Monotone convolution semigroups

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Abstract

We study how a property of a monotone convolution semigroup changes with respect to time parameter. Especially we focus on "time-independent properties": in the classical case, there are many properties of convolution semigroups (or Lévy processes) which are determined at an instant, and moreover, such properties are often characterized by the drift term and Lévy measure. In this paper we show such properties of monotone convolution semigroups; an example is the concentration of the support of a probability measure on the positive real line. Most of them are characterized by the same conditions on drift terms and Lévy measures as known in probability theory. These kinds of properties are mapped bijectively by a monotone analogue of Bercovici-Pata bijection. Finally we compare such properties with classical, free, Boolean cases, which will be important in an approach to unify these notions of independence.

1 Introduction

Muraki defined a monotone convolution as the probability distribution of the sum of two monotone independent random variables [15, 16]. Let $G_\mu(z)$ ($z \in \mathbb{C}\setminus\mathbb{R}$) be the Cauchy transform of a probability measure μ and $H_\mu(z)$ be the reciprocal of $G_\mu(z)$. H_μ is analytic and maps the upper half plane into itself. Moreover, $\inf_{\operatorname{Im} z>0} \frac{\operatorname{Im} H_\mu(z)}{\operatorname{Im} z} = 1$. Consequently, $H_\mu(z)$ can be expressed uniquely in the form

$$H_\mu(z) = z + b + \int_{\mathbb{R}} \frac{1+xz}{x-z}\eta(dx), \tag{1.1}$$

where $b \in \mathbb{R}$ and η is a positive finite measure. The reader is referred to [1]. The monotone convolution $\mu \rhd \nu$ of probability measures μ and ν is characterized by

$$H_{\mu\rhd\nu}(z) = H_\mu(H_\nu(z)). \tag{1.2}$$

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Using this characterization, Muraki classified monotone (or $\rhd$- for short) infinitely divisible distributions with compact supports. The complete classification including probability measures with unbounded supports was proved by Belinschi [4], as follows.

Theorem 1.1. *There is a one-to-one correspondence among the following four objects:*

(1) *a $\rhd$-infinitely divisible distribution μ;*

(2) *a weakly continuous monotone convolution semigroup $\{\mu_t\}$ with $\mu_0 = \delta_0, \mu_1 = \mu$;*

(3) *a composition semigroup of reciprocal Cauchy transforms $\{H_t\}_{t\geq 0}$ ($H_t \circ H_s = H_{t+s}$) with H_0 =id, $H_1 = H_\mu$, where $H_t(z)$ is a continuous function of $t \geq 0$ for any $z \in \mathbb{C} \setminus \mathbb{R}$;*

(4) *a vector field on the upper half plane $A(z) = \lim_{t\searrow 0} \frac{H_t(z)-z}{t}$ which has the form $A(z) = -\gamma + \int_{\mathbb{R}} \frac{1+xz}{x-z} d\tau(x)$, where $\gamma \in \mathbb{R}$ and τ is a positive finite measure.*

The integral representation in (4) is the Lévy-Khintchine formula in monotone probability. The correspondence of (3) and (4) is obtained through the following ordinary differential equation (ODE):

$$\frac{d}{dt} H_t(z) = A(H_t(z)), \quad H_0(z) = z, \tag{1.3}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$. The fact that the solution does not explode in finite time is a consequence of [9]. We call $A(z)$ the associated vector field. When τ has all moments, then the coefficients of the Laurent expansion of A coincide with cumulants [13].

In this paper, we analyze monotone convolution semigroups, especially supports and moments, comparing with results of classical, free and Boolean cases. Results in this paper will be important to clarify similarity and dissimilarity between monotone independence and other kinds of independence. This work is also expected to have connections with an operator-theoretic approach [12] and a categorical approach [11].

The contents of each section are as follows. In Section 2, we prove a condition for a probability measure to be supported on the positive real line, and show how moments change under the monotone convolution. In Section 3, we derive a differential equation about the minimum of support of a monotone convolution semigroup. In Section 4, we study how a property of a monotone convolution semigroup changes with respect to time parameter. Time-independent property is a property of a convolution semigroup which is determined at an instant. We show that the following properties are time-independent: the symmetry around 0; the concentration of a support on the positive real line; the lower boundedness of a support; the finiteness of a moment of even order. All these properties are also time-independent in classical convolution semigroups. In Section 5, a monotone analogue of the Bercovici-Pata bijection is defined. Many time-independent properties in the previous section can be formulated in terms of the Bercovici-Pata bijection. In Section 6, we study convolution semigroups in free probability and Boolean probability. A remarkable point is that the concentration of the support on the positive real line is a time-independent property in the monotone, Boolean and classical cases, but this is not true in free probability.

2 Behavior of supports and moments under monotone convolution

We consider properties of probability measures which are conserved under the monotone convolution. Let μ be a probability measure. Define the minimum and the maximum of the support: $a(\mu) := \inf\{x \in \text{supp}\mu\}$, $b(\mu) := \sup\{x \in \text{supp}\mu\}$. Here $-\infty \leq a(\mu) < \infty$ and $-\infty < b(\mu) \leq \infty$ hold. We say that μ contains an isolated atom at $c \in \mathbb{R}$ if $\mu(\{c\}) > 0$ and $c \notin \overline{(\text{supp}\,\mu)\backslash\{c\}}$. In this paper we occasionally consider analytic continuations of functions such as G_μ or H_μ from $\mathbb{C}\setminus\mathbb{R}$ to an open subset U of $\mathbb{C}$ which intersects $\mathbb{R}$. If there are no confusions, for simplicity, we only say that a function is analytic in U, instead of saying that a function has an analytic continuation.

Lemma 2.1. *Let μ be a probability measure. We use the notation (1.1).*
(1) $(\text{supp}\,\mu)^c \cup (\mathbb{C}\setminus\mathbb{R})$ is the maximal domain in which $G_\mu(z)$ is analytic. Similarly, $(\text{supp}\,\eta)^c \cup (\mathbb{C}\setminus\mathbb{R})$ is the maximal domain in which $H_\mu(z)$ is analytic.
(2) $\{x \in (\text{supp}\,\mu)^c; G_\mu(x) \neq 0\} \subset (\text{supp}\,\eta)^c$. Similarly, $\{x \in (\text{supp}\,\eta)^c; H_\mu(x) \neq 0\} \subset (\text{supp}\,\mu)^c$. In particular, $a(\eta) \geq a(\mu)$ since $G_\mu(x) \neq 0$ for $x \in (-\infty, a(\mu))$.

Proof. These statements easily follow from the Perron-Stieltjes inversion formula. □

A classical infinitely divisible distribution necessarily has a noncompact support, except for a delta measure. This situation is different from monotone, free and Boolean cases. For instance, a centered arcsine law is $\rhd$-infinitely divisible. The study of the maximum or minimum of a support becomes more important for this reason. It is known that if $\lambda = \nu \rhd \mu$ and λ has a compact support, then the support of μ is also compact [16]. We generalize this and prove a basic estimate of supports.

Proposition 2.2. *The following inequalities hold for probability measures ν and μ.*
(1) If $\text{supp}\,\nu \cap (-\infty, 0] \neq \emptyset$ and $\text{supp}\,\nu \cap [0,\infty) \neq \emptyset$, then $a(\mu) \geq a(\nu \rhd \mu)$, $b(\mu) \leq b(\nu \rhd \mu)$.
(2) If $\text{supp}\,\nu \subset (-\infty, 0]$, then $a(\mu) \geq a(\nu \rhd \mu)$, $b(\nu) + b(\mu) \leq b(\nu \rhd \mu)$.
(3) If $\text{supp}\,\nu \subset [0,\infty)$, then $a(\nu) + a(\mu) \geq a(\nu \rhd \mu)$, $b(\mu) \leq b(\nu \rhd \mu)$.

Proof. For a probability measure ρ, we denote by ρ^x the probability measure $\delta_x \rhd \rho$. This is useful since $\nu \rhd \mu$ can be expressed as

$$\nu \rhd \mu(B) = \int_{\mathbb{R}} \mu^x(B)\nu(dx) \tag{2.1}$$

for Borel sets B [16].

Let $\lambda := \nu \rhd \mu$. We prove first the following inequalities for an arbitrary probability measure ρ:

$$\begin{cases} a(\rho^x) \geq a(\rho), \quad b(\rho^x) \leq b(\rho) + x & \text{for all } x > 0, \\ a(\rho^x) \geq a(\rho) - |x|, \quad b(\rho^x) \leq b(\rho) & \text{for all } x < 0. \end{cases}$$

It easy to prove that ρ^x can be characterized by $G_{\rho^x} = \frac{G_\rho}{1 - xG_\rho}$. If $x > 0$, then $1 - xG_\rho(z) \neq 0$ for $z \in \mathbb{C}\setminus[a(\rho), b(\rho) + x]$ and G_ρ is analytic in this domain. Therefore, the first inequality holds. The second is proved similarly.

