Free probabilistic analysis of random matrices converging to compact operators

Takahiro Hasebe (Hokkaido University)

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Joint work with
Benoit Collins (Kyoto University)
Noriyoshi Sakuma (Aichi University of Education)

Main results

Our model is a selfadjoint polynomial

$$X = P(A_1, \dots, A_k, G)$$

where

- Let G be an $n \times n$ GUE (Gaussian unitary ensemble);
- $A_1, \dots, A_k$ are $n \times n$ deterministic Hermitian matrices "converging to compact operators";
- $P(0, \dots, 0, G) = 0$ (e.g. $P(A_1, A_2, G) = A_1 G^2 A_1 + G A_2 G$).

Theorem

Almost surely, the eigenvalues of $P(A_1, \dots, A_k, G)$ converge to the eigenvalues of some compact selfadjoint operator. We compute explicit limiting eigenvalues for several examples of P.

Empirical Eigenvalue Distribution

- X: $n \times n$ Hermitian random matrix
- $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$: Eigenvalues of X

The random probability measure

$$\mu_X := \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(X)}$$

is called the empirical eigenvalue distribution of X.

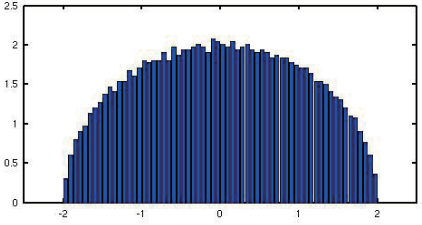


Fig. 1: Histogram of μ_G for GUE ($n = 3000$). Taken from Wikipedia

GUE (Gaussian Unitary Ensemble)

A random mat. $G = G(n) := (g_{ij})_{i,j=1}^n$ is a (normalized) GUE if

- G is Hermitian,
- $\{\mathrm{Re}(g_{ij}), \mathrm{Im}(g_{ij}) : 1 \leq i \leq j \leq n\}$ is a family of independent Gaussian r.v.s
- $\mathbb{E}[g_{ij}] = 0$
- $\mathrm{Var}(\mathrm{Re}(g_{ij})) = \mathrm{Var}(\mathrm{Im}(g_{ij})) = \frac{1}{n}$ if $i \neq j$,
- $\mathrm{Var}(g_{ii}) = \frac{2}{n}$.

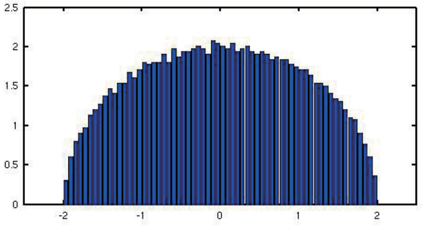


Fig. 2: GUE ($n = 3000$). Taken from Wikipedia

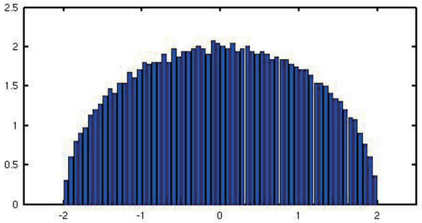
Wigner's semicircle law

- Eugene Wigner ('55, '58) proved that

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i(G)}\right] \Rightarrow \underbrace{\frac{1}{2\pi}\sqrt{(4-x^2)_+}\,dx}_{\text{Wigner's semicircle law}} \quad \text{as } n \to \infty.$$

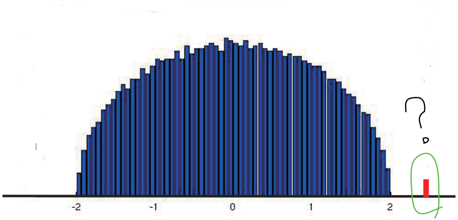
- Ludwig Arnold '67 proved that

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i(G)} \Rightarrow \frac{1}{2\pi}\sqrt{(4-x^2)_+}\,dx \quad \text{as } n \to \infty \text{ almost surely}.$$



Is there an outlier?

An outlier is an eigenvalue of G which lies outside of $[-2, 2]$ (the support of Wigner's semicircle law).



The almost sure weak convergence

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i(G)} \Rightarrow \frac{1}{2\pi}\sqrt{(4-x^2)_+}\,dx$$

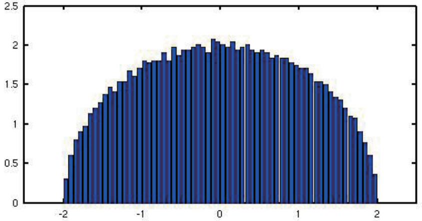
does not say anything about small number of outliers.

