MONOTONE PROBABILITY THEORY

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Abstract. This is an introduction to monotone probability theory, a kind of noncommutative probability based on the notion of monotone independence for noncommutative random variables. The first part, Sections 1–4, introduces basic materials including noncommutative probability spaces, monotone independence, sums and products of independent random variables, monotone cumulants, the central limit theorem, and Cauchy transform. The second part, Section 5–8, focuses on more advanced topics including a detailed study of monotone convolution, noncommutative stochastic processes, connections to dynamics of holomorphic self-maps of the upper half-plane (in particular, Loewner theory), and applications to outliers of random matrices.

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Preface

Noncommutative probability theory is developing a kind of probability theory by regarding operators as random variables. The mathematical basis of noncommutative probability theory comes from quantum physics, where physical observables are modeled as self-adjoint operators on a Hilbert space and the probability distribution of an observable is defined through the spectral measure evaluated by a state.

An example of noncommutative random variable is a random matrix, i.e., a matrix that has random variables as its entries. Two significant contributions have been made in an early stage of research: Wishart in 1920's applied random matrices as the estimator of the covariance matrix of iid data of normally distributed random vectors; Wigner in 1950's introduced random matrices to model the energy levels of nucleons of nuclei. Since then random matrices have discovered numerous connections and applications to other fields: connections to other mathematics include the distribution of prime numbers, integrable systems (e.g. the Peinlevé equations) [107], and random Young diagrams [36, 91, 123]; applications beyond mathematics include log-gas systems [61], wireless communications [51], quantum information [47] and deep learning [126].

Significant progress in noncommutative probability has been made in the context of operator algebras: Voiculescu initiated free probability theory in 1980's motivated by free group factors. The central idea of free probability is "free independence" for noncommutative random variables [144]. A parallelism lies between probability theory and free probability theory, and various concepts are defined accordingly, e.g., free entropy, free convolution, free central limit theorem, freely infinitely divisible distributions and free cumulants. In addition to operator algebras, free probability has found various connections and applications to other fields including random matrices, representation theory, combinatorics, complex analysis, Hopf algebras and quantum groups. In particular, through random matrices, free probability has been applied beyond mathematics to other fields, see e.g. [47, 51, 60].

Fock spaces form an important aspect of noncommutative probability from physical, probabilistic, and operator-algebraic perspectives. Numerous generalized Fock spaces have been proposed so far. In the 1990's, De Giosa, Lu and Muraki introduced monotone Fock spaces, creation and annihilation operators on them, and a monotone Brownian motion as the sum of these operators [55, 104, 114]. Muraki identified the concept of "monotone probability theory" as implicit in these operators [115, 116]. The building block is "monotone independence" for noncommutative random variables. Again, a parallelism exists between monotone probability and free or classical probability. Progress in this field has uncovered connections to various fields as well. The monotone convolution of probability measures is characterized by the composition of holomorphic self-maps of the complex upper half-plane, thus finding a connection to (holomorphic) dynamical systems. In particular, certain noncommutative stochastic processes correspond to dynamics of holomorphic functions called Loewner chains. As for random matrices, Cébron, Dahlqvist and Gabriel recently applied monotone independence to the analysis of eigenvalues of large random matrices with perturbation [40]. Monotone independence also appears in some graph product [1]. Given increasing new aspects of monotone independence, it is now an appropriate time to offer a detailed expository article to these subjects.

The structure of this exposition is as follows. Section 1 offers the definition of noncommutative probability spaces with examples, the definition of monotone independence and the calculation of the distribution of the sum and product of monotonically independent random variables.

Section 2 presents a canonical construction of monotonically independent subalgebras on the free product algebra. Some properties of monotone independence, in particular, positivity and associativity are proved.

Section 3 introduces the concept of monotone cumulants of random variables and demonstrates how to calculate the monotone cumulants from the moments of random variables. The monotone cumulants turn out to be useful for investigating the distribution of iid sums of random variables. As applications, the central limit theorem and Poisson's law of small numbers for monotonically iid random variables are established. In particular, the arcsine distribution appears in the monotone CLT instead of the normal distribution in the classical CLT.

Section 4 is concerned with tools in complex analysis. First we establish an integral formula for Nevanlinna functions, i.e., holomorphic functions on the upper half-plane taking values with nonnegative imaginary part. Then the Cauchy transform and several other transforms of probability measures on the real line are investigated.

Section 5 introduces additive/multiplicative monotone convolutions for arbitrary probability measures on the basis of complex-analytic methods. We then study various aspects of additive monotone convolution, including the support and moments, convolution semigroups and infinite divisibility.

Section 6 discusses monotone convolution hemigroups that describe the marginal distributions of noncommutative stochastic processes with monotonically independent increments (or monotone additive processes). Their relation to Loewner theory is studied in details. We derive an integral equation and an integro-differential equation satisfied by a Loewner chain.

Section 7 is devoted to constructions of monotone additive processes as operator processes on Hilbert spaces. We present two constructions: one is the sum of three kinds of operators on the monotone Fock space, which generalizes Lu and Muraki's construction of monotone Brownian motion. This construction heavily depends on the integral equation for Loewner chains developed in Section 6. We also offer quite a different construction based on certain classical Markov processes.

Lastly, Section 8 addresses applications/connections of monotone independence to random matrices and a graph product. For a large square random matrix, a small number of eigenvalues located away from the other eigenvalues are called outliers. Existence or non-existence of outliers for some random matrix models are analyzed by using monotone independence. In graph theory, there are numerous binary operations on graphs called graph products. One of them is called a comb product and is connected to monotone independence. We apply the monotone CLT to the iterated comb product graph, which allows us to estimate the number of closed paths of the graph.

Some topics that are not included but deserve to be mentioned are: a Hopf algebraic approach to cumulants; refined limit theorems that lead to deep dynamics of iteration of holomorphic self-maps; connections of monotone probability and free probability; C^*-algebras related to monotone probability. Some references to these subjects are provided in Notes 3.6, 3.6, 5.4 and 7.3 respectively. Finally, it is worth noting that the monograph [86] and the expository article [132] are also valuable sources on monotone and other notions of independence written from different perspectives.

I am grateful to Dan Voiculescu who encouraged me to write an introductory exposition of monotone probability theory. I greatly appreciate discussions with Guillaume Cébron and Katsunori Fujie, which improved my understanding of applications of monotone independence to random matrices. I would like to thank Katsunori Fujie and Wojciech Młotkowski for reading the manuscript and giving helpful comments.

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Notation

$\mathbb{N}$: the set of positive integers $\{1, 2, 3, ...\}$

$\mathbb{N}_0$: the set of nonnegative integers $\{0, 1, 2, 3, ...\}$

$[n]$: the set $\{1, 2, ..., n\}$ for any $n \in \mathbb{N}$

$\mathbf{1}_A$: the unit of a unital algebra A

$\mathrm{Sp}(a)$: the spectrum $\{z \in \mathbb{C} : z\mathbf{1}_A - a \text{ is not invertible}\}$ of an element a of a unital C^*-algebra

$\langle S \rangle$: the subalgebra generated by a subset S of an algebra

$\mathrm{C}^*\langle S \rangle$: the C^*-subalgebra generated by a subset S of a C^*-algebra

$\overrightarrow{\prod_{t\in T}} a_t$: the ordered product $a_{t_1}a_{t_2}\cdots a_{t_n}$ for a totally ordered set $T = \{t_1 < t_2 < \cdots < t_n\}$ and elements a_t of an associative algebra A; the ordered product is $\mathbf{1}_A$ if $T = \emptyset$

$\mathbb{B}(H)$: the set of bounded linear operators on a Hilbert space H

$\mathrm{M}_N(\mathbb{C})$: the set of $N \times N$ matrices with complex entries

$C(X)$: the set of $\mathbb{C}$-valued continuous functions on a topological space X

$\mathcal{B}(X)$: the set of the Borel subsets of a topological space X

$\mathbb{D}$: the complex unit disk $\{z \in \mathbb{C} : |z| < 1\}$

$\mathbb{C}^+$: the complex upper half-plane $\{z \in \mathbb{C} : \Im(z) > 0\}$

μ_a: the analytic distribution of a real random variable a; $\varphi(a^n) = \int_{\mathbb{R}} x^n \, \mu_a(dx), n \in \mathbb{N}$

G_μ: the Cauchy transform $\int_{\mathbb{R}} \frac{1}{z-t} \, \mu(dt)$

F_μ: the reciprocal Cauchy transform $1/G_\mu(z)$

ψ_μ: the moment generating function $\int_{\mathbb{R}} \frac{zt}{1-zt} \, \mu(dt) = \frac{1}{z} G_\mu\left(\frac{1}{z}\right) - 1$

η_μ: the η-transform $\frac{\psi_\mu(z)}{1+\psi_\mu(z)} = 1 - zF_\mu\left(\frac{1}{z}\right)$

arg: the argument function defined on $\mathbb{C} \setminus [0, +\infty)$ so that $\arg z \in (0, 2\pi)$

∇_γ: the sector domain $\{z \in \mathbb{C}^+ : \gamma|\Re(z)| < \Im(z)\}, \gamma > 0$

χ_B: the characteristic function of a subset B, i.e., $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ otherwise

$m_n(\mu)$: the nth moment $\int_{\mathbb{R}} t^n \, \mu(dt)$ of a Borel measure μ on $\mathbb{R}$, if it exists

$\mathrm{Var}(\mu)$: the variance of a probability measure μ on $\mathbb{R}$

$\triangle$: the set $\{(s, t) \in \mathbb{R}^2 : 0 \le s \le t < +\infty\}$

1. Monotone independence

The standard Kolmogorov's formulation of probability theory builds upon a probability space that is a triple of a set Ω, a σ-field $\mathcal{F} \subseteq 2^\Omega$ and a probability measure $\mathbb{P}$ defined as a function on $\mathcal{F}$. The idea of noncommutative probability is to shift the focus from the probability space to the function space over it, say $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. We can regard the expectation $\mathbb{E}$ as a linear functional on L^∞. The central idea is to generalize L^∞ to a possibly noncommutative algebra and $\mathbb{E}$ to a linear functional on it, which leads to the concept of noncommutative probability space. Elements of the algebra are called random variables. To contrast, the usual probability theory is often called classical probability theory, just as mechanics in physics before quantum theory is referred to as classical mechanics.