Let $J := \operatorname{supp}\lambda$. In view of the relation $\lambda(A) = \int_{\mathbb{R}} \mu^x(A)d\nu(x)$, we have $\lambda(J^c) = \int_{\mathbb{R}} \mu^x(J^c)d\nu(x) = 0$. Hence we obtain $\mu^x(J^c) = 0$, ν-a.e. $x \in \mathbb{R}$. Take any x_0 such that $\mu^{x_0}(J^c) = 0$. Then we have $a(\mu^{x_0}) \geq a(\lambda)$ and $b(\mu^{x_0}) \leq b(\lambda)$. If $x_0 > 0$, combining the inequalities $a(\rho^x) \geq a(\rho) - |x|$ and $b(\rho^x) \leq b(\rho)$ for $\rho = \mu^{x_0}$ and $x = -x_0 < 0$, we have

$$\begin{aligned} a(\mu) &= a(\mu^{x_0 - x_0}) \geq a(\lambda) - |x_0|, \\ b(\mu) &= b(\mu^{x_0 - x_0}) \leq b(\lambda). \end{aligned}$$

Similarly if $x_0 < 0$,

$$\begin{aligned} a(\mu) &\geq a(\lambda), \\ b(\mu) &\leq b(\lambda) + |x_0|. \end{aligned}$$

Assume that $\operatorname{supp}\nu \subset (-\infty, 0]$. Then we obtain $a(\mu) \geq a(\lambda)$ and $b(\mu) \leq b(\lambda) + |b(\nu)|$ since there is a sequence of such x_0's converging to the point $b(\nu)$. Hence we have proved (2). The statements (1) and (3) are proved in a similar way to (2). □

Corollary 2.3. *Let ν be a probability measure and let $n \geq 1$ be a natural number.*
(1) If $\operatorname{supp}(\nu^{\rhd n}) \subset (-\infty, 0]$, then $\operatorname{supp}\nu \subset (-\infty, 0]$ and $|b(\nu)| \geq \frac{1}{n}|b(\nu^{\rhd n})|$.
(2) If $\operatorname{supp}(\nu^{\rhd n}) \subset [0, \infty)$, then $\operatorname{supp}\nu \subset [0, \infty)$ and $a(\nu) \geq \frac{1}{n}a(\nu^{\rhd n})$.

This corollary puts a restriction on the support of a $\rhd$-infinitely divisible distribution. The continuous time version of (2) will be proved in Section 4.

Proof. Let $\lambda := \nu^{\rhd n}$.
(1) Assume that both $b(\nu) > 0$ and $b(\lambda) = b(\nu^{\rhd n}) \leq 0$ hold, then there are two possible cases: (a) $\operatorname{supp}\nu \cap [0, \infty) \neq \emptyset$ and $\operatorname{supp}\nu \cap (-\infty, 0] \neq \emptyset$; (b) $\operatorname{supp}\nu \subset [0, \infty)$ in Proposition 2.2. We apply Proposition 2.2 replacing λ and μ with $\nu^{\rhd n}$ and $\nu^{\rhd n-1}$, respectively. In both cases (a) and (b), it holds that $b(\nu^{\rhd n-1}) \leq b(\lambda) \leq 0$. Thus we obtain $b(\nu^{\rhd n-1}) \leq 0$. This argument can be repeated and finally we have $b(\nu) \leq 0$, a contradiction. Thereofre, $b(\nu) \leq 0$. By the iterative use of Proposition 2.2 (2) we obtain $b(\nu^{\rhd n}) \geq nb(\nu)$, from which the conclusion follows. A similar argument applies to (2). □

The following theorem is well known. We will need almost the same argument in Proposition 2.5.

Lemma 2.4. *For a finite measure μ, $\lim_{y \searrow 0} iyG_\mu(a + iy) = \mu(\{a\})$ for all $a \in \mathbb{R}$.*

Proof. This claim follows from the dominated convergence theorem. □

Now we prove a condition for a support to be included in the positive real line. A similar result was obtained in [6].

Proposition 2.5. *We use the notation (1.1). Then $\operatorname{supp}\mu \subset [0, \infty)$ if and only if $\operatorname{supp}\eta \subset [0, \infty)$ and $H_\mu(-0) \leq 0$ hold. Moreover, under the condition $\operatorname{supp}\eta \subset [0, \infty)$, the condition $H_\mu(-0) \leq 0$ is equivalent to the following conditions: $(*)$ $\eta(\{0\}) = 0$; $\int_0^\infty \frac{1}{x}d\eta(x) < \infty$; $b + \int_0^\infty \frac{1}{x}d\eta(x) \leq 0$.*

Proof. If $\operatorname{supp}\eta \subset [0,\infty)$ and $H_\mu(-0) \leq 0$, we have $H_\mu(u) < 0$ for all $u < 0$ since H_μ is strictly increasing. Then $G_\mu = \frac{1}{H_\mu}$ is analytic in $\mathbb{C}\setminus[0,\infty)$, which implies $\operatorname{supp}\mu \subset [0,\infty)$. Conversely, we assume $\operatorname{supp}\mu \subset [0,\infty)$. By Lemma 2.1, we have $\operatorname{supp}\eta \subset [0,\infty)$. If $H_\mu(-0)$ were greater than 0, there would exist $u_0 < 0$ such that $H_\mu(u_0) = 0$. Then μ has an atom at $u_0 < 0$, which contradicts the assumption. Therefore, $H_\mu(-0) \leq 0$.

We show the equivalence in the last claim. It is not difficult to prove that $(*)$ implies $H_\mu(-0) \leq 0$. Now we shall prove the converse statement. Assume that $\lambda := \eta(\{0\}) > 0$. By a similar argument to Lemma 2.4, we can prove that $\lim_{u\nearrow 0} uH_\mu(u) = -\lambda$. Therefore, for $u < 0$ sufficiently close to 0, we have $H_\mu(u) > -\frac{\lambda}{2u} > 0$, which contradicts the condition $H_\mu(-0) \leq 0$. Then we have $\eta(\{0\}) = 0$. Since $f_u(x) := \frac{1+xu}{x-u}$ is increasing with respect to u, we can apply the monotone convergence theorem and obtain the two inequalities $\int_0^\infty \frac{1}{x} d\eta(x) < \infty$ and $b + \int_0^\infty \frac{1}{x} d\eta(x) \leq 0$. □

Corollary 2.6. *The monotone convolution preserves the set $\{\mu; \operatorname{supp}\mu \subset [0,\infty)\}$ of probability measures.*

Proof. If $\operatorname{supp}\mu \subset [0,\infty)$ and $\operatorname{supp}\nu \subset [0,\infty)$, $H_{\mu\rhd\nu} = H_\mu \circ H_\nu$ is analytic in $\mathbb{C}\setminus[0,\infty)$. Since $H_{\mu\rhd\nu}$ is increasing in $(-\infty, 0)$, we have $H_{\mu\rhd\nu}(-0) = H_\mu \circ H_\nu(-0) \leq H_\mu(-0) \leq 0$. By Proposition 2.5, we obtain $\operatorname{supp}(\mu \rhd \nu) \subset [0,\infty)$. □

Remark 2.7. The above property is also true for Boolean convolution. The proof goes similarly. We note that the corollary follows immediately if we use the operator-theoretic realization of monotone independent random variables in [12].

Next we consider moments. Let $m_n(\mu) := \int_{\mathbb{R}} x^n \mu(dx)$ be the n-th moment of a proba-blility measure μ.

Proposition 2.8. *Let μ be a probability measure and let $n \geq 1$ be a natural number. Then the following conditions are equivalent.*

(1) $m_{2n}(\mu) < \infty$,

(2) H_μ *has the expression* $H_\mu(z) = z + a + \int_{\mathbb{R}} \frac{\rho(dx)}{x-z}$, *where $a \in \mathbb{R}$ and ρ is a positive finite measure satisfying $m_{2n-2}(\rho) < \infty$,*

(3) there exist $a_1, \cdots, a_{2n} \in \mathbb{R}$ such that

$$H_\mu(z) = z + a_1 + \frac{a_2}{z} + \cdots + \frac{a_{2n}}{z^{2n-1}} + o(|z|^{-(2n-1)}) \tag{2.2}$$

for $z = iy$ $(y \to \infty)$.

If (3) holds, for any $\delta > 0$ the expansion (2.2) holds for $z \to \infty$ satisfying $\operatorname{Im} z > \delta|\operatorname{Re} z|$. Moreover, we have $a_{k+2} = -m_k(\rho)$ $(0 \leq k \leq 2n-2)$.

Proof. The equivalence (1) ⇔ (3) follows from Theorem 3.2.1 in [1] by calculating the reciprocals. The implication (2) ⇒ (3) is not difficult. The proof of (3) ⇒ (2) runs by the same technique as in Theorem 3.2.1 in the book [1]. □

Proposition 2.9. *Let μ and ν be probability measures and let $n \geq 1$ be a natural number. If $m_{2n}(\mu) < \infty$ and $m_{2n}(\nu) < \infty$, then $m_{2n}(\mu \rhd \nu) < \infty$. Moreover, we have*

$$m_l(\mu \rhd \nu) = m_l(\mu) + m_l(\nu) + \sum_{k=1}^{l-1} \sum_{\substack{j_0+j_1+\cdots+j_k=l-k, \\ 0\leq j_p,\ 0\leq p\leq k}} m_k(\mu)m_{j_0}(\nu)\cdots m_{j_k}(\nu) \tag{2.3}$$

for $1 \leq l \leq 2n$.