Is there an outlier?

Geman '80 proved that there is no outlier, that is,

Theorem (Geman, Ann. Probab. '80)

The largest eigenvalue $\lambda_1(G)$ *converges to 2 as* $n \to \infty$ *almost surely.*



Remark

Haagerup and Thorbjørnsen '05 proved a similar result for polynomials of independent GUEs.

Near Edge

We know that $\lambda_1(G(n)) \to 2$ almost surely (Law of large numbers). Is there a sequence $\{a_n\}_{n\geq 1}, a_n \uparrow \infty$ such that

$$a_n(\lambda_1(G(n)) - 2)$$

converges to a non zero limit?

- Central limit theorem
- Law of iterated logarithms

Tracy-Widom distribution (CLT)

Theorem (Tracy-Widom, CMP '94)

For any $x \in \mathbb{R}$

$$\mathbb{P}\left[n^{2/3}(\lambda_1(G) - 2) \leq x\right] \to F(x), \qquad n \to \infty,$$

where

$$F(x) = \exp\left(-\int_x^\infty (y-x)^2 q(y)^2\, dy\right),$$

and q *is the unique solution to the Peinlevé II equation*

$$\frac{d^2 q}{dx^2} = xq + 2q^3$$

with the boundary condition $q(x) \sim \mathrm{Ai}(x), x \to \infty$.

Law of non-iterated logarithm

Theorem (Paquette, Zeitouni, arXiv, June '15)

$$\varlimsup_{n\to\infty} \frac{n^{2/3}(\lambda_1(G)-2)}{(\log n)^{2/3}} = \frac{1}{4^{2/3}} \quad a.s.$$

Conjecture

$$\varliminf_{n\to\infty} \frac{n^{2/3}(\lambda_1(G)-2)}{(\log n)^{1/3}} = \frac{1}{4^{1/3}} \quad \text{a.s.}$$

Perturbation of GUE

Péché '06 considered the following deformation of a GUE:

$$X = G + A,$$

where A is an $n \times n$ deterministic Hermitian matrix (with bounded rank, for simplicity).
A phase transition of the largest eigenvalue $\lambda_1(X)$ appears.

Perturbation of GUE

Theorem (Péché, PTRF '06)

Suppose that for a fixed $r \in \mathbb{N}$ and $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_r$,

$$A(n) = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_r, 0, \ldots, 0).$$

Then almost surely

$$\lim_{n\to\infty} \lambda_1(G(n) + A(n)) = \begin{cases} 2, & \theta_1 \leq 1, \\ \theta_1 + \frac{1}{\theta_1}, & \theta_1 > 1. \end{cases}$$

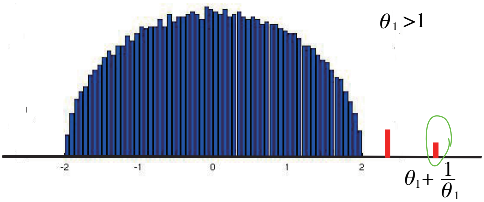


Fig. 4: Empirical eigenvalue distribution of $G(n) + A(n)$

Perturbation of GUE

Remark

(1) Péché also obtained a Tracy-Widom type distribution.

(2) Baik, Ben Arous, Péché '05 considered the multiplicative perturbation $(I + A)G^2(I + A)$ and found a phase transition.

(3) Benaych-Georges and Nadakuditi '11 generalized the results of Baik, Ben Arous, Péché '05 and Péché '06.

(4) General polynomials of A's and G's have not been investigated.

Our Model

Our model (in a simple case) is

$$X = P(A_1, \dots, A_k, G)$$

such that

- $P(0, \dots, 0, G) = 0$,
- $(A_1, \dots, A_k)$ are $n \times n$ deterministic matrices converging to trace class operators, i.e. there exists a tuple $(X_1, \dots, X_k)$ of trace class operators on a Hilbert space H such that

$$\lim_{n\to\infty} \mathrm{Tr}_n(A_{i_1}^{\varepsilon_1} \cdots A_{i_p}^{\varepsilon_p}) = \mathrm{Tr}_H(X_{i_1}^{\varepsilon_1} \cdots X_{i_p}^{\varepsilon_p})$$

  for any $p \in \mathbb{N}$, $(i_1, \dots, i_p) \in \{1, \dots, k\}^p$, $(\varepsilon_1, \dots, \varepsilon_p) \in \{1, *\}^p$.