Numerous notions in probability theory can naturally be formulated in the noncommutative setting. In this section, we will consider the distribution of a random variable, independence of random variables, and the sum and product of independent random variables. A striking feature is that the notion of independence is not unique. What we focus on is the one called "monotone independence".

1.1. **Noncommutative probability spaces.** We start by collecting basic materials in algebra and functional analysis. We quote some results on C^*-algebras, which can be found e.g. in [50].

Definition 1.1. Let A be an associative algebra over $\mathbb{C}$, possibly not having a unit.

(i) A is called a $*$-algebra if there exists a map $A \ni a \mapsto a^* \in A$ that satisfies $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ for all $\lambda, \mu \in \mathbb{C}$ and all $a, b \in A$.

(ii) A is called a Banach algebra if A is a Banach space with respect to a norm $\|\cdot\|$ and the inequality $\|ab\| \leq \|a\|\|b\|$ holds for all $a, b \in A$.

(iii) A is called a C^*-algebra if A is a $*$-algebra and also a Banach algebra such that $\|a^*a\| = \|a\|^2$ holds for all $a \in A$.

For an algebra A, especially when it is non-unital, we define its **unitization** $\tilde{A} := \mathbb{C} \oplus A$ with unit $\mathbf{1}_{\tilde{A}} := (1, 0)$ and multiplication $(\lambda, a)(\mu, b) := (\lambda\mu, \lambda b + \mu a + ab)$. If A is a $*$-algebra then $\tilde{A}$ is a unital $*$-algebra with involution $(\lambda, a)^* := (\overline{\lambda}, a^*)$.

Definition 1.2. (i) A pair (A, φ) of an associative algebra A over $\mathbb{C}$ and a linear functional $\varphi \colon A \to \mathbb{C}$ is called a **noncommutative probability space** (nc-probability space for short). If A is a unital algebra and φ is a unital linear functional, i.e., $\varphi(\mathbf{1}_A) = 1$, then (A, φ) is called a unital nc-probability space.

(ii) Let A be a unital $*$-algebra. A linear functional $\varphi \colon A \to \mathbb{C}$ is called a **state** if $\varphi(\mathbf{1}_A) = 1$ and φ is positive, i.e., $\varphi(a^*a) \geq 0$ for all $a \in A$. Such a pair (A, φ) is called a **unital $*$-probability space**.

(iii) Let A be a (possibly non-unital) $*$-algebra. A linear functional $\varphi \colon A \to \mathbb{C}$ is called a **restricted state** if the extended linear functional $\tilde{\varphi} \colon \tilde{A} \to \mathbb{C}$ with $\tilde{\varphi}(\mathbf{1}_{\tilde{A}}) := 1$ is a state in the sense of (ii). A pair (A, φ) of a $*$-algebra and a restricted state on it is called a **$*$-probability space**.

(iv) Let A be a unital C^*-algebra. If a linear functional φ on A is a state in the sense of (ii) above, then the pair (A, φ) is called a **unital C^*-probability space.** It is known that φ automatically becomes continuous with norm 1.

In any setting above, an element $a \in A$ is called a **random variable** and $\varphi(a)$ is called the **expectation** of a. In a $*$-probability space, an element a with $a^* = a$ is called a **real random variable**.

Remark 1.3. (a) On a unital $*$-probability space (A, φ), φ is self-adjoint, i.e., $\varphi(a^*) = \overline{\varphi(a)}$ holds for all $a \in A$. Moreover, the Cauchy-Schwarz inequality

$$|\varphi(a^*b)| \leq \sqrt{\varphi(a^*a)}\sqrt{\varphi(b^*b)}$$

holds for all $a, b \in A$. These can be proved from the fact that the quadratic function $\mathbb{C} \ni \lambda \mapsto \varphi((a+\lambda b)^*(a+\lambda b))$ is nonnegative.

(b) On a $*$-algebra A, the mere positivity $\varphi(a^*a) \geq 0, a \in A$ does not imply that φ is a restricted state. For example, let $\mathbb{C}_0[x]$ be the $*$-algebra of polynomials without constant terms, equipped with the involution $(a_1 x + a_2 x^2 + \cdots a_n x^n)^* := \overline{a_1} x + \overline{a_2} x^2 + \cdots + \overline{a_n} x^n$. Let $\varphi \colon \mathbb{C}_0[x] \to \mathbb{C}$ be defined linearly by $\varphi(x) := \alpha$ and $\varphi(x^n) := 0$ for all $n \geq 2$. For any $\alpha \in \mathbb{C}$ the positivity condition $\varphi(a^*a) \geq 0$ holds, but φ fails to be a restricted state as soon as $\alpha \neq 0$ because $\tilde{\varphi}((\lambda 1 + x)^*(\lambda 1 + x)) = |\lambda|^2 + 2\alpha \Re(\lambda)$, which fails to be nonnegative for some $\lambda \in \mathbb{C}$.

(c) Let A be a unital $*$-algebra and $\varphi \colon A \to \mathbb{C}$ be a unital linear functional. Then φ is a state if and only if φ is a restricted state. First, it is obvious that if φ is a restricted state then it is a state. Conversely, suppose that φ is a state. Let $\mathbf{1}_{\tilde{A}}$ denote the unit of $\tilde{A}$ and $\mathbf{1}_A$ denote that of A. For the unital extension $\tilde{\varphi} \colon \tilde{A} \to \mathbb{C}$

and $a \in A$ we have

$$\tilde{\varphi}((\lambda \mathbf{1}_{\tilde{A}} + a)^*(\lambda \mathbf{1}_{\tilde{A}} + a)) = |\lambda|^2 + \lambda\varphi(a^*) + \overline{\lambda}\varphi(a) + \varphi(a^* a).$$

This is exactly the same as $\varphi((\lambda \mathbf{1}_A + a)^*(\lambda \mathbf{1}_A + a))$, which is nonnegative.

(d) We could also define non-unital C^*-probability spaces by requiring φ to be a positive continuous linear functional with norm 1, but we will not need this general setting.

Fundamental examples of noncommutative probability spaces are provided below. The first three correspond to classical probability theory.

Example 1.4. (a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathbb{E}$ denote the expectation, i.e., the linear functional $X \mapsto \int_\Omega X(\omega)\,\mathbb{P}(d\omega)$. Then the pair $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ is a unital C^*-probability space with unit χ_Ω, involution $X \mapsto X^*$ defined by $X^*(\omega) := \overline{X(\omega)}$, and the L^∞ norm $\|\cdot\|$. Alternatively, if we consider the larger space $L^{\infty-} := \bigcap_{1 \le p < +\infty} L^p(\Omega, \mathcal{F}, \mathbb{P})$, then $(L^{\infty-}, \mathbb{E})$ is a unital $*$-probability space.

(b) Let Ω be a compact topological space and $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{B}(\Omega))$. The set $C(\Omega)$ of the $\mathbb{C}$-valued continuous functions on Ω is a unital C^*-algebra equipped with the same unit and involution as above and the supremum norm $\|\cdot\|$. The pair $(C(\Omega), \mathbb{E})$ is a unital C^*-probability space.

(c) Let $\mathbb{C}[x]$ be the polynomial algebra containing the unit. Let $(\alpha_n)_{n \ge 0}$ be any sequence of complex numbers. The linear functional $\varphi\colon \mathbb{C}[x] \to \mathbb{C}$ defined by $\varphi(x^n) := \alpha_n, n \in \mathbb{N}_0$, provides a nc-probability space $(\mathbb{C}[x], \varphi)$. Moreover, we introduce an involution on $\mathbb{C}[x]$ by $(x^n)^* := x^n$ and extending it by antilinearity. Let μ be a probability measure on $\mathbb{R}$ that has finite moments of all orders. Then the linear function $\varphi_\mu\colon \mathbb{C}[x] \to \mathbb{C}$,

$$\varphi_\mu(P(x)) := \int_{\mathbb{R}} P(t)\,\mu(dt)$$

gives a unital $*$-probability space $(\mathbb{C}[x], \varphi_\mu)$.

(d) Let H be a Hilbert space and $\xi \in H$ be a unit vector, i.e., $\|\xi\| = 1$. Let $\mathbb{B}(H)$ be the C^*-algebra of bounded linear operators on H equipped with operator norm and involution being the adjoint. Let $\varphi_\xi(a) := \langle \xi, a\xi \rangle$ called the vector state. Then $(\mathbb{B}(H), \varphi_\xi)$ is a unital C^*-probability space.

(e) Let $\mathrm{M}_N(\mathbb{C})$ be the unital $*$-algebra of $N \times N$ matrices of complex numbers with involution being the conjugate transpose. Let Tr be the canonical trace, i.e., $\mathrm{Tr}(a)$ is the sum of the diagonal entries of a. Then $(\mathrm{M}_N(\mathbb{C}), \frac{1}{N}\mathrm{Tr})$ is a unital $*$-probability space. We can also make it a unital C^*-probability space by naturally identifying $\mathrm{M}_N(\mathbb{C})$ with $\mathbb{B}(\mathbb{C}^N)$.

(f) We generalize example (e) to random matrices. Let $\mathrm{M}_N(L^{\infty-})$ be the unital $*$-algebra of $N \times N$ matrices with entries in $L^{\infty-}$ defined in (a). With the natural unit $\mathbf{1} = (\delta_{i,j})_{i,j \in [N]}$ and involution $a^* := (X_{j,i}^*)_{i,j \in [N]}$ for $a = (X_{i,j})_{i,j \in [N]}$, the pair $(\mathrm{M}_N(L^{\infty-}), \frac{1}{N}\mathbb{E} \circ \mathrm{Tr})$ becomes a unital $*$-probability space.