Proof. We note that $\operatorname{Im} H_\nu(z) \geq \operatorname{Im} z$. For any $\delta > 0$, there exists $M = M(\delta) > 0$ such that

$$\operatorname{Im} H_\nu(iy) \geq y > \delta|\operatorname{Re} H_\nu(iy)| \text{ for } y > M. \tag{2.4}$$

By (2.2), we obtain

$$H_\mu(H_\nu(iy)) = H_\nu(iy) + a_1 + a_2G_\nu(iy) + \cdots + a_{2n}G_\nu(iy)^{2n-1} + R(H_\nu(iy)), \tag{2.5}$$

where $z^{2n-1}R(z) = \int_\mathbb{R} \frac{x^{2n-1}}{x-z}\rho(dx) \to 0$ as $z \to \infty$ satisfying $\operatorname{Im} z > \delta|\operatorname{Re} z|$ for a fixed $\delta > 0$. We have

$$y^{2n-1}|R(H_\nu(iy))| \leq |H_\nu(iy)|^{2n-1}|R(H_\nu(iy))| \to 0$$

as $y \to \infty$ by the condition (2.4). Thus $R(H_\nu(iy)) = o(y^{-(2n-1)})$. Expanding $H_\nu(z)$ in the form (2.2), we can see that there exist $c_1, \cdots, c_{2n} \in \mathbb{R}$ such that $H_\mu(H_\nu(z)) = z + c_1 + \frac{c_2}{z} + \cdots + \frac{c_{2n}}{z^{2n-1}} + o(|z|^{-(2n-1)})$ for $z = iy$ $(y \to \infty)$. Then the $2n$-th moment of $\mu \rhd \nu$ is finite by Proposition 2.8. The equality (2.3) is obtained by the expansion of $G_{\mu\rhd\nu}(z) = G_\mu(\frac{1}{G_\nu(z)})$. □

3 Differential equations arising from monotone convolution semigroups

Let $\{\mu_t\}_{t\geq 0}$ be a weakly continuous $\rhd$-convolution semigroup with $\mu_0 = \delta_0$. We denote H_{μ_t} by H_t for simplicity. We sometimes write $H(t, z)$ to express explicitly that $H_t(z)$ is a function of two variables. By (1.1), H_t can be expressed as

$$H_t(z) = b_t + z + \int_\mathbb{R} \frac{1 + xz}{x - z} d\eta_t(x), \tag{3.1}$$

where, for each $t > 0$, a_t is a real number and η_t is a finite positive measure. We denote by $A(z)$ the associated vector field throughout this paper.

Throughout this section, we will prove the following properties of the minimum of the support of a convolution semigroup.

Theorem 3.1. *Let $\{\mu_t\}_{t\geq 0}$ be a weakly continuous $\rhd$-convolution semigroup with $\mu_0 = \delta_0$. We assume that for every $t > 0$ μ_t is not a delta measure. We have such a form $\mu_t = \lambda(t)\delta_{\theta(t)} + \nu_t$ with $\theta(t) \notin \operatorname{supp} \nu_t$, $\theta(t) = a(\mu_t)$ and $\lambda(t) \geq 0$.*
(1) *Assume $a(\tau) > 0$. Then there are four cases:*

(A) *If* $A(u_0) = 0$ *for some* $u_0 \in [-\infty, 0)$ *and* $A(u) < 0$ *on* $(-\infty, u_0)$ *and* $A(u) > 0$ *on* $(u_0, 0)$ *(when* $u_0 = -\infty$*, we understand the condition as* $A > 0$*), then* $\lambda(t) > 0$*. Moreover, the inequality* $u_0 < \theta(t) < 0$ *holds for all* $t > 0$*.*

(B) *If* $A(u) < 0$ *on* $(-\infty, 0)$ *and* $A(0) = 0$*, then* $\theta(t) = 0$ *and* $\lambda(t) > 0$ *for all* $t > 0$*.*

(C) *If there exists* $u_0 \in (0, a(\tau))$ *such that* $A(u) < 0$ *on* $(-\infty, u_0)$ *and* $A(u) > 0$ *on* $(u_0, a(\tau))$*, then it follows that* $\theta(t) \in (0, u_0)$ *and* $\lambda(t) > 0$ *for* $0 < t < \infty$ *and* $\lambda(t) > 0$ *for* $t > 0$*.*

(D) *If* $A(u) < 0$ *on* $(-\infty, a(\tau))$*, then there exists* $t_0 \in (0, \infty]$ *such that* $\lambda(t) > 0$ *for all* $0 < t < t_0$ *and* $\lambda(t) = 0$ *for* $t_0 \le t < \infty$*.*

If $A(0) \neq 0$ *and* $\lambda(t) > 0$*, the weight of the delta measure is written as* $\lambda(t) = \frac{A(\theta(t))}{A(0)}$*. If* $A(0) = 0$ *(case(B)), then we have* $\lambda(t) = e^{-A'(0)t}$*. Concerning the position of the delta measure, the following ODE holds:*

$$\begin{cases} \frac{d}{dt}\theta(t) = -A(\theta(t)), \\ \theta(0) = 0. \end{cases} \tag{3.2}$$

(2) *We assume* $a(\tau) > -\infty$*. There are three cases in terms of the signs of the associated vector field:*

(a) $A(u) > 0$ *on* $(-\infty, a(\tau))$*;*

(b) $A(u_0) = 0$ *for some* $u_0 \in (-\infty, a(\tau))$ *and* $A(u) < 0$ *on* $(-\infty, u_0)$ *and* $A(u) > 0$ *on* $(u_0, a(\tau))$*;*

(c) $A(u) < 0$ *on* $(-\infty, a(\tau))$*.*

In case (a) and case (b), we have the following ODE for $a(\nu_t)$*:*

$$\begin{cases} \frac{d}{dt}a(\nu_t) = -A(a(\nu_t)), \\ a(\nu_0) = a(\tau). \end{cases} \tag{3.3}$$

In case (c), the equality $a(\nu_t) = a(\tau)$ *holds for a.e.* t *and* $a(\nu_t) \ge a(\tau)$ *for all* $t \in [0, \infty)$*. Moreover, if* $\lim_{u \nearrow a(\tau)} A(u) < 0$*, we have* $a(\nu_t) = a(\tau)$ *for all* t*.*

Example 3.2. We can confirm the validity of the ODEs of $\theta(t)$ and $a(\nu_t)$, and the validity of the formula of the weight of a delta measure in each example.

- Arcsine law: $\mu_t = \frac{1}{\pi\sqrt{2t - x^2}} 1_{(-\sqrt{2t}, \sqrt{2t})}(x)dx$, $A(z) = -\frac{1}{z}$, $a(\tau) = 0$, $a(\mu_t) = -\sqrt{2t}$.

- A deformation of α-strictly stable distributions ($0 < \alpha < 2$) with parameter $c \in \mathbb{C}$, $\operatorname{Im} c = 0, \operatorname{Re} c \ge 0$ (see [14]): $\mu_t = \mu_{t,ac}$, $\operatorname{supp} \mu_{t,ac} = (-\infty, c + t^{\frac{1}{\alpha}}]$, $A(z) = -\frac{1}{\alpha}(z - c)^{1-\alpha}$. We can check that the solution of the ODE (3.3) is $c + t^{\frac{1}{\alpha}}$ (the same ODE (3.3) holds for $b(\mu_t)$).

- The monotone Poisson distribution with parameter $\lambda > 0$: $\mu_t(dx) = \mu_{t,ac} + \mu_{t,sing}$, $A(z) = \frac{\lambda z}{1-z}$, where $\mu_{t,sing}$ is a delta measure at 0. , and hence, it holds that $A(0) = 0$ and $A'(0) = \lambda$. This is the case (B). Therefore, we have $\mu_{t,sing} = e^{-\lambda t}\delta_0$.

3.1 Differential equation of delta measure

We summarize three equalities, some of which were used by Muraki in [16].

Lemma 3.3. *Let $\{\mu_t\}_{t\geq 0}$ be a weakly continuous $\triangleright$-convolution semigroup with $\mu_0 = \delta_0$. Then we have three equalities on $\mathbb{C}\setminus\mathbb{R}$:*
(1) $A(H_t(z)) = A(z)\frac{\partial H_t}{\partial z}(z)$;
(2) $\frac{\partial}{\partial t}G_t(z) = A(z)\frac{\partial}{\partial z}G_t(z)$;
(3) $\frac{\partial}{\partial t}H_t(z) = A(z)\frac{\partial}{\partial z}H_t(z)$.

Proof. Since $H_t(z)$ is a flow in $\mathbb{C}\setminus\mathbb{R}$, $H_t \circ H_s = H_{t+s}$ for $t, s \geq 0$. (1) follows from the derivative $\frac{\partial}{\partial s}|_{s=0}$. (3) follows from (1) and (1.3). (2) follows from (3) immediately. □

First we treat a distribution which contains a delta measure at the minimum of the support. Suppose that $\{\mu_t\}_{t\geq 0}$ is a weakly continuous $\triangleright$-convolution semigroup with $\mu_0 = \delta_0$. Then μ can be written as $\mu = \lambda\delta_\theta + \nu$ with $\theta \in (\operatorname{supp}\nu)^c$ and $0 < \lambda < 1$. We use the integral representation in Theorem 1.1 (4) for the associated vector field $A(z)$. Throughout this subsection, we assume that A is not a real constant which means that μ_t is not a delta measure for any $t > 0$ and that $a(\tau) > 0$. We shall show that there exists a delta measure at the minimum point of the support for some (finite or infinite) time interval. Moreover, the weight of a delta measure is calculated.

The derivative of A satisfies $A'(u) > 0$ for all $u \in (-\infty, 0)$. This implies that there are five possible cases:

(A) $A(u) > 0$ on $(-\infty, 0)$;

(A') $A(u_0) = 0$ for some $u_0 \in (-\infty, 0)$ and $A(u) < 0$ on $(-\infty, u_0)$ and $A(u) > 0$ on $(u_0, 0)$;

(B) $A(u) < 0$ on $(-\infty, 0)$ and $A(0) = 0$;

(C) there exists $u_0 \in (0, a(\tau))$ such that $A(u) < 0$ on $(-\infty, u_0)$ and $A(u) > 0$ on $(u_0, a(\tau))$;

(D) $A(u) < 0$ on $(-\infty, a(\tau))$;

We consider the solution of the ODE (1.3) also on the real line as well as on $\mathbb{C}\setminus\mathbb{R}$.

