research Projects Grants JPJSBP120209921 and JPJSBP120203202. T.H. is supported by JSPS Grant-in-Aid for Young Scientists 19K14546; K.N. is supported by JSPS Grant-in-Aid for Young Scientists 21K13807 and received research environment support from CIMAT as a JSPS Overseas Research Fellow; N.S. is supported by JSPS Grant-in-Aid for Scientific Research (C) JP19K03515 and JP23K03133; Y.U. is supported by JSPS Grant-in-Aid for Young Scientists 22K13925. Moreover, the last two authors are supported by JSPS Grant-in-Aid for Scientific Research (B) 19H01791.

References

[AH13] O. Arizmendi and T. Hasebe, *On a class of explicit Cauchy–Stieltjes transforms related to monotone stable and free Poisson laws*, Bernoulli 19 (2013), 2750–2767.

[BNT02] O. E. Barndorff-Nielsen and S. Thorbjørnsen, *Self-decomposability and Lévy processes in free probability*, Bernoulli 8 (2002), 323–366.

[BBLS11] S. T. Belinschi, M. Bożejko, F. Lehner and R. Speicher, *The normal distribution is $\boxplus$-infinitely divisible*, Adv. Math. 226 (2011), 3677–3698.

[BN08] S. T. Belinschi and A. Nica, *On a remarkable semigroup of homomorphisms with respect to free multiplicative convolution*, Indiana Univ. Math. J. 57 (2008), 1679–1713.

[BP99] H. Bercovici and V. Pata, *Stable laws and domains of attraction in free probability theory*, with an appendix by P. Biane, Ann. of Math. (2) 149 (1999), 1023–1060.

[BP00] H. Bercovici and V. Pata, *A free analogue of Hinčin's characterization of infinite divisibility*, Proc. Amer. Math. Soc. 128 (2000), 1011–1015.

[BV93] H. Bercovici and D. Voiculescu, *Free convolution of measures with unbounded support*, Indiana Univ. Math. J. 42 (1993), 733–773.

[BCDM20] F. Bracci, M. D. Contreras and S. Díaz-Madrigal, *Continuous Semigroups of Holomorphic Self-Maps of the Unit Disc*, Springer, Cham, 2020.

[CG08] G. P. Chistyakov and F. Götze, *Limit theorems in free probability theory. I*, Ann. Probab. 36 (2008), 54–90.

[FHS20] U. Franz, T. Hasebe and S. Schleißinger, *Monotone increment processes, classical Markov processes, and Loewner chains*, Dissertationes Math. 552 (2020), 119 pp.

[GK54] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Cambridge, MA, 1954.

[HS15] T. Hasebe and N. Sakuma, *Unimodality of Boolean and monotone stable distributions*, Demonstr. Math. 48 (2015), 424–439.

[HST19] T. Hasebe, N. Sakuma and S. Thorbjørnsen, *The normal distribution is freely self-decomposable*, Int. Math. Res. Notices 2019, 1758–1787.

[HT16] T. Hasebe and S. Thorbjørnsen, *Unimodality of the freely selfdecomposable probability laws*, J. Theoret. Probab. 29 (2016), 922–940.

[HO07] A. Hora and N. Obata, *Quantum Probability and Spectral Analysis of Graphs*, Springer, Berlin, 2007.

[K98] S. Kerov, *Interlacing measures*, in: Kirillov's Seminar on Representation Theory, Amer. Math. Soc. Transl. (2) 181, Adv. Math. Sci. 35, Amer. Math. Soc., Providence, RI, 1998, 35–83.

[MSU20] W. Młotkowski, N. Sakuma and Y. Ueda, *Free self-decomposability and unimodality for the Fuss–Catalan distributions*, J. Statist. Phys. 178 (2020), 1055–1075.

[M03] N. Muraki, *The five independences as natural products*, Infin. Dimens. Anal. Quantum Probab. Related Topics 6 (2003), 337–371.

[Sat13] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, Cambridge, 2013.

[Sch12] K. Schmüdgen, *Unbounded Self-Adjoint Operators on Hilbert Space*, Grad. Texts in Math. 265, Springer, Dordrecht, 2012.

[SW97] R. Speicher and R. Woroudi, *Boolean convolution*, in: Free Probability Theory, Fields Inst. Commun. 12, Amer. Math. Soc., Providence, RI, 1997, 267–279.

[W08] J.-C. Wang, *Limit laws for Boolean convolutions*, Pacific J. Math. 237 (2008), 349–371.

Takahiro Hasebe
Department of Mathematics
Hokkaido University
Sapporo, 060-0810 Hokkaido, Japan
E-mail: thasebe@math.sci.hokudai.ac.jp

Kei Noba
Department of Statistical Inference and Mathematics
The Institute of Statistical Mathematics
Tachikawa, 190-8562 Tokyo, Japan
E-mail: knoba@ism.ac.jp

Noriyoshi Sakuma
Graduate School of Natural Sciences
Nagoya City University
Nagoya, 467-8501 Aichi, Japan
E-mail: sakuma@nsc.nagoya.ac.jp

Yuki Ueda
Department of Mathematics
Hokkaido University of Education
Asahikawa, 070-8621 Hokkaido, Japan
E-mail: ueda.yuki@a.hokkyodai.ac.jp