1.2. **Distributions of random variables.** Now we come to the consideration of distributions of random variables. For an $\mathbb{R}$-valued random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the distribution of X is the probability measure μ_X on $\mathbb{R}$ defined by

$$\mu_X(B) := \mathbb{P}[\{\omega \in \Omega : X(\omega) \in B\}], \qquad B \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ is the set of the Borel subsets of $\mathbb{R}$. By the change-of-variable formula, for any bounded continuous function $f\colon \mathbb{R} \to \mathbb{C}$, we have

$$\mathbb{E}[f(X)] = \int_{\mathbb{R}} f(x)\mu_X(dx).$$

We take this formula as a starting point to define a distribution of a real random variable in the noncommutative setting.

1.2.1. *The case of C^*-probability spaces.* Let a be real random variable in a unital C^*-probability space (A, φ). Recall that the spectrum

$$\mathrm{Sp}(a) := \{z \in \mathbb{C} : z\mathbf{1}_A - a \text{ is not invertible}\}$$

is a compact subset and is contained in the interval $[-\|a\|, \|a\|]$, and a admits continuous functional calculus, i.e., there is an isometric unital $*$-homomorphism $F_a\colon C(\mathrm{Sp}(a)) \to A$ such that $F_a(P) = P(a)$ for any polynomial P, where $C(\mathrm{Sp}(a))$ is the C^*-algebra endowed with supremum norm. The notation $f(a) := F_a(f)$ is used for all $f \in C(\mathrm{Sp}(a))$.

Proposition 1.5. *Let (A, φ) be a unital C^*-probability space and $a \in A$ be a real random variable. There exists a unique probability measure μ_a on* $\mathrm{Sp}(a)$ *such that*

$$\varphi(f(a)) = \int_{\mathrm{Sp}(a)} f(x)\mu_a(dx)$$

for all continuous functions $f\colon \mathrm{Sp}(a) \to \mathbb{C}$. The probability measure μ_a is called the **distribution** *of a. Sometimes we call it the* **analytic distribution** *of a to distinguish it from the algebraic one in Definition 1.7 below.*

Proof. This is a consequence of Riesz-Markov-Kakutani's theorem applied to the continuous positive linear functional $f \mapsto \varphi(f(a))$. ☐

Since μ_a is supported on the compact set $\mathrm{Sp}(a)$, μ_a is a unique probability measure on $\mathbb{R}$ such that

$$\varphi(a^n) = \int_{\mathbb{R}} x^n \mu_a(dx), \qquad n \in \mathbb{N}, \tag{1.1}$$

see Proposition A.3.

Example 1.6. (a) Let $a \in \mathrm{M}_N(\mathbb{C})$ be Hermitian. Then $a = udu^*$ for some unitary matrix u and the real diagonal matrix $d = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_N)$ of eigenvalues of a, so that

$$\frac{1}{N}\mathrm{Tr}(a^n) = \frac{1}{N}\mathrm{Tr}(d^n) = \frac{1}{N}\sum_{i=1}^{N} \lambda_i^n = \int_{\mathbb{R}} x^n \,\mu_a(dx), \qquad n \in \mathbb{N},$$

where $\mu_a := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ is called the empirical eigenvalue distribution of a.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider the unital C^*-probability space $(\mathbb{B}(H), \varphi_{\chi_\Omega})$, where $H := L^2(\Omega, \mathcal{F}, \mathbb{P})$ and φ_{χ_Ω} is the vector state. Let X be a bounded real random variable. Let $m_X \in \mathbb{B}(H)$ denote the multiplication operator $f \mapsto Xf$ on H. Then

$$\varphi_{\chi_\Omega}(m_X^n) = \langle \chi_\Omega, m_X^n \chi_\Omega \rangle = \int_{\Omega} X(\omega)^n \,\mathbb{P}(d\omega) = \int_{\mathbb{R}} x^n \,\mu_X(dx), \qquad n \in \mathbb{N},$$

where $\mu_X(\cdot) := \mathbb{P}[X \in \cdot\,]$ is the distribution of X. Therefore, μ_{m_X} coincides with μ_X.

(c) Suppose that (A, φ) is a unital C^*-probability space and $a \in A$ is a real random variable. If φ is a homomorphism on $\langle a \rangle$ (the algebra generated by a), then the analytic distribution μ_a is the delta measure $\delta_{\varphi(a)}$, as the nth moment of μ_a equals $\varphi(a)^n$.

1.2.2. *The cases of $*$- and nc-probability spaces.* We now turn to the setting of a $*$-probability space (A, φ). Let $(\tilde{A}, \tilde{\varphi})$ be its unital extension. Let $a \in A$ be a real random variable. Then the sequence $s_n := \tilde{\varphi}(a^n), n = 0, 1, 2, ...$ is **positive semi-definite**, i.e., for every $n \in \mathbb{N}_0$ and $c_0, c_1, ..., c_n \in \mathbb{R}$ we have

$$\sum_{i,j=0}^{n} c_i c_j s_{i+j} = \tilde{\varphi}\left(\left(\sum_{i=0}^{n} c_i a^i\right)^* \left(\sum_{j=0}^{n} c_j a^j\right)\right) \geq 0.$$

This guarantees the existence of a probability measure μ_a on $\mathbb{R}$ having finite moments of all orders such that (1.1) holds (Hamburger's moment problem); see e.g. [3, Theorem 2.1.1] and [134, Theorem 3.8]. In general, however, the probability measure μ_a is not unique, see Example A.2. For this reason, we will call the sequence $(\varphi(a^n))_{n\in\mathbb{N}}$ itself the distribution of a instead of μ_a. More generally, we extend this term to elements of a nc-probability space.

Definition 1.7. Let (A, φ) be a nc-probability space. For any $a \in A$, the sequence $(\varphi(a^n))_{n\in\mathbb{N}}$ is called the **distribution** of a. Each number $\varphi(a^n)$ is called the **nth moment of a.**

Relying on this definition, we can say that a and b have an identical distribution if $\varphi(a^n) = \varphi(b^n)$ for all $n \in \mathbb{N}$. More generally, given two nc-probability spaces (A, φ) and (B, ψ), we may say that $a \in A$ and $b \in B$ have an identical distribution if $\varphi(a^n) = \psi(b^n)$ for all $n \in \mathbb{N}$.

1.2.3. *Spectral measures and analytic distributions.* For bounded self-adjoint operators on complex Hilbert spaces, we can show that the analytic distribution is the evaluation of the spectral measure by a state. Although this fact is not essential in the following, this is a core idea of noncommutative probability and of quantum physics, and so it is worth noting here. Let us first recall the concept of spectral measure.

Definition 1.8. Let a be a bounded self-adjoint operator on a complex Hilbert space H. The **spectral measure of** a is a function E_a defined on $\mathcal{B}(\mathrm{Sp}(a))$ taking values in the set of orthogonal projections on H, such that

(i) $E_a(\mathrm{Sp}(a)) = I_H$,

(ii) for disjoint sets $B_n \in \mathcal{B}(\mathrm{Sp}(a)), n \in \mathbb{N}$, the identity

$$E_a\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} E_a(B_n)$$

holds, where the infinite sum converges in the sense of strong operator topology,

(iii) it holds for all $\xi \in H$ that

$$\langle \xi, a\xi \rangle = \int_{\mathrm{Sp}(a)} x \, d\langle \xi, E_a(x)\xi \rangle,$$

where the right-hand side is the integral against the Borel measure $B \mapsto \langle \xi, E_a(B)\xi \rangle$. This formula is often written more simply as

$$a = \int_{\mathrm{Sp}(a)} x \, E_a(dx)$$

and is called the spectral decomposition of a.

It is known that a spectral measure exists; the reader is referred to e.g. [133, Theorem 5.1 and Proposition 5.10] or [50, Theorem IX, 2.2].

The spectral measure offers Borel functional calculus: for any bounded (Borel) measurable function $f \colon \mathrm{Sp}(a) \to \mathbb{C}$ we can define $f(a)$ by the **spectral integral**

$$f(a) := \int_{\mathrm{Sp}(a)} f(x) \, E_a(dx). \tag{1.2}$$

This satisfies several properties e.g., $\chi_{\mathrm{Sp}(a)}(a) = I_H, (fg)(a) = f(a)g(a)$, and

$$\langle \xi, f(a)\xi \rangle = \int_{\mathrm{Sp}(a)} f(x) \, d\langle \xi, E_a(x)\xi \rangle, \qquad \xi \in H. \tag{1.3}$$

One can also interpret that (1.3) is the definition of the spectral integral, i.e., the spectral integral $f(a)$ in (1.2) is a unique bounded operator such that (1.3) holds; recall the well known fact that $\langle \xi, b\xi \rangle = 0$ for all $\xi \in H$ implies $b = 0$ if H is a complex Hilbert space.

Proposition 1.9. *Let a be a bounded self-adjoint operator on a Hilbert space H and $\xi \in H$ be a unit vector. Let E_a be the spectral measure of a. Then the analytic distribution of a in the C^*-probability space $(\mathbb{B}(H), \varphi_\xi)$ is given by $\mu_a = \varphi_\xi \circ E_a$.*

Proof. The definition of spectral measure implies that $\mu := \varphi_\xi \circ E_a$ is a probability measure on $\mathbb{R}$. By Borel functional calculus we have

$$\varphi_\xi(a^n) = \langle \xi, a^n \xi \rangle = \int_{\mathrm{Sp}(a)} x^n \, d\langle \xi, E_a(x)\xi \rangle = \int_{\mathrm{Sp}(a)} x^n \, \mu(dx), \qquad n \in \mathbb{N}_0,$$

showing $\mu = \mu_a$. □

Remark 1.10. The spectral measure exists also for unbounded self-adjoint operators on Hilbert spaces. Then we can define the distribution of the operator to be the evaluation of the spectral measure by a state. This is a standard method in noncommutative probability to handle arbitrary probability measures on the real line.

The analytic distributions in Example 1.6 can be understood via spectral measures.