Case (A) and case (A')

Case (A) is reduced to case (A') if we define $u_0 := -\infty$. Since $H(t, u)$ is an increasing function of $u \in (\operatorname{supp}\eta_t)^c$, there is a unique point $\theta(t)$ satisfying $u_0 < \theta(t) < 0$ and

$$H(t, \theta(t)) = 0. \tag{3.4}$$

$\theta(t)$ is a zero point of H_t of degree 1 since $\partial_u H(t, u) \geqq 1$. Therefore, by lemma 2.4, there is a delta measure $\lambda(t)\delta_{\theta(t)}$ in μ_t with $u_0 < \theta(t) < 0$. By the implicit function theorem, $\theta(t)$ is in C^ω class. Differentiating the equation $H(t, \theta(t)) = 0$ and using Lemma 3.3, we obtain

$$\theta'(t) = -\frac{\frac{\partial H}{\partial t}(t, \theta(t))}{\frac{\partial H}{\partial z}(t, \theta(t))} = -A(\theta(t)). \tag{3.5}$$

The initial condition is $\theta(0) = 0$.

Case (B)

In case (B), the same differential equation (3.5) holds. Since $A(0) = 0$, we have $\theta(t) = 0$ for all t. This is true for a monotone Poisson distribution.

Case (C) and case (D)

Case (C) and case (D) can be treated at the same time. We define

$$u_1 := \begin{cases} u_0, & \text{in case (C)}, \\ a(\tau), & \text{in case (D)}, \end{cases}$$

to treat the two cases at the same time. In the cases (C) and (D), H_t is analytic in $\mathbb{C} \setminus [u_1, \infty)$ (see Subsection 3.2 for details). Then there exists $t_0 \in (0, \infty]$ such that μ_t includes a delta measure in $(0, u_1)$ for $0 < t < t_0$. We can prove that $t_0 = \infty$ in case (C). In case (D), we have an example, where $t_0 < \infty$ holds (see the section of Example in [14]). $t_0 = \infty$ may occur if $\lim_{u \nearrow a(\tau)} A(u) = 0$. μ_t has the form

$$\mu_t = \begin{cases} \lambda(t)\delta_{\theta(t)} + \nu_t, & 0 \leq t < t_0, \\ \nu_t, & t_0 \leq t < \infty, \end{cases}$$

where it holds that $0 < \lambda(t) \leq 1$ and $0 \leq \theta(t) < a(\tau)$ for $0 \leq t < t_0$, and $a(\nu_t) \geq a(\tau)$ for all $0 < t < \infty$. The differential equation (3.5) holds also in this case.

Weight $\lambda(t)$ in the cases (A), (A'), (C) and (D)

It is possible to calculate the weight $\lambda(t)$. First we exclude case (B). Then we have $A(0) \neq 0$. We expand $H_t(z)$ in a Taylor series around $\theta(t)$ as $H_t(z) = \sum_{n=1}^{\infty} a_n(t)(z - \theta(t))^n$ with $a_1(t) = \frac{1}{\lambda(t)}$. Also we expand $A(z)$ as $\sum_{n=0}^{\infty} b_n z^n$ with $b_n \in \mathbb{R}$. If we compare the coefficients of the constant term in the ODE (1.3), we obtain $-\theta'(t)a_1(t) = b_0 = A(0)$. Hence it holds that

$$\lambda(t) = \frac{A(\theta(t))}{A(0)}.$$

Weight $\lambda(t)$ in the case (B)

In case (B), we express the Taylor expansions of H_t and $A(z)$ at 0 respectively by $H_t(z) = \sum_{n=1}^{\infty} a_n(t) z^n$ and $A(z) = \sum_{n=1}^{\infty} b_n z^n$ with $a_1(t) = \frac{1}{\lambda(t)}$ and $b_1 = A'(0) > 0$. Comparing the coefficients of z^n in the ODE (1.3), we obtain the equation $a_1'(t) = A'(0)a_1(t)$. Therefore, we get $a_1(t) = e^{A'(0)t}$ because of the initial condition $a_1(0) = 1$. Thus we obtain

$$\lambda(t) = e^{-A'(0)t}.$$

3.2 Differential equation of non-atomic part

In the previous subsection we considered the case $a(\tau) > 0$. Now we consider a more general case. We investigate $a(\mu_t)$ including the case where there is no isolated delta measure at $a(\mu_t)$. Assume that the lower bound $a(\tau)$ of the Lévy measure τ is finite: $-\infty < a(\tau)$. There are three cases:

(a) $A(u) > 0$ on $(-\infty, a(\tau))$;

(b) $A(u_0) = 0$ for some $u_0 \in (-\infty, a(\tau))$ and $A(u) < 0$ on $(-\infty, u_0)$ and $A(u) > 0$ on $(u_0, a(\tau))$;

(c) $A(u) < 0$ on $(-\infty, a(\tau))$.

μ_t may contain an isolated delta measure at $a(\mu_t)$. If so, we write as $\mu_t = \lambda(t)\delta_{\theta(t)} + \nu_t$. We can understand that $\lambda(t) = 0$ if μ_t does not contain an atom at $a(\mu_t)$, or if μ_t contains an atom at $a(\mu_t)$ but it is not isolated. The motion of the position $\theta(t)$ of a delta measure was clarified in the previous subsection. To investigate $a(\nu_t)$, we introduce a function E: $[0, \infty) \longrightarrow (-\infty, a(\tau)]$ by

$$E(t) := \begin{cases} \sup\{u \le a(\tau); H_t(u) = a(\tau)\} & \text{in case (a) and case (b),} \\ a(\tau) & \text{in case (c)} \end{cases}$$

for $t \in [0, \infty)$. The definition in the cases (a) and (b) may seem to be unclear since $H_t(z)$ was only defined in $\mathbb{C} \setminus \mathbb{R}$. The precise definition is as follows. Since case (a) and case (b) can be treated in the same way, we explain only case (b). If u is in the interval $(u_0, a(\tau))$, let $R(u)$ be defined so that $H_t(u)$ exists for all $t \in (0, R(u))$ and $\lim_{t \nearrow R(u)} H_t(u) = a(\tau)$. We observe that R is a function of u which satisfies $0 < R(u) < \infty$ on $(u_0, a(\tau))$. R is a bijection from $(u_0, a(\tau))$ to $(0, \infty)$. Therefore, we can define a bijection $E(t) := R^{-1}(t)$, which we have denoted simply as $\sup\{u \le a(\tau); H_t(u) = a(\tau)\}$.

$a(\nu_t)$ is characterized by the following result.

Lemma 3.4. *Let μ be a $\triangleright$-infinitely divisible distribution. μ can be expressed in the form $\mu = \lambda\delta_\theta + \nu$, where $\theta = a(\mu)$ is an isolated atom. We understand that $\mu = \nu$ or $\lambda = 0$ if μ does not contain an atom at $a(\mu)$ or if μ contains an atom at $a(\mu)$ but it is not isolated. Then the equalities*

$$a(\nu) = a(\eta) = \sup\{x \in \mathbb{R}; H_\mu \text{ has an analytic continuation to } \mathbb{C} \setminus [x, \infty)\}$$

hold under the notation (1.1).

Proof. The latter equality follows from Lemma 2.1 (1) immediately and we only need to prove that $a(\nu) = a(\eta)$. First, if $\lambda = 0$ we can easily prove $a(\mu) = a(\eta)$ by Lemma 2.1 (2). Second, we assume that $\lambda > 0$. We show that $a(\nu) \neq a(\eta)$ causes a contradiction. We notice first that the difference $a(\nu) \neq a(\eta)$ comes from the zero points of $H_\mu(x)$ or $G_\mu(x)$ by Lemma 2.1 (2). If $a(\nu) < a(\eta)$, then $H_\nu(a(\nu)) = 0$. This implies, however, G_μ contains two atoms at $a(\nu)$ and θ. This contradicts infinite divisibility (see Theorem 3.5 in [14]). If $a(\nu) > a(\eta)$, then $G_\nu(a(\eta)) = 0$. Since $\frac{d}{dx}H_\mu(x) \ge 1$ in $(\operatorname{supp}\mu)^c \subset \mathbb{R}$, $H_\mu(x)$ is increasing. Therefore $\lim_{x \nearrow a(\eta)} H_\mu(x) = \infty$ and $\lim_{x \searrow a(\eta)} H_\mu(x) = -\infty$. Also, $\lim_{x \to -\infty} H_\mu(x) = -\infty$. These imply that there exist $x_1 < a(\eta)$ and $x_2 > a(\eta)$ such that $H_\mu(x_1) = H_\mu(x_2)$. By Rouche's theorem, there exist distinct points $z_1, z_2 \in \mathbb{C}$ with positive imaginary parts such that $H_\mu(z_1) = H_\mu(z_2)$ (this argument is similar to the proof of Theorem 3.5 in [14]); this contradicts the infinite divisibility again since the solution of (1.3) defines a flow of injective mappings. □

Remark 3.5. If μ is not $\triangleright$-infinitely divisible, the above property does not hold. For instance, if $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$, $a(\nu) = 1$ but $a(\eta) = 0$.

We define $a(\nu_0) := a(\tau)$ in order that $a(\nu_t)$ becomes a continuous function around 0.

Theorem 3.6. *In case (a) and case (b), the equality $E(t) = a(\nu_t)$ holds for all $t \in [0, \infty)$. In case (c), the equality holds under the further assumption $\lim_{u \nearrow a(\tau)} A(u) < 0$.*

Proof. We can prove this equality by considering the region in which $H_t(z)$ is analytic. We first consider case (a) and case (b). We prove that

$$E(t) = \sup\{x \in \mathbb{R}; H_t \text{ has an analytic continuation to } \mathbb{C} \setminus [x, \infty)\}. \tag{3.6}$$

By *reductio ad absurdum* we show that H_t never has an analytic continuation beyond $E(t)$. If $H_t(z)$ has an analytic continuation to $\mathbb{C} \setminus [E(t) + \delta, \infty)$ for some $t > 0$ and $\delta > 0$, then we find the following three facts: the image of $H_t(u)$ includes the point $a(\tau)$ since $\frac{\partial H}{\partial u} \geq 1$ and $H(t, E(t)) = a(\tau)$; H_t is injective in $\mathbb{C} \setminus [E(t) + \delta, \infty)$; we can take $\delta > 0$ small enough so that $A(z)$ is analytic in $\mathbb{C} \setminus [E(t) + \delta, \infty)$ since $E(t) < a(\tau)$. Then by the equality $A(H_t(z)) = A(z) \frac{\partial H_t}{\partial z}(z)$ in $\mathbb{C} \setminus \mathbb{R}$, we conclude that $A(z)$ has an analytic continuation to the image of H_t. In particular, A is analytic around the point $a(\tau)$; this is a contradiction. Therefore, H_t cannot have an analytic continuation beyond $E(t)$.