Model (Examples)

Suppose that $\{\mu_n\}_{n=1}^{\infty}, \{\mu'_n\}_{n=1}^{\infty} \in \ell^1(\mathbb{R})$ and

- G: $n \times n$ GUE,
- $A = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n), A' = \mathrm{diag}(\mu'_1, \mu'_2, \ldots, \mu'_n)$

For example we can consider

- $P(A, A', G) = A + GA'G$,
- $P(A, A', G) = AGA + GA'G$,
- $P(A, G) = AG + GA$,
- $P(A, G) = \mathrm{i}(AG - GA)$.

Main theorem 1: Discrete eigenvalues

Our model is an $n \times n$ Hermitian random matrix

$$X = P(A_1, \dots, A_k, G)$$

such that

- $P(0, \dots, 0, G) = 0$,
- $(A_1, \dots, A_k)$ are $n \times n$ deterministic matrices converging to trace class operators.

Roughly, "$\lim_{n\to\infty} G(n)$" is a "bounded operator", and "$\lim_{n\to\infty} A_i(n)$" is a "trace class operator", so we can expect that $P(A_1, \dots, A_k, G)$ is also a "trace class operator".

Theorem (Collins, Sakuma, H.)

For any such polynomial $P(A_1, \dots, A_k, G)$, its eigenvalues $\{\lambda_i(n)\}_{i=1}^{n}$ converge "pointwise" to a deterministic sequence $\{\lambda_i\}_{i=1}^{\infty} \in \ell^2(\mathbb{R})$ almost surely.

Main Theorem 1: Discrete eigenvalues

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Definition

"Pointwise" convergence means that if we order $\{\lambda_i(n)\}_{i=1}^n$ in the way

$$\lambda_1^+(n) \geq \lambda_2^+(n) \geq \cdots \geq 0 \geq \cdots \geq \lambda_2^-(n) \geq \lambda_1^-(n),$$

then

$$\lim_{n\to\infty} \lambda_i^+(n) = \lambda_i^+, \qquad i \in \mathbb{N},$$
$$\lim_{n\to\infty} \lambda_i^-(n) = \lambda_i^-, \qquad i \in \mathbb{N}.$$

Main Theorem 2: Explicit computation

- $\mathrm{EV}(X)$: the multiset of eigenvalues of a Hermitian matrix or a selfadjoint compact operator X. $\mathrm{EV}(X) \cup \mathrm{EV}(Y)$ counts the multiplicity, e.g.

$$\{2, 1, 0, 0, \dots\} \cup \{3, 2, 1, 0, \dots\} = \{3, 2, 2, 1, 1, 0, \dots\}.$$

Theorem (Collins, Sakuma, H., simplified version)

Suppose that $\{\mu_i\}_{i \geq 1}, \{\mu_i'\}_{i \geq 1} \in \ell^1(\mathbb{R})$ *and*

- $A = A(n) = \mathrm{diag}(\mu_1, \mu_2, \dots, \mu_n)$,
- $A' = A'(n) = \mathrm{diag}(\mu_1', \mu_2', \dots, \mu_n')$,
- $G = G(n)$ *is a GUE.*

Then

(1) $\lim_{n\to\infty} \mathrm{EV}(A + GA'G) = \{\mu_i\}_{i \geq 1} \cup \{\mu_i'\}_{i \geq 1}$,

(2) $\lim_{n\to\infty} \mathrm{EV}(AG^2A + A'G^2A') = \{\mu_i^2 + (\mu_i')^2\}_{i \geq 1}$,

Main theorem 2: Explicit computation

Theorem (Collins, Sakuma, H., simplified version)

Suppose that $\{\mu_i\}_{i\geq 1}, \{\mu'_i\}_{i\geq 1} \in \ell^1(\mathbb{R})$ *and*

- $A = A(n) = \mathrm{diag}(\mu_1, \mu_2, \dots, \mu_n)$,
- $A' = A'(n) = \mathrm{diag}(\mu'_1, \mu'_2, \dots, \mu'_n)$,
- $G = G(n)$ *is a GUE.*