Example 1.11. In the setting of Example 1.6 (a), we identify $\mathrm{M}_N(\mathbb{C})$ with $\mathbb{B}(\mathbb{C}^N)$. Let $\lambda'_1, \lambda'_2, ..., \lambda'_M$ $(1 \le M \le N)$ be the eigenvalues of a without counting multiplicities. The spectrum of a equals the finite set

$$\mathrm{Sp}(a) = \{\lambda'_1, ..., \lambda'_M\}.$$

Let E_i be the orthogonal projection from $\mathbb{C}^N$ onto the eigenspace of the eigenvalue λ'_i, i.e.,

$$E_i \xi = \sum_{j=1}^{m_i} \langle u_{i,j}, \xi \rangle u_{i,j},$$

where $\{u_{i,1}, u_{i,2}, ..., u_{i,m_i}\}$ is an orthonormal basis of the eigenspace. We show that

$$E(B) := \sum_{i \colon \lambda'_i \in B} E_i, \qquad B \subseteq \mathrm{Sp}(a)$$

is the spectral measure of a. First, as is well known in linear algebra, the vectors $\{u_{i,j} : 1 \le j \le m_i, 1 \le i \le M\}$ form a basis of $\mathbb{C}^N$. Since every vector $\xi \in \mathbb{C}^N$ can be expressed as the linear combination

$$\xi = \sum_{i=1}^{M} \sum_{j=1}^{m_i} \langle u_{i,j}, \xi \rangle u_{i,j},$$

one can easily see that

$$E(\mathrm{Sp}(a)) = \sum_{i=1}^{M} E_i = I_{\mathbb{C}^N}.$$

Second, observing $\langle \xi, E(B)\xi\rangle = \sum_{i:\, \lambda_i' \in B} \sum_{j=1}^{m_i} |\langle \xi, u_{i,j}\rangle|^2$ we see that the Borel measure $\langle \xi, E(\cdot)\xi\rangle$ can be expressed as $\langle \xi, E(\cdot)\xi\rangle = \sum_{i=1}^{M} (\sum_{j=1}^{m_i} |\langle \xi, u_{i,j}\rangle|^2)\delta_{\lambda_i'}$. We thus obtain

$$\begin{aligned}\int_{\mathrm{Sp}(a)} x\, d\langle \xi, E(x)\xi\rangle &= \sum_{i=1}^{M}\left(\sum_{j=1}^{m_i} |\langle \xi, u_{i,j}\rangle|^2\right)\lambda_i' = \sum_{i=1}^{M}\sum_{j=1}^{m_i}\langle \xi, a u_{i,j}\rangle\langle u_{i,j}, \xi\rangle \\ &= \sum_{i=1}^{M}\sum_{j=1}^{m_i}\langle a^*\xi, u_{i,j}\rangle\langle u_{i,j}, \xi\rangle = \langle a^*\xi, \xi\rangle = \langle \xi, a\xi\rangle,\end{aligned}$$

where we used the basic relations $a u_{i,j} = \lambda_i' u_{i,j}$ and $\langle \xi, \eta\rangle = \sum_{k=1}^{N} \langle \xi, v_k\rangle\langle v_k, \eta\rangle$ for any orthonormal basis $(v_k)_{k\in[N]}$. Therefore we have verified that E is the spectral measure.

Finally, the analytic distribution μ_a in the C^*-probability space $(\mathrm{M}_N(\mathbb{C}), \frac{1}{N}\mathrm{Tr})$ can be recovered as

$$\mu_a(B) = \frac{1}{N}\mathrm{Tr}(E(B)) = \frac{1}{N}\sum_{i:\, \lambda_i' \in B} \mathrm{Tr}(E_i) = \frac{1}{N}\sum_{i:\, \lambda_i' \in B} m_i = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}(B).$$

Example 1.12. In the setting of Example 1.6 (b), observe first that the spectrum of m_X is exactly the support S of μ_X (see Section 4.1 for the definition of support). We show that the spectral measure of the multiplication operator m_X is given by

$$E(B) := m_{\chi_B(X)}, \qquad B \in \mathcal{B}(S).$$

First, it is obvious that $E(S) = I_H$ as $\mathbb{P}[X \in S] = 1$. Second, $E(B)$ is an orthogonal projection as one can easily check $E(B) = E(B)^2 = E(B)^*$. Third, for disjoint Borel subsets $B_n \in \mathcal{B}(S)$ and $\xi \in H$ we have

$$E\left(\bigcup_{n=1}^{\infty} B_n\right)\xi = \chi_{\bigcup_{n\in\mathbb{N}} B_n}(X)\xi = \sum_{n=1}^{\infty} \chi_{B_n}(X)\xi = \sum_{n=1}^{\infty} E(B_n)\xi,$$

where the second equality holds (in the L^2 sense) by the dominated convergence theorem. Finally, for $\xi \in H$, the measure $B \mapsto \langle \xi, E(B)\xi\rangle$ is given by $\mathbb{E}[\chi_B(X)|\xi|^2] = \mathbb{E}[\delta_X(B)|\xi|^2]$. If $f = \chi_B$ with $B \in \mathcal{B}(S)$, then by the definition of integral

$$\int_S f(x)\, d\langle \xi, E(x)\xi\rangle = \mathbb{E}[\chi_B(X)|\xi|^2] = \mathbb{E}[f(X)|\xi|^2] = \langle \xi, f(X)\xi\rangle.$$

For a nonnegative bounded Borel measurable function f, we approximate it by simple functions from below. By the monotone convergence theorem, the same formula $\int_S f(x)\, d\langle \xi, E(x)\xi\rangle = \langle \xi, f(X)\xi\rangle$ still holds. By linearity, the same holds for any bounded Borel measurable function f. In particular, selecting $f = \mathrm{id}$ we conclude that E is indeed the spectral measure of m_X. Note hat the above arguments actually show that $f(m_X) = m_{f(X)}$.

Using the spectral measure, we recover the analytic distribution of m_X in Example 1.6 (b)

$$\mu_{m_X}(B) = \varphi_{\chi_\Omega}(E(B)) = \langle \chi_\Omega, \chi_B(X)\chi_\Omega\rangle = \mathbb{P}[X \in B],$$

so that μ_{m_X} coincides with the distribution of X in the sense of classical probability.

1.3. **Independence of two subalgebras.** On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the independence of two σ-subfields $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ is defined by the condition

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B), \qquad A \in \mathcal{G}, B \in \mathcal{H}.$$

This is also equivalent to the condition that

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \tag{1.4}$$

for all $X \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ and $Y \in L^\infty(\Omega, \mathcal{H}, \mathbb{P})$. The latter condition can be regarded as a certain relation between the subalgebras $L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ and $L^\infty(\Omega, \mathcal{H}, \mathbb{P})$ of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

Let us generalize the independence of the form (1.4) to a nc-probability space (A, φ). It is a natural viewpoint that subalgebras of A should play the role of σ-subfields in probability theory. On the basis of this viewpoint, let us consider analogues of (1.4). Let B, C be subalgebras of A. Because random variables are allowed to be noncommuting, focusing only on the expectation $\varphi(bc), b \in B, c \in C$ is not sufficient; it is natural to discuss the more general quantities

$$\begin{aligned}&\varphi(b_1c_1b_2\cdots c_{n-1}b_n), &&\varphi(b_1c_1b_2\cdots c_{n-1}b_nc_n), \\ &\varphi(c_0b_1c_1b_2\cdots c_{n-1}b_n), &&\varphi(c_0b_1c_1b_2\cdots c_{n-1}b_nc_n), \qquad b_i \in B, c_i \in C,\end{aligned}$$

which we call **mixed moments of** B **and** C. To discuss the above four kinds of alternating words in a unified way, we consider $\varphi(c_0b_1c_1\cdots c_{n-1}b_nc_n)$ allowing $c_0 = \mathbf{1}_{\tilde{A}}$ or $c_n = \mathbf{1}_{\tilde{A}}$ in the unitized algebra $\tilde{A}$ (if A is unital then we can just take $c_0 = \mathbf{1}_A$ or $c_n = \mathbf{1}_A$). Inspired by (1.4), we consider a "universal rule" for calculating the mixed moments of B and C, and call it an independence of B and C in noncommutative probability theory.

A direct generalization of the classical stochastic independence (1.4) to a nc-probability space is as follows.

Definition 1.13. Let (A,φ) be a nc-probability space. Subalgebras $B, C \subseteq A$ are called **tensor independent** if

$$\varphi(c_0b_1c_1b_2c_2\cdots b_nc_n) = \varphi(b_1b_2\cdots b_n)\varphi(c_0c_1c_2\cdots c_n)$$

holds for all $n \geq 1$, $b_1, b_2, \dots, b_n \in B$ and $c_1, c_2, \cdots, c_{n-1} \in C$ and $c_0, c_n \in C \cup \{\mathbf{1}_{\tilde{A}}\}$. In case $n = 1$ and $c_0 = c_n = \mathbf{1}_{\tilde{A}}$ we understand $\varphi(c_0c_n) = 1$.

Two subsets $S, T \subseteq A$ are said to be tensor independent if the subalgebras $B := \langle S\rangle$ and $C := \langle T\rangle$ are tensor independent. If (A,φ) is a $*$-probability space, then two subsets $S, T \subseteq A$ are called $*$-tensor independent if the $*$-subalgebras $B := \langle b, b^* : b \in S\rangle$ and $C := \langle c, c^* : c \in T\rangle$ are tensor independent.

Example 1.14 (A canonical model for the tensor independence)**.** Let $(A_i, \varphi_i), i = 1, 2$ be two nc-probability spaces. Let $A := A_1 \otimes A_2$ and $\varphi := \varphi_1 \otimes \varphi_2$. Then the subalgebras $B := A_1 \otimes \mathbf{1}_{A_2}$ and $C := \mathbf{1}_{A_1} \otimes A_2$ are tensor independent in (A,φ). To see this, for example for $b_i = x_i \otimes \mathbf{1}_{A_2}, i = 1, 2$ and $c_i = \mathbf{1}_{A_2} \otimes y_i, i = 1, 2, 3$, we have

$$c_0b_1c_1b_2c_2 = (x_1x_2) \otimes (y_0y_1y_2)$$

and hence

$$\varphi(c_0b_1c_1b_2c_2) = \varphi_1(x_1x_2)\varphi_2(y_0y_1y_2) = \varphi(b_1b_2)\varphi(c_0c_1c_2).$$

Here we introduce monotone independence as another factorization formula for mixed moments.