Conversely, for any $u < E(t)$, $H_t(z)$ has an analytic continuation to the region $\mathbb{C} \setminus [u + \delta, \infty)$ for some $\delta > 0$ by the solution of the ODE (1.3). Then the equality (3.6) holds.

The proof of the equality $E(t) = a(\nu_t)$ in case (c) under the assumption $\lim_{u \nearrow a(\tau)} A(u) < 0$ is similar to the above. For all $t > 0$, we have $\lim_{u \nearrow a(\tau)} H_t(u) < a(\tau)$. Assume that $H_t(z)$ has an analytic continuation to $\mathbb{C} \setminus [E(t) + \delta, \infty)$ for some $t > 0$ and $\delta > 0$. We can take δ small enough such that $H_t(u) \in (-\infty, a(\tau))$ for all $u \in (-\infty, a(\tau) + \delta)$. This contradicts the equality $A(H_t(z)) = A(z) \frac{\partial H_t}{\partial z}(z)$. □

In case (c), if $\lim_{u \nearrow a(\tau)} A(u) = 0$, the question as to whether the relation $E(t) = a(\nu_t)$ holds for all $t > 0$ or not, has not been clarified yet. A partial answer is shown in the following proposition.

Proposition 3.7. *We consider the case (c). Then $a(\nu_t) = a(\tau)$ a.e. with respect to the Lebesgue measure on $[0, \infty)$ and $a(\nu_t) \geq a(\tau)$ for all $t > 0$.*

Proof. Step 1. First, we prove the following fact: if $\limsup_{t \to t_0, t \neq t_0} a(\nu_t) \geq a(\nu_{t_0})$, then $A(z)$ is analytic in the region $(-\infty, a(\nu_{t_0}))$ and moreover, $a(\nu_{t_0}) = a(\tau)$ $(= E(t_0)$. Fix an arbitrary number $\epsilon \in (0, 1)$. Take a sequence $\{t_n\}_{n=1}^{\infty}$ such that $a(\nu_{t_n}) \geq a(\nu_{t_0}) - \frac{\epsilon}{2}$ for all $n \geq 1$ and define the sequence of analytic functions in $(-\infty, a(\nu_{t_0}) - \epsilon)$ by

$$A_n^{\epsilon}(z) := \frac{H_{t_n}(z) - H_{t_0}(z)}{t_n - t_0}$$

for $n \geq 1$. For any compact set $K \subset \mathbb{C} \setminus [a(\tau) - \epsilon, \infty)$, we can prove that the sequence $\{A_n^{\epsilon}\}$ is uniformly bounded on K for sufficiently large n. Hence we obtain the analyticity of $\partial_t H(t_0, z)$ in $(-\infty, a(\nu_{t_0}) - \epsilon)$. Since $1 > \epsilon > 0$ is arbitrary, we conclude that $\partial_t H(t_0, z)$ is analytic in $(-\infty, a(\nu_{t_0}))$. $A(z)$ has an analytic continuation from $\mathbb{C} \setminus \mathbb{R}$ to $\mathbb{C} \setminus [a(\nu_{t_0}), \infty)$ by the equality $A(z) = \frac{\partial_t H(t_0, z)}{\partial_z H(t_0, z)}$. Now we show $a(\tau) = a(\nu_{t_0})$. As explained before, the solution $H_t(z)$ of the ODE exists for all time and for any initial position $z \in \mathbb{C} \setminus [a(\tau), \infty)$. Therefore, we obtain $a(\nu_t) \geq a(\tau)$ for all $t \in [0, \infty)$. Moreover, we can prove that $a(\tau) \geq a(\nu_{t_0})$ by the analyticity of $A(z)$ in $(-\infty, a(\nu_{t_0}))$.

Step 2. We note that $a(\nu_t)$ is Borel measurable. This is easy since the coefficients of the Taylor expansion of H_t is measurable (by the Cauchy integral formula), and $a(\nu_t)$ can be expressed by the limit supremum of them. We define a Borel set B by

$$B := \{t \in [0, \infty);\ \text{there exist } \epsilon = \epsilon(t) > 0 \text{ and } \eta = \eta(t) > 0 \text{ such that } |a(\nu_t) - a(\nu_s)| > \epsilon \text{ for all } s \text{ satisfying } 0 < |s - t| < \eta\}.$$

If $t \in B^c$, $a(\nu_t) = E(t)$ by Step 1. It is known that a Borel measurable function on an interval is continuous except for an open set with arbitrary small Lebesgue measure by Lusin's theorem (see [10]). Therefore, the Lebesgue measure of the set B is 0. $a(\nu_t) \geq a(\tau)$ was already mentioned in the proof of Step 1. □

So far we have proved that $E(t) = a(\nu_t)$ in generic cases. Next we show an ODE for the function $E(t)$. Define by

$$E_\epsilon(t) := \sup\{u \leq a(\tau); H_t(u) = a(\tau) - \epsilon\}$$

an approximate family for $\epsilon > 0$. This approximation is needed to use the implicit function theorem in the proof of Theorem 3.9.

Lemma 3.8. *In case* (*a*) *and case* (*b*), E_ϵ *and* E *enjoy the following properties.*
(1) $E_\epsilon < E$ *for all* $\epsilon \in (0, 1)$. *In addition,* E_ϵ *converges to* E *pointwise as* $\epsilon \to 0$.
(2) $\sup_{\epsilon>0, t\in I} |E_\epsilon(t)| < \infty$ *for any compact set* $I \subset [0, \infty)$

The above lemma is easily proved and we omit its proof.

Theorem 3.9. *We consider case* (*a*) *and case* (*b*). *Then* $E(t)$ *satisfies the ODE*

$$\begin{cases} \frac{d}{dt}E(t) = -A(E(t)) & \text{for } 0 < t < \infty, \\ E(0) = a(\tau). \end{cases}$$

In particular, E *is in* $C^\omega(0, \infty) \cap C[0, \infty)$.

Proof. We note that the inequality $\frac{\partial H}{\partial u} \geq 1$ holds. Then Implicit Function Theorem is applicable to the equation $H = a(\tau) - \epsilon$ because H is defined in the open set $\{(t, u); 0 < t < \infty, -\infty < u < E(t)\}$ which contains $(t, E_\epsilon(t))$ for all t. Therefore, E_ϵ is in class $C^\omega(0, \infty)$ and its derivative is

$$\frac{dE_\epsilon}{dt}(t) = -\frac{\partial_t H(t, E_\epsilon(t))}{\partial_u H(t, E_\epsilon(t))} = -A(E_\epsilon(t))$$

by Lemma 3.3. After integrating the above, we take the limit $\epsilon \to 0$ using Lemma 3.8, to obtain

$$E(t) = \int_t^{t_1} A(E(s))ds + E(t_1).$$

This implies that E is in class $C^\omega(0, \infty)$ and the ODE holds. The right continuity of E at 0 follows from the fact $\lim_{t\searrow 0} H_t(z) = z$. □

4 Time-dependent and time-independent properties of monotone convolution semigroup

In classical probability theory, it is often true that a property of a convolution semigroup μ_t is completely determined at an instant. Such a property is called a time-independent property. In this section, we prove such properties for monotone convolution semigroups.

Lemma 4.1. *Let $\{\mu_t\}_{t\geq 0}$ be a weakly continuous $\triangleright$-convolution semigroup with $\mu_0 = \delta_0$, and $A(z)$ be the associated vector field. If there exists $t_0 > 0$ such that $\operatorname{supp}\mu_{t_0} \subset [0,\infty)$, then $A(z)$ is analytic in $\mathbb{C}\setminus[0,\infty)$.*

Proof. We have $\operatorname{supp}\mu_{\frac{t_0}{n}} \subset [0,\infty)$ by Corollary 2.3 (1). Let $A_n(z)$ be defined by $A_n(z) := (H_{\frac{t_0}{n}}(z) - z)/\frac{t_0}{n}$. A_n is analytic in $\mathbb{C}\setminus[0,\infty)$. By definition $A(z) = \lim_{n\to\infty} A_n(z)$ for $z \in \mathbb{C}\setminus\mathbb{R}$. By Montel's theorem, it suffices to show that the RHS is uniformly bounded on each compact subset of $\mathbb{C}\setminus[0,\infty)$. Fix an arbitrary compact set $K \subset \mathbb{C}\setminus[0,\infty)$. By Lemma 2.1, $\operatorname{supp}\eta_{\frac{t_0}{n}} \subset [0,\infty)$. Since $H_t(i) = b_t + i(1+\eta_t(\mathbb{R}))$ is differentiable, there exist $M, M' > 0$ such that $\frac{\eta_t(\mathbb{R})}{t} \leq M$ and $\left|\frac{b_t}{t}\right| \leq M'$ for all $t \in [0, t_0]$. Then

$$\begin{aligned}|A_n(z)| &\leq \left|\frac{n}{t_0} b_{\frac{t_0}{n}}\right| + \left|\int_0^\infty \frac{1+xz}{x-z}\frac{n}{t_0}\eta_{\frac{t_0}{n}}(x)\right| \\ &\leq M' + L'\end{aligned}$$

for all n and $z \in K$. $L' > 0$ is a constant dependent only on K. □

Using Proposition 2.5 and Lemma 4.1, one can prove the monotone analogue of subordinator theorem. For the classical version, the reader is referred to Theorem 24.11 of [17].