Then

$$(1)\quad \lim_{n\to\infty} \mathrm{EV}(A + GA'G) = \{\mu_i\}_{i\geq 1} \cup \{\mu'_i\}_{i\geq 1},$$

$$(2)\quad \lim_{n\to\infty} \mathrm{EV}(AG^2A + A'G^2A') = \{\mu_i^2 + (\mu'_i)^2\}_{i\geq 1},$$

$$(3)\quad \lim_{n\to\infty} \mathrm{EV}(AG + GA) = \lim_{n\to\infty} \mathrm{EV}(\mathrm{i}(AG - GA)) = \{\mu_i\}_{i\geq 1} \cup \{-\mu_i\}_{i\geq 1},$$

$$(4)\quad \lim_{n\to\infty} \mathrm{EV}(AG^2 + G^2A) = \{(\sqrt{2}+1)\mu_i\}_{i\geq 1} \cup \{(1-\sqrt{2})\mu_i\}_{i\geq 1},$$

Proof of Main Theorem 1

Step 1: Convergence of traces implies convergence of eigenvalues.

Proposition

Suppose that $X, X(n)$ are trace class selfadjoint operators on Hilbert spaces $H, H(n)$ for $n \in \mathbb{N}$. If

$$\lim_{n\to\infty} \mathrm{Tr}_{H(n)}(X(n)^\ell) = \mathrm{Tr}_H(X^\ell) \qquad \forall \ell \in \mathbb{N},$$

then $\lim_{n\to\infty} \mathrm{EV}(X(n)) = \mathrm{EV}(X)$.

Corollary

Suppose that X, Y are trace class selfadjoint operators on Hilbert spaces H, K. If $\mathrm{Tr}_H(X^\ell) = \mathrm{Tr}_K(Y^\ell)$ for every $\ell \in \mathbb{N}$ then $\mathrm{EV}(X) = \mathrm{EV}(Y)$.

Proof of Main Theorem 1

Step 2: $\exists X \in S^2(H)_{\mathrm{sa}}, \lim_{n\to\infty} \mathrm{Tr}_n(P(A_1, \ldots, A_k, G)^\ell) = \mathrm{Tr}_H(X^\ell)$ (almost surely)

- First we take the expectation.
- We look at a monomial.

The problem reduces to study of $\mathbb{E}[\mathrm{Tr}_n(A_1 G^{\ell_1} A_2 G^{\ell_2} \cdots A_k G^{\ell_k})]$.

Proposition (Shlyakhtenko, arXiv '15, Sep 29)

For any $p_1, \ldots, p_k \in \mathbb{N} \cup \{0\}$,

$$\mathbb{E}[\mathrm{Tr}_n(A_1 G^{\ell_1} A_2 G^{\ell_2} \cdots A_k G^{\ell_k})] = \mathrm{Tr}_n(A_1 \cdots A_k) \prod_{i=1}^{k} \mathbb{E}[\mathrm{tr}_n(G^{\ell_i})] + O(n^{-1}),$$

where $\mathrm{tr}_n = \frac{1}{n}\mathrm{Tr}_n$. *It is known that*

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{tr}_n(G^\ell)] = \int_{-2}^{2} x^\ell \frac{1}{2\pi} \sqrt{4 - x^2}\, dx.$$

Proof of Main Theorem 1

We know that

$$\mathbb{E}[\mathrm{Tr}_n(A_1 G^{\ell_1} A_2 G^{\ell_2} \cdots A_k G^{\ell_k})] = \mathrm{Tr}_n(A_1 \cdots A_k) \prod_{i=1}^{k} \mathbb{E}[\mathrm{tr}_n(G^{\ell_i})] + O(n^{-1}),$$

where $\mathrm{tr}_n = \frac{1}{n}\mathrm{Tr}_n$. We will remove $\mathbb{E}$.

Lemma

Let $Y := \mathrm{Tr}_n(A_1 G^{\ell_1} A_2 G^{\ell_2} \cdots A_k G^{\ell_k})$. *Then*

$$\mathbb{E}[|Y - \mathbb{E}[Y]|^4] = O(n^{-2}).$$

By Borel-Cantelli's lemma we have the almost sure convergence

$$\lim_{n\to\infty} \mathrm{Tr}_n(A_1 G^{\ell_1} A_2 G^{\ell_2} \cdots A_k G^{\ell_k}) = \lim_{n\to\infty} \mathrm{Tr}_n(A_1 \cdots A_k) \prod_{i=1}^{k} \mathbb{E}[\mathrm{tr}_n(G^{\ell_i})].$$

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