Definition 1.15. Let (A,φ) be a nc-probability space. Subalgebras $B, C \subseteq A$ are called **monotonically independent** if

$$\varphi(c_0b_1c_1b_2c_2\cdots b_nc_n) = \varphi(b_1b_2\cdots b_n)\varphi(c_0)\varphi(c_1)\varphi(c_2)\cdots\varphi(c_n) \tag{1.5}$$

for all $n \geq 1$, $b_1, b_2, \dots, b_n \in B$ and $c_1, c_2, \cdots, c_{n-1} \in C$ and $c_0, c_n \in C \cup \{\mathbf{1}_{\tilde{A}}\}$. Here we understand $\varphi(\mathbf{1}_{\tilde{A}}) = 1$.

Two subsets $S, T \subseteq A$ are said to be monotonically independent if the subalgebras $B := \langle S\rangle$ and $C := \langle T\rangle$ are monotonically independent. If (A,φ) is a $*$-probability space, then subsets $S, T \subseteq A$ are said to be $*$-monotonically independent if the $*$-subalgebras $B := \langle b, b^* : b \in S\rangle$ and $C := \langle c, c^* : c \in T\rangle$ are monotonically independent.

Remark 1.16. (a) Monotone independence has an "asymmetric" nature: B and C being monotonically independent does not imply that C and B are monotonically independent.

(b) If B, C are monotonically independent and (A,φ) is unital, then one can show that B and $\langle \mathbf{1}_A, C\rangle$ are also monotonically independent, i.e., (1.5) holds for $c_0, c_1, ..., c_n \in C \cup \{\mathbf{1}_A\}$ too. However, if one takes $b_i = \mathbf{1}_A$ for some i then (1.5) holds only in trivial cases. Suppose for example that $\varphi(c_0b_1c_1) = \varphi(b_1)\varphi(c_0)\varphi(c_1)$ were the case for $b_1 = \mathbf{1}_A$. Then we would have $\varphi(c_0c_1) = \varphi(c_0)\varphi(c_1)$ for all $c_0, c_1 \in C$, i.e., φ would be a homomorphism on C. In the setting of C^*-probability spaces, this implies that any real random variable $c \in C$ has the trivial distribution $\delta_{\varphi(c)}$, see Example 1.6 (c).

(c) Remark (b) suggests that the role of the unit $\mathbf{1}_A$ is rather different from classical probability theory. Here are two more remarks on the unit. In probability theory, a constant random variable is independent of any other random variable. In monotone probability, it is easy to see that a and $\mathbf{1}_A$ are monotonically independent for any $a \in A$; however, typically $\mathbf{1}_A$ and a are not monotonically independent. Also we can observe that the monotone independence of b and c does not imply that of $b + \lambda\mathbf{1}_A$ and c for $\lambda \in \mathbb{C} \setminus \{0\}$.

(d) Suppose (A,φ) is a unital C^*-probability space. If B, C are $*$-subalgebras of A that are monotonically independent, then we can prove (1.5) for all $n \geq 1$, $b_1, b_2, \dots, b_n \in \mathrm{C}^*\langle B\rangle = \overline{B}$ and $c_0, c_1, \cdots, c_{n-1} \in \mathrm{C}^*\langle \mathbf{1}_A, C\rangle$. Moreover, when $c_0 = c_n = \mathbf{1}_A$, the left- and right-most letters b_1 and b_n are allowed to be the unit, i.e., the formula

$$\varphi(b_1c_1b_2c_2\cdots c_{n-1}b_n) = \varphi(b_1b_2\cdots b_n)\varphi(c_1)\varphi(c_2)\cdots\varphi(c_{n-1}) \tag{1.6}$$

holds for all $n \geq 2$, $b_1, b_n \in \mathrm{C}^*\langle \mathbf{1}_A, B\rangle$, $b_2, \dots, b_{n-1} \in \mathrm{C}^*\langle B\rangle$ and $c_1, c_2, \cdots, c_{n-1} \in \mathrm{C}^*\langle \mathbf{1}_A, C\rangle$.

Example 1.17 (A canonical model for monotone independence)**.** Let $H_i, i = 1, 2$ be two Hilbert spaces. We fix arbitrary unit vectors ξ_i in H_i, $i = 1, 2$. Let $H := H_1 \otimes H_2$ be the tensor product Hilbert space and $\xi := \xi_1 \otimes \xi_2$ and $p \in \mathbb{B}(H_2)$ be the orthogonal projection onto the one-dimensional subspace $\mathbb{C}\xi_2$. We consider the unital C^*-probability space (A,φ), where $A = \mathbb{B}(H)$ and $\varphi(a) := \langle \xi, a\xi\rangle, a \in A$, and the $*$-subalgebras $B := \mathbb{B}(H_1) \otimes p$ and $C := I_{H_1} \otimes \mathbb{B}(H_2)$ of A. Note that B is a $*$-subalgebra because of $p^2 = p = p^*$, and moreover, B, C are both C^*-subalgebras (i.e., closed with respect to the operator norm) because $\|x \otimes y\| = \|x\|\|y\|$ holds for all $x \in \mathbb{B}(H_1)$ and $y \in \mathbb{B}(H_2)$. Then B and C are monotonically independent in (A,φ). To see this, for example for $b_i = b_i' \otimes p, i = 1, 2$ and $c_i = I_{H_1} \otimes c_i', i = 1, 2, 3$, we have

$$c_0b_1c_1b_2c_2 = (b_1'b_2') \otimes (c_0'pc_1'pc_2')$$

and hence, with notation $\varphi_i(\cdot) = \langle \xi_i, \cdot\xi_i\rangle_{H_i}$,

$$\varphi(c_0b_1c_1b_2c_2) = \varphi_1(b_1'b_2')\varphi_2(c_0'pc_1'pc_2').$$

Note here that, since $p\xi_2 = \xi_2$, we have

$$\varphi_2(c_0'pc_1'pc_2') = \varphi_2(pc_0'pc_1'pc_2'p).$$

Straightforward calculations yield $pc'p = \varphi_2(c')p$ for any $c' \in \mathbb{B}(H_2)$. Hence, we arrive at

$$\varphi_2(c_0'pc_1'pc_2') = \varphi_2(c_0')\varphi_2(c_1')\varphi_2(c_2') = \varphi(c_0)\varphi(c_1)\varphi(c_2)$$

and finally

$$\varphi(c_0b_1c_1b_2c_2) = \varphi(b_1b_2)\varphi(c_0)\varphi(c_1)\varphi(c_2).$$

Example 1.18. A simpler example can be constructed on a single Hilbert space H equipped with a unit vector $\xi \in H$. Let p be the orthogonal projection onto $\mathbb{C}\xi$ and let $\varphi := \langle \xi, \cdot\xi\rangle$. We show that the sets $\{p\}$ and $\mathbb{B}(H)$ are monotonically independent in $(\mathbb{B}(H), \varphi)$. Since the $*$-algebra generated by p is just $\mathbb{C}p$, it suffices to compute $\varphi(a_1pa_2p\cdots pa_n)$, where $a_1, a_2, ..., a_n \in \mathbb{B}(H)$. Using the relations $pap = \varphi(a)p$, $p\xi = \xi$ and $\varphi(p^n) = 1, n \in \mathbb{N}$, we can see that

$$\varphi(a_1pa_2p\cdots pa_n) = \varphi(pa_1pa_2\cdots a_np) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)\varphi(p)$$

as desired.

1.4. **Independence of several subalgebras.** We extend the definition of tensor and monotone independence to the case of several subalgebras. For the tensor case, the definition comes from the following natural extension of Example 1.14.

Example 1.19. Let $(A_i, \varphi_i), i = 1, 2, ..., N$ be unital nc-probability spaces. Let

$$A := A_1 \otimes A_2 \otimes \cdots \otimes A_N,$$
$$\varphi := \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_N.$$

We consider the subalgebras

$$B_i := \mathbf{1}_{A_1} \otimes \mathbf{1}_{A_2} \otimes \cdots \otimes \mathbf{1}_{A_{i-1}} \otimes A_i \otimes \mathbf{1}_{A_{i+1}} \otimes \cdots \otimes \mathbf{1}_{A_N}$$

for $i = 1, 2, ..., N$. Then for any $i_1, i_2, ..., i_n \in [N]$ and $b_1 \in B_{i_1}, b_2 \in B_{i_2}, ..., b_n \in B_{i_n}$ we have

$$\varphi(b_1b_2\cdots b_n) = \varphi\left(\overrightarrow{\prod_{p:\, i_p=1}} b_p\right)\varphi\left(\overrightarrow{\prod_{p:\, i_p=2}} b_p\right)\cdots\varphi\left(\overrightarrow{\prod_{p:\, i_p=N}} b_p\right).$$

The above example can be abstracted to any family of subalgebras of any nc-probability space as follows.

Definition 1.20. Let (A, φ) be a nc-probability space. A family of subalgebras $(A_i)_{i\in I}$ of A is called **tensor independent** if for any $i_1, i_2, \ldots, i_n \in I$ and $a_1 \in A_{i_1}, a_2 \in A_{i_2}, \ldots, a_n \in A_{i_n}$, we have

$$\varphi(a_1a_2\cdots a_n) = \prod_{j\in I}\varphi\left(\overrightarrow{\prod_{p:\, i_p=j}} a_p\right).$$

Moreover, a family $(S_i)_{i\in I}$ of subsets of A is said to be tensor independent if so is $(A_i)_{i\in I}$, where $A_i = \langle S_i\rangle$ is the subalgebra generated by S_i. Independence of random variables $(x_i)_{i\in I}$ can be defined by regarding each x_i as the set of single element.