Theorem 4.2. . *Let $\{\mu_t\}_{t\geq 0}$ be a weakly continuous $\triangleright$-convolution semigroup with $\mu_0 = \delta_0$. Then the following statements are equivalent:*

(1) there exists $t_0 > 0$ such that $\operatorname{supp}\mu_{t_0} \subset [0,\infty)$;

(2) $\operatorname{supp}\mu_t \subset [0,\infty)$ for all $0 \leq t < \infty$;

(3) $\operatorname{supp}\tau \subset [0,\infty)$, $\tau(\{0\}) = 0$, $\int_0^\infty \frac{1}{x}d\tau(x) < \infty$ and $\gamma \geq \int_0^\infty \frac{1}{x}d\tau(x)$.

Remark 4.3. (i) The equality $\tau(\{0\}) = 0$ in condition (3) means that there is no component of a Brownian motion in the Lévy-Khintchine formula.
(ii) The equivalence also holds in the classical and Boolean Lévy-Khintchine formulae. In the free case, however, (1) and (2) are not equivalent (see Section 6).

Proof. We note that (3) is equivalent to (3'): A is analytic in $\mathbb{C}\setminus[0,\infty)$ and $A < 0$ on $(-\infty, 0)$, by an argument in Proposition 2.5.

(1) ⇒ (2), (3'): If $\{\mu_t\}$ is a delta measure, then the statement follows immediately. We assume that μ_t is not a delta measure for some $t > 0$. This is equivalent to assuming that

μ_t is not a delta measure for all $t > 0$. Then τ is a nonzero positive finite measure. $A(z)$ is analytic in $\mathbb{C} \setminus [0, \infty)$ by Lemma 4.1, and hence, $\operatorname{supp} \tau \in [0, \infty)$. There are three possible cases: (a) $A(u) > 0$ on $(-\infty, 0)$; (b) $A(u_0) = 0$ for some $u_0 \in (-\infty, 0)$ and $A(u) < 0$ on $(-\infty, u_0)$ and $A(u) > 0$ on $(u_0, 0)$; (c) $A(u) < 0$ on $(-\infty, 0)$.

In case (a) and case (b), we have $a(\mu_t) < 0$ for all $t > 0$ by Theorem 3.1 (2). In case (c), we have $a(\mu_t) \geq a(\tau) \geq 0$ again by Theorem 3.1 (2). Hence only case (c) has no contradiction to the assumption.

$(3') \Rightarrow (1)$: This proof was actually done in the end of the proof of $(1) \Rightarrow (2)$. □

We can prove that the lower boundedness of the support is determined at one instant.

Theorem 4.4. *Let $\{\mu_t\}_{t \geq 0}$ be a weakly continuous $\rhd$-convolution semigroup with $\mu_0 = \delta_0$. Then the following statements are equivalent:*

(1) *there exists $t_0 > 0$ such that $\operatorname{supp} \mu_{t_0}$ is bounded below;*

(2) *$\operatorname{supp} \mu_t$ is bounded below for all $0 \leq t < \infty$;*

(3) *$\operatorname{supp} \tau$ is bounded below.*

Remark 4.5. The same kind of theorem also holds in the free and Boolean cases. The classical case is exceptional since the condition (3) needs to be replaced by $\operatorname{supp} \tau \subset [0, \infty)$, $\tau(\{0\}) = 0$ and $\int_{-1}^{1} \frac{1}{|x|} d\tau(x) < \infty$ [17]. Therefore, the boundedness below is not mapped bijectively by the monotone analogue of Bercovici-Pata bijection defined in Section 5.

Proof. $(1) \Rightarrow (3)$: When $a(\mu_{t_0}) \geq 0$, the claim follows from Theorem 4.2. We consider the case $a(\mu_{t_0}) < 0$. By Proposition 2.2, we have $a(\mu_t) \geq a(\mu_{t_0}) > -\infty$ for all $t \leq t_0$. By the same argument as in Lemma 4.1, one can show that A is analytic in $(-\infty, a(\mu_{t_0}))$.
$(3) \Rightarrow (2)$: The lower boundedness of the support of μ_t for all $t \geq 0$ comes from Theorem 3.1. □

Next we consider the symmetry around the origin. We say that a measure μ on the real line is symmetric if $\mu(dx) = \mu(-dx)$. The proof depends on the assumption of compact support. We could not prove the result for all probability measures.

Theorem 4.6. *Let $\{\mu_t\}_{t \geq 0}$ be a weakly continuous $\rhd$-convolution semigroup with $\mu_0 = \delta_0$. We assume that the support of each μ_t is compact (this is a time-independent property). Then the following statements are all equivalent.*

(1) *There exists $t_0 > 0$ such that μ_{t_0} is symmetric.*

(2) *μ_t is symmetric for all $t > 0$.*

(3) *$\gamma = 0$ and τ is symmetric.*

Proof. We prove this theorem in terms of moments. We use the representation of the vector field $A(z) = -\gamma + \int \frac{1}{x-z} d\sigma(x)$, $d\sigma(x) = (1 + x^2) d\tau(x)$, where σ has a compact support. We use the notation $m_n(t) = m_n(\mu_t)$ for simplicity. We notice that the symmetry is equivalent to the vanishment of odd moments for a compactly supported measure. Define a sequence

$\{r_n\}_{n=1}^{\infty}$ by $r_1 := \gamma$, $r_n := m_{n-2}(\sigma)$ for $n \geq 2$. Then $A(z) = -\sum_{n=1}^{\infty} \frac{r_n}{z^{n-1}}$. By Lemma 3.3 (2), we get differential equations $\frac{dm_0(t)}{dt} = 0$ and

$$\frac{dm_n(t)}{dt} = \sum_{k=1}^{n} k r_{n-k+1} m_{k-1}(t) \quad \text{for } n \geq 1 \tag{4.1}$$

with initial conditions $m_0(0) = 1$ and $m_n(0) = 0$ for $n \geq 1$.

Now we prove the implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). We can easily prove that $m_{2n+1}(t_0) = 0$ and $r_{2n+1} = 0$ for $n \geq 0$, and then $m_{2n+1}(t) = 0$ for all $t > 0$ and $n \geq 0$. Then σ and μ_t are both symmetric for all $t > 0$. The proof of the implication (3) $\Rightarrow$ (2) runs by a similar argument. □

We show some time-dependent properties.

Proposition 4.7. (1) *Absolute continuity is a time-dependent property.*
(2) *Existence of an atom is a time-dependent property.*

Proof. There is an example [14]. Let $\{\mu_t\}_{t\geq 0}$ be the monotone convolution semigroup defined by

$$H_t^{(\alpha,1,c)}(z) = c + \{(z-c)^\alpha + t\}^{\frac{1}{\alpha}} \quad \text{for } 0 < \alpha < 1. \tag{4.2}$$

Then μ_t contains an atom for $0 \leq t < |c|^\alpha$ and μ_t is absolutely continuous for $t \geq |c|^\alpha$. □

The property $m_{2n}(\mu) = \int_{\mathbb{R}} x^{2n}\mu(dx) < \infty$ is also time-independent. That is, we prove the following theorem which is also true in classical and free probabilities [5, 18]. In addition, this also extends Theorem 4.9 in [16] to higher order moments.

Theorem 4.8. *Let $\{\mu_t\}_{t\geq 0}$ be a weakly continuous $\rhd$-convolution semigroup with $\mu_0 = \delta_0$ and let $n \geq 1$ be a natural number. Then the following statements are equivalent:*

(1) *there exists $t_0 > 0$ such that $m_{2n}(t_0) < \infty$;*

(2) *$m_{2n}(t) < \infty$ for all $0 < t < \infty$;*

(3) *$m_{2n}(\tau) < \infty$.*

Proof. (1) $\Rightarrow$ (2): We use the notation $\mu_t^y := \delta_y \rhd \mu_t$ introduced in (2.1). For $0 \leq t \leq t_0$, we set $\lambda = \mu_{t_0-t}$ and $\nu = \mu_t$. Then we obtain $\int\int x^{2n}\mu_t^y(dx)\mu_{t_0-t}(dy) = \int_{\mathbb{R}} x^{2n}\mu_{t_0}(dx) < \infty$, which implies $m_{2n}(\mu_t^y) < \infty$ for some $y \in \mathbb{R}$. By Proposition 2.8, we obtain $m_{2n}(t) < \infty$ for $0 \leq t \leq t_0$. For arbitrary $0 < s < \infty$, we can write $s = kt_0 + t$ with $k \in \mathbb{N}$ and $0 \leq t < t_0$. Then we have $m_{2n}(s) < \infty$ by Proposition 2.9.
(2) $\Rightarrow$ (3): We first note that $m_k(t)$ is a Borel measurable function of $t \leq 0$ since μ_t is weakly continuous. Moreover, we show that there exist $r_1, \cdots, r_{2n} \in \mathbb{R}$ such that

$$m_l(t) = \sum_{k=1}^{l} \sum_{1=i_0<i_1<\cdots<i_{k-1}<i_k=l+1} \frac{t^k}{k!} \prod_{p=1}^{k} i_{p-1} r_{i_p - i_{p-1}} \tag{4.3}$$

for $1 \le l \le 2n$. For the proof we use the equality

$$m_l(t+s) = m_l(t) + m_l(s) + \sum_{k=1}^{l-1} \sum_{\substack{j_0+j_1+\cdots+j_k=l-k, \\ 0\le j_p,\ 0\le p\le k}} m_k(t) m_{j_0}(s) \cdots m_{j_k}(s) \tag{4.4}$$

for $1 \le l \le 2n$. For $l = 1$, (4.4) becomes $m_1(t+s) = m_1(t) + m_1(s)$. This is Cauchy's functional equation and there exists $r_1 \in \mathbb{R}$ such that $m_1(t) = r_1 t$ by the measurability (for a simple proof of Cauchy's functional equation, see [2]). We assume that there exist $r_1, \cdots, r_q \in \mathbb{R}$ such that (4.3) holds for $1 \le l \le q$. For an arbitrary $r'_{q+1} \in \mathbb{R}$, we define

$$\widetilde{m}_{q+1}(t) := r'_{q+1} t + \sum_{k=2}^{q+1} \sum_{1=i_0<i_1<\cdots<i_{k-1}<i_k=q+2} \frac{t^k}{k!} \prod_{p=1}^{k} i_{p-1} r_{i_p - i_{p-1}}. \tag{4.5}$$

Then the equality

$$\widetilde{m}_{q+1}(t+s) = \widetilde{m}_{q+1}(t) + \widetilde{m}_{q+1}(s) + \sum_{k=1}^{q} \sum_{\substack{j_0+j_1+\cdots+j_k=q+1-k, \\ 0\le j_l,\ 0\le l\le k}} m_k(t) m_{j_0}(s) \cdots m_{j_k}(s) \tag{4.6}$$

holds; this will be proved soon later in Proposition 4.10. Therefore, (4.4) and (4.6) imply that $m_{q+1}(t+s) - \widetilde{m}_{q+1}(t+s) = m_{q+1}(t) - \widetilde{m}_{q+1}(t) + m_{q+1}(s) - \widetilde{m}_{q+1}(s)$. This is again Cauchy's functional equation, and hence, there exists $r''_{q+1} \in \mathbb{R}$ such that $m_{q+1}(t) = \widetilde{m}_{q+1}(t) + r''_{q+1} t$. The above argument runs until $q = 2n-1$, and then we conclude that there exist $r_1, \cdots, r_{2n} \in \mathbb{R}$ such that (4.3) holds for $1 \le l \le 2n$.

By the equality $\frac{\partial G}{\partial t}(t,z) = A(z) \frac{\partial G}{\partial z}(t,z)$ we obtain $A(z) = \frac{G(1,z) - \frac{1}{z}}{\int_0^1 \frac{\partial G}{\partial z}(s,z) ds}$, which implies

$$A(z) = -\frac{\frac{m_1(1)}{z^2} + \cdots + \frac{m_{2n}(1)}{z^{2n+1}} + o(|z|^{-(2n+1)})}{\frac{1}{z^2} + \frac{2\int_0^1 m_1(s)ds}{z^3} + \cdots + \frac{(2n+1)\int_0^1 m_{2n}(s)ds}{z^{2n+2}} + \int_0^1 R_s(z) ds}, \tag{4.7}$$

where $R_s(z)$ is defined by $R_s(z) = \frac{2n+1}{z^{2n+2}} \int_{\mathbb{R}} \frac{x^{2n+1}}{z-x} \mu_s(dx) + \frac{1}{z^{2n+1}} \int_{\mathbb{R}} \frac{x^{2n+1}}{(z-x)^2} \mu_s(dx)$. We prove a property of $R_s(z)$ here. Since $m_{2n}(s)$ is a polynomial, x^{2n} is integrable with respect to the measure $\mu_s(dx)ds$ on $\mathbb{R} \times [0,t]$. Easily we can show that $\int_0^1 R_s(iy) ds = o(y^{-(2n+2)})$ by the dominated convergence theorem. Therefore, there exist $u_1, \cdots, u_{2n} \in \mathbb{R}$ such that $A(iy) = u_1 + \frac{u_2}{iy} + \cdots + \frac{u_{2n}}{(iy)^{2n-1}} + o(y^{-(2n-1)})$. By Proposition 2.8, we have $m_{2n}(\tau) < \infty$ (the equivalence between (2) and (3) in Proposition 2.8 is true for $A(z)$. The proof needs no changes).

(3) $\Rightarrow$ (2): Since $m_{2n}(\tau) < \infty$, we have the expansion $A(z) = u_1 + \frac{u_2}{z} + \cdots + \frac{u_{2n}}{z^{2n-1}} + Q(z)$, where $Q(z) := \frac{1}{z^{2n-1}} \int_{\mathbb{R}} \frac{x^{2n-1}}{x-z} (1+x^2) \tau(dx)$. We obtain

$$H_t(z) = z + u_1 t + \int_0^t \frac{u_2}{H_s(z)} ds + \cdots + \int_0^t \frac{u_{2n}}{H_s(z)^{2n-1}} ds + \int_0^t Q(H_s(z)) ds \tag{4.8}$$

from the equality $\frac{d}{dt} H_t(z) = A(H_t(z))$. We can prove that $\sum_{k=p}^{2n-1} \int_0^t \frac{u_{k+1}}{H_s(iy)^k} ds + \int_0^t Q(H_s(iy)) ds = o(y^{-(p-1)})$ since $|\int_0^t \frac{1}{H_s(iy)^k} ds| \le \frac{t}{y^k}$. In addition, $\int_0^t Q(H_s(iy)) ds = o(y^{-(2n-1)})$ for any $t > 0$

by the dominated convergence theorem. Now we show by induction that there exist polynomials $c_k(t)$ of t $(1 \leq k \leq 2n)$ such that

$$H_t(z) = z + c_1(t) + \frac{c_2(t)}{z} + \cdots + \frac{c_{2n}(t)}{z^{2n-1}} + o(|z|^{-(2n-1)}) \quad (z = iy,\ y \to \infty) \tag{4.9}$$

for any $t > 0$. First $H_t(iy) = iy + u_1 t + \frac{u_2 t}{iy} + o(\frac{1}{y})$ holds by (4.8). Next we assume that there exist polynomials $c_k(t)$ of t $(1 \leq k \leq 2q)$ such that

$$H_t(z) = z + c_1(t) + \frac{c_2(t)}{z} + \cdots + \frac{c_{2q}(t)}{z^{2q-1}} + P_t(z), \tag{4.10}$$

where $P_t(iy) = o(y^{-(2q-1)})$ for any $t > 0$. We can write $P_t(z) = \frac{1}{z^{2q-1}} \int_{\mathbb{R}} \frac{x^{2q-1}}{x-z} \rho_t(dx)$, where ρ_t is the positive finite measure in Proposition 2.8 (2). Then we obtain the asymptotic behavior $\int_0^t P_s(iy)ds = o(y^{-(2q-1)})$. Substituting (4.10) into the right hand side of (4.8), we obtain the expansion

$$H_t(z) = z + b_1(t) + \frac{b_2(t)}{z} + \cdots + \frac{b_{2q+2}(t)}{z^{2q+1}} + o(|z|^{-(2q+1)}), \tag{4.11}$$

where $b_k(t)$ is a polynomial of t (we note that $b_k(t) = c_k(t)$ holds for $1 \leq k \leq 2q$ by the uniqueness of the expansion). This induction goes until $q = n - 1$ and we obtain (4.9). The conclusion follows from Proposition 2.8. □

Remark 4.9. We have proved that $m_k(t)$ is a polynomial of t in the proof of (2) ⇒ (3). This property might seem to be too strong: what we needed was the integrability of $m_k(t)$ in a finite interval. The author however could not find an alternative proof of the integrability.

The following result completes the above theorem.

Proposition 4.10. *For any complex numbers r_n, $n \geq 1$, $m_n(t)$ defined by*

$$m_n(t) = \sum_{k=1}^{n} \sum_{1=i_0<i_1<\cdots<i_{k-1}<i_k=n+1} \frac{t^k}{k!} \prod_{p=1}^{k} i_{p-1} r_{i_p - i_{p-1}} \tag{4.12}$$

satisfy the equality

$$m_n(t+s) = m_n(t) + m_n(s) + \sum_{k=1}^{n-1} \sum_{\substack{j_0+j_1+\cdots+j_k=n-k, \\ 0 \leq j_p,\ 0 \leq p \leq k}} m_k(t) m_{j_0}(s) \cdots m_{j_k}(s) \tag{4.13}$$

for any $n \geq 1$.

Proof. Every series in this proof is a formal power series. We define $A(z) = -\sum_{z=1}^{\infty} \frac{r_n}{z^{n-1}}$. We solve the differential equation (1.3) in the sense of formal power series. Then the solution $H_t(z)$ of the form $H_t(z) = \sum_{n=-1}^{\infty} \frac{a_n(t)}{z^n}$ uniquely exists. It is easy to prove that $H_{t+s}(z) = H_t(H_s(z))$ in the sense of formal power series with respect to t, s, z. If we define $G_t(z)$ by $\frac{1}{H_t(z)}$, then Lemma 3.3 holds by the same proof. We can easily prove that $m_n(t)$ are given by $G_t(z) = \sum_{n=0}^{\infty} \frac{m_n(t)}{z^{n+1}}$ using the equality (2) in Lemma 3.3. (4.13) follows from the power series expansion of $G_{t+s}(z) = G_t(\frac{1}{G_s(z)})$. □

5 Connection to infinite divisibility in classical probability theory

Now we consider the correspondence between classical probability theory and monotone probability. The usual Lévy-Khintchine formula is given by

$$\widehat{\mu}(u) = \exp\Big(i\gamma u + \int_{\mathbb{R}} \big(e^{ixu} - 1 - \frac{ixu}{1+x^2}\big)\frac{1+x^2}{x^2}\tau(dx)\Big), \tag{5.1}$$

where $\gamma \in \mathbb{R}$ and τ is a positive finite measure. We show that identification of (γ, τ) in Theorem 1.1 and in (5.1) is important. For instance, the support of a classical infinitely divisible distribution is concentrated on the positive real line in and only if (see Theorem 24.11 in [17])

$$\operatorname{supp}\tau \subset [0,\infty),\ \int_0^1 \frac{1}{x}\tau(dx) < \infty,\ \tau(\{0\}) = 0,\ \gamma \geq \int_0^\infty \frac{1}{x}\tau(dx). \tag{5.2}$$

These conditions are completely the same as in Theorem 4.2. Then it is natural to define the monotone analogue of the Bercovici-Pata bijection (for the details of the Bercovici-Pata bijection in free probability, the reader is referred to [7]). Let $ID(\rhd)$ be the set of all $\rhd$-infinitely divisible distributions; let $ID(*)$ be the set of all classical infinitely divisible distributions. We define a map $\Lambda_M : ID(*) \to ID(\rhd)$ by sending the pair (γ,τ) in (5.1) to the pair (γ,τ) in Theorem 1.1 (4). This map enjoys nice properties. Let D_λ be the dilation operator defined by

$$\int_{\mathbb{R}} f(x) D_\lambda\mu(dx) = \int_{\mathbb{R}} f(\lambda x)\mu(dx)$$

for all probability measures μ and all bounded continuous functions f.