Example 1.17 can also be extended to an arbitrary number of subalgebras. Given Hilbert spaces H_i with unit vectors ξ_i $(1 \le i \le N)$, we set

$$H := H_1 \otimes H_2 \otimes \cdots \otimes H_N,$$
$$\xi := \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_N,$$

and $A := \mathbb{B}(H), \varphi(a) := \langle \xi, a\xi\rangle_H$, which form a unital C^*-probability space (A, φ). Let $p_i \in \mathbb{B}(H_i)$ be the orthogonal projection onto $\mathbb{C}\xi_i$. Then we consider the $*$-subalgebras

$$\begin{aligned}
A_1 &:= \mathbb{B}(H_1) \otimes p_2 \otimes p_3 \otimes \cdots \otimes p_N,\\
A_2 &:= I_{H_1} \otimes \mathbb{B}(H_2) \otimes p_3 \otimes \cdots \otimes p_N,\\
A_3 &:= I_{H_1} \otimes I_{H_2} \otimes \mathbb{B}(H_3) \otimes \cdots \otimes p_N,\\
&\ \ \vdots\\
A_N &:= I_{H_1} \otimes I_{H_2} \otimes \cdots \otimes I_{H_{N-1}} \otimes \mathbb{B}(H_N).
\end{aligned} \tag{1.7}$$

This operator model leads to the following definition; see also Example 1.23.

Definition 1.21. Let (A, φ) be a nc-probability space and I be a totally ordered set. A family of subalgebras $(A_i)_{i\in I}$ of A is called **monotonically independent** if for any $i_1, i_2, \ldots, i_n \in I$ and $a_1 \in A_{i_1}, a_2 \in A_{i_2}, \ldots, a_n \in A_{i_n}$, we have

$$\varphi(a_1a_2\cdots a_n) = \begin{cases}\varphi(a_k)\varphi(a_1a_2\cdots a_{k-1}a_{k+1}\cdots a_n) & \text{if } 2 \le k \le n-1, i_{k-1} < i_k > i_{k+1},\\ \varphi(a_1)\varphi(a_2a_3\cdots a_n) & \text{if } i_1 > i_2,\\ \varphi(a_n)\varphi(a_1a_2\cdots a_{n-1}) & \text{if } i_{n-1} < i_n.\end{cases}$$

Moreover, a family $(S_i)_{i\in I}$ of subsets of A is said to be monotonically independent if so is $(A_i)_{i\in I}$, where $A_i = \langle S_i\rangle$ is the subalgebra generated by S_i. If (A,φ) is a $*$-probability space, then a family $(S_i)_{i\in I}$ of subsets of A is said to be $*$-monotonically independent if the $*$-subalgebras A_i generated by S_i are monotonically independent.

Remark 1.22. This is a recursive definition of monotone independence. Applying the definition repeatedly, the mixed moment $\varphi(a_1a_2\cdots a_n)$ will eventually be of the form

$$\prod_{j=1}^{m} \varphi(a_{k_1(j)}a_{k_2(j)}...a_{k_{i(j)}(j)}),$$

where $k_1(j) < k_2(j) < \cdots < k_{i(j)}(j)$ for each j and $a_{k_1(j)}, a_{k_2(j)}, ...$ belong to a common subalgebra $A_{p(j)}$; see also Example 1.24. Although writing down the general expression is complicated, we will later do so for $I = \{1,2,3\}$ in the proof of Proposition 2.1.

Example 1.23. The sequence $(A_i)_{i=1}^N$ of $*$-subalgebras of $\mathbb{B}(H)$ defined in (1.7) is monotonically independent. To see this, let $i_1, i_2, ..., i_n \in [N], a_j \in A_{i_j}$ and suppose $k \in [n]$ be such that $i_{k-1} < i_k > i_{k+1}$. We can see $a_{k-1}a_ka_{k+1} = \varphi(a_k)a_{k-1}a_{k+1}$ because of $p_ixp_i = \varphi_i(x)p_i$ for any $x \in \mathbb{B}(H_i)$, and hence

$$\varphi(a_1a_2\cdots a_n) = \varphi(a_k)\varphi(a_1a_2\cdots a_{k-1}a_{k+1}\cdots a_n).$$

The cases $i_1 > i_2$ and $i_{n-1} < i_n$ are handled with a trick: if $i_1 > i_2$ then we may insert the projection $p = p_1 \otimes p_2 \otimes \cdots \otimes p_N$ as

$$\varphi(a_1a_2\cdots a_n) = \langle \xi, a_1a_2\cdots a_n\xi\rangle = \langle p\xi, a_1a_2\cdots a_n\xi\rangle = \langle \xi, pa_1a_2\cdots a_n\xi\rangle;$$

the last equality is because of $p = p^*$. Then we can use the operator identity $pa_1a_2 = \varphi(a_1)pa_2$ to show

$$\varphi(a_1a_2\cdots a_n) = \varphi(a_1)\varphi(a_2\cdots a_n).$$

The case $i_{n-1} < i_n$ is similar.

Example 1.24. Suppose that subsets $\{a,a'\}, \{b,b'\}, \{c,c'\}$ are monotonically independent in (A,φ). Then

$$\begin{aligned}
&\varphi(ab) = \varphi(ba) = \varphi(a)\varphi(b),\\
&\varphi(aba') = \varphi(aa')\varphi(b), \qquad \varphi(bab') = \varphi(b)\varphi(a)\varphi(b'),\\
&\varphi(abcb'c'a') = \varphi(abcb'a')\varphi(c') = \varphi(abb'a')\varphi(c)\varphi(c') = \varphi(aa')\varphi(bb')\varphi(c)\varphi(c').
\end{aligned}$$

The formula for $\varphi(bab')$ is already different from the case of tensor independence.

Proposition 1.25. *Let (A,φ) be a nc-probability space and I be a totally ordered set. Suppose that a family of subalgebras $(A_i)_{i\in I}$ of A is monotonically independent. Then the restriction of φ to the subalgebra $\langle A_i : i \in I\rangle$ is determined by $\varphi|_{A_i}, i \in I$.*

Proof. This is a consequence of Remark 1.22. □

Remark 1.26. In free probability, unital $*$- or C^*-probability spaces (A,φ) often satisfy the conditions that

- φ is faithful, i.e., $\varphi(a^*a) = 0$ implies $a = 0$,
- φ is tracial, i.e., $\varphi(ab) = \varphi(ba)$ for $a, b \in A$.

In monotone probability, however, these conditions hold only in rather trivial cases. Let a, b be monotonically independent real random variables in a unital C^*-probability space (A,φ).

(i) If φ is tracial then either $\mu_a = \delta_0$ or $\mu_b = \delta_{\varphi(a)}$. Suppose that $\mu_a \neq \delta_0$. We first observe that $\varphi((ba^2)b) = \varphi(a^2)\varphi(b)^2$, while $\varphi(b(ba^2)) = \varphi(a^2)\varphi(b^2)$. The traciality of φ therefore implies $\varphi(b^2) = \varphi(b)^2$ as $\varphi(a^2) > 0$. This means that the analytic distribution μ_b has vanishing variance, so that $\mu_b = \delta_{\varphi(b)}$.

(ii) If φ is faithful then again either $\mu_a = \delta_0$ or $\mu_b = \delta_{\varphi(b)}$. Since $a, b - \varphi(b)\mathbf{1}_A$ are monotonically independent, we may assume that $\varphi(b) = 0$ from the beginning. Because $\varphi((ab)^*(ab)) = \varphi(ba^2b) = \varphi(a^2)\varphi(b)^2 = 0$, the faithfulness of φ implies $ab = 0$. If $\mu_a \neq \delta_0$ then $\varphi(a^2) > 0$ and so $0 = \varphi(a^2b^n) = \varphi(a^2)\varphi(b^n)$, i.e., $\varphi(b^n) = 0$ for all $n \in \mathbb{N}$. This implies $\mu_b = \delta_0$.

1.5. **Additive monotone convolution.** For classically independent $\mathbb{R}$-valued random variables X, Y, the distribution of $X + Y$ is called the convolution of μ_X and μ_Y and is given by

$$(\mu_X * \mu_Y)(B) := \mu_{X+Y}(B) = \int_{\mathbb{R}^2} \chi_B(s+t)\mu_X(ds)\mu_Y(dt), \qquad B \in \mathcal{B}(\mathbb{R}).$$

It is well known that the exponential moment generating function (essentially equivalent to the characteristic function) is useful to calculate the convolution. For simplicity, assuming $X \in L^\infty$, let

$$E_X(z) := \mathbb{E}[e^{zX}] = \sum_{n\geq 0} \frac{\mathbb{E}[X^n]}{n!}z^n, \qquad z \in \mathbb{C}.$$

Due to the independence we have $E_{X+Y}(z) = \mathbb{E}[e^{zX}e^{zY}] = \mathbb{E}[e^{zX}]\mathbb{E}[e^{zY}] = E_X(z)E_Y(z)$.

Here we consider the distribution of $x+y$ when x and y are monotonically independent real random variables in a unital C^*-probability space (A,φ). Instead of the exponential moment generating function, a more useful function is the shifted moment generating function

$$M_x(z) := z\varphi((1-zx)^{-1}) = \sum_{n\geq 0}\varphi(x^n)z^{n+1}, \qquad z\in\mathbb{C},\ |z|<1/\|x\|,$$

where $1/0$ is to be interpreted as $+\infty$.

Theorem 1.27. *Let (A,φ) be a unital C^*-probability space. Suppose that $x,y\in A$ are monotonically independent. Then for all $z\in\mathbb{C}$ with $|z|<1/(\|x\|+\|y\|)$ we have $|M_y(z)|<1/\|x\|$ and*

$$M_{x+y}(z) = M_x(M_y(z)). \tag{1.8}$$

Proof. First we check that the functions $M_{x+y}(z)$ and $M_x(M_y(z))$ make sense. The assumption $|z|<1/(\|x\|+\|y\|)$ implies $|z|<1/\|x+y\|$ and $|z|<1/\|y\|$ so that $M_{x+y}(z)$ and $M_y(z)$ are well defined. Moreover, one can check $|M_y(z)|<1/\|x\|$ by using the estimate

$$\begin{aligned}|M_y(z)| &\leq \sum_{n=0}^{\infty}|\varphi(y^n)||z|^{n+1} \leq \sum_{n=0}^{\infty}\|y\|^n|z|^{n+1} = \frac{|z|}{1-\|y\||z|}\\ &< \frac{\frac{1}{\|x\|+\|y\|}}{1-\|y\|\frac{1}{\|x\|+\|y\|}} = \frac{1}{\|x\|}.\end{aligned}$$

Because x,y are noncommuting in general, the expansion of $(x+y)^n$ contains 2^n terms. The following expression is useful for us:

$$(x+y)^n = \sum_{\ell=0}^{n}\ \sum_{\substack{k_0,k_1,\dots,k_\ell\geq 0,\\ k_0+k_1+\cdots+k_\ell=n-\ell}} y^{k_0}xy^{k_1}x\cdots xy^{k_\ell}.$$

Evaluating this by φ and applying the definition of monotone independence together with Remark 1.16 (b) yields

$$\varphi((x+y)^n) = \sum_{\ell=0}^{n}\ \sum_{\substack{k_0,k_1,\dots,k_\ell\geq 0,\\ k_0+k_1+\cdots+k_\ell=n-\ell}} \varphi(x^\ell)\varphi(y^{k_0})\varphi(y^{k_1})\cdots\varphi(y^{k_\ell}). \tag{1.9}$$

Then we can proceed as follows. First we perform formal calculations and later discuss analytic issues:

$$\begin{aligned}M_{x+y}(z) &= \sum_{n\geq 0} z^{n+1}\sum_{\ell=0}^{n}\ \sum_{\substack{k_0,k_1,\dots,k_\ell\geq 0,\\ k_0+k_1+\cdots+k_\ell=n-\ell}} \varphi(x^\ell)\varphi(y^{k_0})\varphi(y^{k_1})\cdots\varphi(y^{k_\ell}) &(1.10)\\ &= \sum_{\ell\geq 0}\sum_{n=\ell}^{\infty}\ \sum_{\substack{k_0,k_1,\dots,k_\ell\geq 0,\\ k_0+k_1+\cdots+k_\ell=n-\ell}} \varphi(x^\ell)\varphi(y^{k_0})z^{k_0+1}\varphi(y^{k_1})z^{k_1+1}\cdots\varphi(y^{k_\ell})z^{k_\ell+1} &(1.11)\\ &= \sum_{\ell\geq 0}\ \sum_{k_0,k_1,\dots,k_\ell\geq 0} \varphi(x^\ell)\varphi(y^{k_0})z^{k_0+1}\varphi(y^{k_1})z^{k_1+1}\cdots\varphi(y^{k_\ell})z^{k_\ell+1} &(1.12)\\ &= \sum_{\ell\geq 0}\varphi(x^\ell)M_y(z)^{\ell+1} &(1.13)\\ &= M_x(M_y(z)). &(1.14)\end{aligned}$$

Calculations (1.11)–(1.13) can be justified with Fubini's theorem because the sum (1.12) is absolutely convergent:

$$\begin{aligned}&\sum_{\ell\geq 0}\ \sum_{k_0,k_1,\dots,k_\ell\geq 0}\left|\varphi(x^\ell)\varphi(y^{k_0})z^{k_0+1}\varphi(y^{k_1})z^{k_1+1}\cdots\varphi(y^{k_\ell})z^{k_\ell+1}\right|\\ &\leq \sum_{\ell\geq 0}\ \sum_{k_0,k_1,\dots,k_\ell\geq 0}|z|^{\ell+1}\|x\|^\ell(|z|\|y\|)^{k_0+k_1+\cdots+k_\ell}\\ &= \frac{|z|}{1-|z|\|y\|}\sum_{\ell\geq 0}\left(\frac{|z|\|x\|}{1-|z|\|y\|}\right)^\ell < +\infty.\end{aligned}$$

□

Suppose that x is a real random variable in a unital C^*-probability space. Since the moment sequence $\varphi(x^n), n=1,2,3,\ldots$ is encoded in $M_x(z)$ as the Taylor coefficients, the analytic distribution μ_x can be determined from

M_x (later we show a more straightforward formula that recovers μ_x from M_x called the Stieltjes inversion, see Proposition 4.30). Conversely, M_x can be computed from μ_x by the formula

$$M_x(z) = \int_{\mathrm{Sp}(x)} \frac{z}{1-zt}\,\mu_x(dt).$$

Thus, we can identify M_x with μ_x. Therefore, formula (1.8) gives a binary operation on the set of compactly supported probability measures, which is called **additive monotone convolution**. Later we extend additive monotone convolution to arbitrary probability measures on $\mathbb{R}$, see Theorem 5.1.

1.6. **Multiplicative monotone convolution.** In probability theory, multiplication of independent random variables is another natural operation. If X, Y are $\mathbb{R}$-valued random variables defined on a probability space, then the law μ_{XY} is called the multiplicative convolution of μ_X and μ_Y and is given by

$$\mu_{XY}(B) = \int_{\mathbb{R}^2} \chi_B(st)\mu_X(ds)\mu_Y(dt), \qquad B \in \mathcal{B}(\mathbb{R}).$$

In the particular case where X, Y are both positive, the use of $\log(XY) = \log X + \log Y$ allows us to reduce the calculation of multiplicative convolution to the additive convolution.

Here we will consider the multiplication of monotonically independent random variables. In the setting of unital C^*-probability space, for real random variables x, y, the product xy is not self-adjoint in general. To recover the self-adjointness, we consider $\sqrt{x}y\sqrt{x}$ or $\sqrt{y}x\sqrt{y}$ assuming x or y are positive. However, the result turns out to be rather trivial.

Proposition 1.28. *Let (A, φ) be a unital C^*-probability space. Let $x, y \in A$ be monotonically independent real random variables. Let $\alpha := \varphi(y)$.*

(i) *If $x \geq 0$ then $\mu_{\sqrt{x}y\sqrt{x}} = \mu_{\alpha x}$.*

(ii) *If $y \geq 0$ and $\alpha > 0$ then $\mu_{\sqrt{y}x\sqrt{y}} = (1-\beta)\delta_0 + \beta\mu_{\alpha x}$, where $\beta := \varphi(\sqrt{y})^2/\alpha$. If $\alpha = 0$ then $\mu_{\sqrt{y}x\sqrt{y}} = \delta_0$.*

Proof. (i) First note that $\mathrm{C}^*\langle x\rangle$ and $\mathrm{C}^*\langle y\rangle$ are monotonically independent, and that $\sqrt{x} \in \mathrm{C}^*\langle x\rangle$. We can therefore obtain

$$\varphi((\sqrt{x}y\sqrt{x})^n) = \varphi(\sqrt{x}yxyx\cdots y\sqrt{x}) = \varphi(\sqrt{x}x^{n-1}\sqrt{x})\varphi(y)^n = \varphi((\alpha x)^n).$$

(ii) If $\alpha = \varphi(y) > 0$ then for $n \geq 1$ we have

$$\varphi((\sqrt{y}x\sqrt{y})^n) = \varphi(\sqrt{y}xyx\cdots x\sqrt{y}) = \varphi(x^n)\varphi(\sqrt{y})^2\varphi(y)^{n-1} = \beta\varphi((\alpha x)^n).$$

Note that $\beta \leq 1$ holds by the Cauchy-Schwarz inequality. Moreover, $\beta > 0$ holds because the analytic distribution of y is supported on $[0, +\infty)$ and not equal to δ_0, so that $\varphi(\sqrt{y}) = \int_0^\infty \sqrt{t}\,\mu_y(dt) > 0$. The conclusion follows by the fact

$$\int_{\mathbb{R}} t^n((1-\beta)\delta_0 + \beta\mu_{\alpha x})(dt) = \beta\varphi((\alpha x)^n), \qquad n \geq 1.$$

If $\alpha = \varphi(y) = 0$ then the Cauchy-Schwarz inequality implies $\varphi(\sqrt{y}) = 0$, and so $\varphi((\sqrt{y}x\sqrt{y})^n) = \varphi(x^n)\varphi(\sqrt{y})^2\varphi(y)^{n-1} = 0$ for $n \geq 1$. $\square$

A more nontrivial distribution of $\sqrt{x}y\sqrt{x}$ can be obtained by assuming the monotone independence of $x - \mathbf{1}_A$ and $y - \mathbf{1}_A$; recall from Remark 1.16 that this assumption is different from the monotone independence of x and y. Currently, in the literature, this is taken as the standard definition of multiplicative monotone convolution although the definition might look strange. There are several reasons why we assume the independence of $x - \mathbf{1}_A$ and $y - \mathbf{1}_A$; one practical reason is that this is useful in a later application to random matrices, see Theorem 8.11. Another reason is that this multiplicative monotone convolution appears in free probability theory in the form of "subordination functions", see Notes 5.4.

To describe the multiplicative monotone convolution, useful transforms are the following **ψ-transform** (also called the **moment generating function**) and the **η-transform**

$$\psi_x(z) := \frac{1}{z}M_x(z) - 1 = \varphi(zx(\mathbf{1}_A - zx)^{-1}) = \sum_{n\geq 1}\varphi(x^n)z^n,$$

$$\eta_x(z) := \frac{\psi_x(z)}{1+\psi_x(z)},$$

which are holomorphic in a neighborhood of zero.