Theorem 5.1. *Λ_M satisfies following properties.*

(1) *Λ_M is continuous;*

(2) *$\Lambda_M(\delta_a) = \delta_a$ for all $a \in \mathbb{R}$;*

(3) *$D_\lambda \circ \Lambda_M = \Lambda_M \circ D_\lambda$ for all $\lambda > 0$.*

(4) *Λ_M maps the Gaussian with mean 0 and variance σ^2 to the arcsine law with mean 0 and variance σ^2;*

(5) *Λ_M maps the Poisson distribution with parameter λ to the monotone Poisson distribution with parameter λ;*

(6) *Λ_M gives a one-to-one correspondence between the set $\{\mu \in ID(*); \operatorname{supp}\mu \subset [0,\infty)\}$ and the set $\{\nu \in ID(\rhd); \operatorname{supp}\nu \subset [0,\infty)\}$.*

(7) *For all $\alpha \in (0,2)$, Λ_M gives a one-to-one correspondence between strictly α-stable distributions and monotone strictly α-stable distributions.*

(8) *If* $\operatorname{supp}\tau$ *is compact, the symmetry of* $\mu \in ID(*)$ *is equivalent to the symmetry of* $\Lambda_M(\mu)$.

(9) *For each* $n \geq 1$, Λ_M *gives a one-to-one correspondence between the set* $\{\mu \in ID(*); \int_{\mathbb{R}} x^{2n}\mu(dx) < \infty\}$ *and the set* $\{\nu \in ID(\rhd); \int_{\mathbb{R}} x^{2n}\nu(dx) < \infty\}$.

Remark 5.2. Since monotone convolution is non-commutative, Λ_M does not preserve the structure of convolutions: $\Lambda_M(\mu * \lambda) \neq \Lambda_M(\mu) \rhd \Lambda_M(\lambda)$ for some $\mu,\ \lambda$.

Proof. (1) It is known that the convergence of a sequence $\{\mu_n\} \subset ID(*)$ to some μ implies the convergence of the corresponding pair (γ_n, τ_n) to some (γ, τ). Now we have the family of ODEs driven by

$$A_n(z) = -\gamma_n + \int_{\mathbb{R}} \frac{1+xz}{x-z} d\tau_n(x);$$

we denote the flow by $\{H_{n,t}\}$. Since (γ_n, τ_n) converges to (γ, τ), A_n converges locally uniformly to A. By the basic result of the theory of ODE, it holds that $H_{n,1}(z) \to H_1(z)$ locally uniformly, which implies that μ_n converges weakly to μ.

(2), (4) and (5) are proved easily by using the Lévy-Khintchine formulae [16, 17].

(3) and (7) follow from direct computations of the Lévy-Khintchine formula. See [14] and [17].

The property (6) follows from Theorem 4.2.

(8) and (9) are direct consequences of the theorems 4.6 and 4.8. □

6 Time-independent properties of free and Boolean convolution semigroups

We prepare tools to study convolution semigroups in free and and Boolean probabilities. Notation is chosen in order that the correspondence becomes clear among the Bercovici-Pata bijections in free, monotone and Boolean probability theories. We define

$$K_\mu(z) := z - H_\mu(z) = \gamma - \int \frac{1+xz}{x-z} d\tau(x). \tag{6.1}$$

As proved in [19], the Boolean convolution $\mu \uplus \nu$ of probability distributions μ and ν is characterized by

$$K_{\mu \uplus \nu} = K_\mu + K_\nu. \tag{6.2}$$

Every probability measure is Boolean infinitely divisible.

We summarize results of infinitely divisible distributions in free probability (see [3, 8] for instance). For a probability measure μ, there exists some $\eta > 0$ and $M > 0$ such that H_μ has an analytic right inverse H_μ^{-1} defined on the region

$$\Gamma_{\eta,M} := \{z \in \mathbb{C}; |\operatorname{Re} z| < \eta |\operatorname{Im} z|,\ |\operatorname{Im} z| > M\}.$$

The Voiculescu transform ϕ_μ is defined by $\phi_\mu(z) := H_\mu^{-1}(z) - z$ in a region where H_μ^{-1} is defined. For probability measures μ and ν, the free convolution of μ and ν is characterized by the relation

$$\phi_{\mu \boxplus \nu} = \phi_\mu + \phi_\nu. \tag{6.3}$$

Theorem 6.1. *Let μ be a probability measure on $\mathbb{R}$. μ is $\boxplus$-infinitely divisible if and only if there exist a finite measure τ and a real number γ such that*

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+xz}{z-x} d\tau(x) \quad \text{for } z \in \mathbb{C}\setminus\mathbb{R}. \tag{6.4}$$

In this section we prove time-independent properties of free and Boolean convolution semigroups to clarify similarity and dissimilarity of the Bercovici-Pata bijections for the free, Boolean and monotone convolutions. First we show that the subordinator theorem is valid in the Boolean case but is not valid in the free case.

Theorem 6.2. *Let $\{\mu_t\}_{t\geq 0}$ be a weakly continuous Boolean convolution semigroup with $\mu_0 = \delta_0$. Then the following statements are equivalent:*

(1) there exists $t_0 > 0$ such that $\operatorname{supp}\mu_{t_0} \subset [0,\infty)$;

(2) $\operatorname{supp}\mu_t \subset [0,\infty)$ for all $0 \leq t < \infty$;

(3) $\operatorname{supp}\tau \subset [0,\infty)$, $\tau(\{0\}) = 0$, $\int_0^\infty \frac{1}{x} d\tau(x) < \infty$ and $\gamma \geq \int_0^\infty \frac{1}{x} d\tau(x)$.

This type of theorem does not hold in free probability: Condition (1) is not equivalent to condition (2).

Proof. In the Boolean case, the proof is easy by Proposition 2.5. In free probability, we show an example of a convolution semigroup where (1) does not imply (2). Since the problem is symmetric around the origin, we show a counter example concerning the condition $\operatorname{supp}\mu_t \subset (-\infty, 0]$. We define $\phi_\mu(z) := a-(z-c)^{\frac{1}{2}}$ with $a, c \in \mathbb{R}$. Then the corresponding convolution semigroup $\{\mu_t\}_{t\geq 0}$ with $\mu_1 = \mu$, $\mu_0 = \delta_0$ is characterized by

$$H_t(z) = z - at + \frac{t^2}{2} + t\sqrt{z - \left(at - \frac{t^2}{4} + c\right)}. \tag{6.5}$$

It is easy to show that $\operatorname{supp}\mu_t \subset (-\infty, 0]$ for sufficiently large t, but $\operatorname{supp}\mu_t \not\subseteq (-\infty, 0]$ for small t. □

The symmetry around the origin is a time-independent property also in the cases of Boolean and free independence. The proof is easy.

Proposition 6.3. *Let $\{\mu_t\}_{t\geq 0}$ be a weakly continuous Boolean (free) convolution semigroup with $\mu_0 = \delta_0$. Then the following statements are all equivalent.*

(1) *There exists $t_0 > 0$ such that μ_{t_0} is symmetric.*

(2) *μ_t is symmetric for all $t > 0$.*

(3) *$\gamma = 0$ and τ is symmetric.*

We can also show that the property $\int_{\mathbb{R}} x^{2n} d\mu_t(x) < \infty$ is time-independent in Boolean case. In free probability, this result is recently obtained in [5].

Proposition 6.4. *Let $n \geq 1$ be a natural number. For a weakly continuous Boolean convolution semigroup $\{\mu_t\}_{t\geq 0}$, the following statements are equivalent.*

(1) $\int_{\mathbb{R}} x^{2n} d\mu_t(x) < \infty$ *for some* $t > 0$.

(2) $\int_{\mathbb{R}} x^{2n} d\mu_t(x) < \infty$ *for all* $t > 0$.

(3) $\int_{\mathbb{R}} x^{2n} d\tau(x) < \infty$.

Proof. The proof follows from Proposition 2.8. □

Now we can compare the properties of Bercovici-Pata bijections in free, monotone and Boolean probability theories. Boolean (strictly) stable distributions have been classified in [19], and they have the same characterization as the monotone case. Considering the contents in this section, we obtain the Boolean analogue of properties (1)-(9) in Theorem 5.1. It might be fruitful to consider the validity of property (6) in the Boolean and monotone cases in terms of the embeddings into tensor independence [11]. In free probability, most of the results of Theorem 5.1 are already known (see [3, 7]) except for the failure of free analog of property (6).

Another similarity between free and monotone independences is that the number of atoms in a $\boxplus$-infinitely divisible distribution is restricted in a similar way to the case of a $\rhd$-infinitely divisible distribution (see Theorem 3.5 in [14] and Proposition 2.8 in [4]).

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