Theorem 1.29. *Let (A, φ) be a unital C^*-probability space. Let $x, y \in A$ be real random variables such that $x \geq 0$ and that $x - \mathbf{1}_A, y - \mathbf{1}_A$ are monotonically independent. Then for all $z \in \mathbb{C}$ sufficiently close to zero we have*

$$\eta_{xy}(z) = \eta_{yx}(z) = \eta_{\sqrt{x}y\sqrt{x}}(z) = \eta_x(\eta_y(z)).$$

Proof. Let $x_0 := x - \mathbf{1}_A$. Recall from Remark 1.16 (d) that $\mathrm{C}^*\langle x_0\rangle$ and $\mathrm{C}^*\langle y\rangle$ are monotonically independent. We first expand $(\sqrt{x}y\sqrt{x})^n$ as

$$\begin{aligned}(\sqrt{x}y\sqrt{x})^n &= \sqrt{x}y(x_0+\mathbf{1}_A)y(x_0+\mathbf{1}_A)\cdots(x_0+\mathbf{1}_A)y\sqrt{x}\\ &= \sum_{k=0}^{n-1}\sum_{\substack{j_1,j_2,\dots,j_{k+1}\geq 1\\ j_1+j_2+\cdots+j_{k+1}=n}} \sqrt{x}y^{j_1}x_0y^{j_2}x_0\cdots x_0y^{j_{k+1}}\sqrt{x},\end{aligned}$$

where k stands for the number of x_0's selected from $(x_0+\mathbf{1}_A)$'s and j_i-1 is the number of consecutive $\mathbf{1}_A$'s selected from the $(i-1)$th x_0 and ith x_0. Evaluating the above by φ yields

$$\varphi((\sqrt{x}y\sqrt{x})^n) = \sum_{k=0}^{n-1}\sum_{\substack{j_1,j_2,\dots,j_{k+1}\geq 1\\ j_1+j_2+\cdots+j_{k+1}=n}} \varphi(\sqrt{x}x_0^k\sqrt{x})\varphi(y^{j_1})\varphi(y^{j_2})\cdots\varphi(y^{j_{k+1}}) \tag{1.15}$$

$$= \sum_{k=0}^{n-1}\sum_{\substack{j_1,j_2,\dots,j_{k+1}\geq 1\\ j_1+j_2+\cdots+j_{k+1}=n}} \varphi(xx_0^k)\varphi(y^{j_1})\varphi(y^{j_2})\cdots\varphi(y^{j_{k+1}}). \tag{1.16}$$

When obtaining line (1.15), we applied monotone independence of the form (1.6) thanks to the fact $\sqrt{x}\in \mathrm{C}^*\langle \mathbf{1}_A, x_0\rangle$. Note also that formula (1.16) holds because x_0 and $\sqrt{x}$ commute. Formula (1.16) leads to the following:

$$\begin{aligned}\psi_{\sqrt{x}y\sqrt{x}}(z) &= \varphi(z\sqrt{x}y\sqrt{x}(\mathbf{1}_A - z\sqrt{x}y\sqrt{x})^{-1}) = \sum_{n=1}^{\infty} z^n\varphi((\sqrt{x}y\sqrt{x})^n)\\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\sum_{\substack{j_1,j_2,\dots,j_{k+1}\geq 1\\ j_1+j_2+\cdots+j_{k+1}=n}} z^n\varphi(xx_0^k)\varphi(y^{j_1})\varphi(y^{j_2})\cdots\varphi(y^{j_{k+1}})\\ &= \sum_{k=0}^{\infty}\sum_{j_1,j_2,\dots,j_{k+1}\geq 1}\varphi(xx_0^k)\varphi((zy)^{j_1})\varphi((zy)^{j_1})\cdots\varphi((zy)^{j_{k+1}}) \qquad (1.17)\\ &= \sum_{k=0}^{\infty}\varphi(xx_0^k)\psi_y(z)^{k+1} = \varphi\left(x\sum_{k=0}^{\infty}(\psi_y(z)x_0)^k\right)\psi_y(z)\\ &= \varphi\left(x(\mathbf{1}_A-\psi_y(z)x_0)^{-1}\right)\psi_y(z) = \varphi\left(x\eta_y(z)(\mathbf{1}_A-\eta_y(z)x)^{-1}\right)\\ &= \psi_x(\eta_y(z)).\end{aligned}$$

The expression in (1.17) is absolutely convergent for sufficiently small $|z|$, so that the above calculations can be justified by Fubini's theorem. The obtained formula $\psi_{\sqrt{x}y\sqrt{x}}(z) = \psi_x(\eta_y(z))$ is equivalent to the desired $\eta_{\sqrt{x}y\sqrt{x}}(z) = \eta_x(\eta_y(z))$ for small $|z|$. A slight modification of the above calculations of $\varphi((\sqrt{x}y\sqrt{x})^n)$ shows $\varphi((\sqrt{x}y\sqrt{x})^n) = \varphi((xy)^n) = \varphi((yx)^n)$. For example,

$$(xy)^n = xy(x_0+\mathbf{1}_A)y(x_0+\mathbf{1}_A)\cdots(x_0+\mathbf{1}_A)y$$

can be used to show $\varphi((\sqrt{x}y\sqrt{x})^n) = \varphi((xy)^n)$. ☐

Remark 1.30. (a) The attentive reader might have noticed that the assumption $x\geq 0$ is unnecessary to show the formulas $\eta_{xy}(z) = \eta_{yx}(z) = \eta_x(\eta_y(z))$.

(b) One could also consider $\sqrt{y}x\sqrt{y}$ by assuming $y\geq 0$, which, however, would result in a more complicated formula; see [65, Section 9] and [66, Theorem 3.18].

Analogously to additive monotone convolution, Theorem 1.29 gives rise to a binary operation on probability measures with compact support (one is required to be supported on $[0,+\infty)$ as it comes from nonnegative elements $x\geq 0$). This operation is called **multiplicative monotone convolution** and it can also be generalized to probability measures with unbounded support, see Theorem 5.5.

1.7. **Notes.** Our definition of $*$-probability space in Definition 1.2 is used e.g. by Muraki [119], Gerhold [70], Gerhold, Hasebe and Ulrich [71] and Lachs [97]. The term "restricted state" is used in [70]. It is called "strongly positive linear functional" in [97] and simply "state" in [71]. Unital C^*-probability spaces and W^*-probability spaces are widely used in free probability [120, 143]. Hora and Obata's book [86] uses the setup of unital $*$-probability spaces and calls them algebraic probability spaces.

Muraki gave an abstract definition of monotone independence in [115] that had been implicit in earlier works on creation and annihilation operators on monotone Fock spaces [55, 104, 113, 114]. The original definition was slightly different from Definition 1.21. A definition equivalent to ours was given e.g. by Franz [63]. The operator model (1.7) for monotone independence is equivalent to the model given by Muraki [115] but the original model appeared more analogous to the operator model for free independence on the free product Hilbert space.

Rank one perturbations of operators are intensively studied in mathematical physics, see e.g. [62, 136, 137]. Example 1.18 is connected to such works; the reader is referred to [75, Section 9] for further information. In general, higher-rank perturbations are not directly connected to monotone independence, but higher-rank perturbations and unitarily invariant random matrices show monotone independence asymptotically in the large size limit. This will be discussed in Section 8.

The formula for additive monotone convolution in Theorem 1.27 was given by Muraki [115]. Our proof is different and is adopted from [127, Theorem 3.2] and [83, Proposition 4.1]. The formula for multiplicative monotone convolution in Theorem 1.29 was given by Bercovici [30, Theorem 2.2] and Franz [65, Corollary 4.3]. One can also consider the multiplication of unitary elements that is omitted in this article; the interested reader is referred to [30, 65].

Attempts are being made to unify or establish connections between different notions of independence. An incomplete list of those related to monotone independence is the following: Arizmendi, Mendoza and Vazquez-Becerra introduced "BMT independence" by naturally generalizing the operator models in Examples 1.19 and 1.23 [13]; Cébron, Dahlqvist, Gabriel and Gilliers found that monotone independence arises naturally from "cyclic-monotone independence" [40] and more generally from "cyclic-conditional freeness" [42]; Cébron, Dahlqvist and Male observed monotone independence in the context of "traffic independence" that captures asymptotic features of permutation-invariant random matrices [41]; Franz observed that "conditional freeness" of Bożejko, Leinert and Speicher contains monotone independence as a special case [65]; Jekel and Liu defined "tree independence" building upon the structure of trees [89]; Hasebe constructed "conditionally monotone independence" with respect to two states [78] and a further generalization with respect to three states [77]; Mingo and Tseng showed a construction of monotone independence within the framework of "infinitesimal freeness" [109]; Skoufranis derived monotone independence from "bi-free independence" of Voiculescu [138]; Wysochanski considered "bm-independence" for subalgebras indexed by partially ordered sets [150, 151, 152].

2. Universal construction of monotone independence

In probability theory, there is a canonical way to construct independent random variables. Let $(\Omega_i, \mathcal{F}_i, P_i), i \in I$ be a family of probability spaces. We set $\Omega := \prod_{i\in I} \Omega_i$ be the product set, $\mathcal{F} := \bigotimes_{i\in I} \mathcal{F}_i$ be the product σ-field and $P := \bigotimes_{i\in I} P_i$ be the product measure. Given random variables $X_i \colon \Omega_i \to \mathbb{C}$ $(i \in I)$, we define $Y_i \colon \Omega \to \mathbb{C}$ by

$$Y_i(\omega_1, \omega_2, ...) := X_i(\omega_i).$$

Then $(Y_i)_{i\in I}$ is an independent family of random variables defined in $(\Omega, \mathcal{F}, P)$ and the distribution of Y_i coincides with the distribution of given X_i.

A natural generalization of the above construction can be given for nc-probability spaces using the tensor product of algebras, which yields tensor independence. This is exactly Example 1.19, in which the index set was a finite set $I = \{1, 2, ..., N\}$.

For monotone independence, a much bigger algebra, called the coproduct or the free product (without identification of units), is useful to define a canonical model of independent subalgebras. An advantage of free product algebra is that it has a universality property that allows us to construct other types of independence on the same algebra just by selecting different linear functionals, see Notes 2.4.

2.1. **Free product of algebras.** Let $(A_i)_{i\in I}$ be a family of algebras. Let A be the vector space over $\mathbb{C}$ defined by the algebraic direct sum

$$A = \bigoplus_{\substack{n\in\mathbb{N}, i_1,\dots,i_n\in I,\\ i_1\neq i_2, i_2\neq i_3,\dots,i_{n-1}\neq i_n}} A_{i_1,i_2,\dots,i_n},$$

where $A_{i_1,i_2,\dots,i_n} := A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_n}$. In this vector space, we define a multiplication called the concatenation: for $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A_{i_1,i_2,\dots,i_n}$ and $b_1 \otimes b_2 \otimes \cdots \otimes b_m \in A_{j_1,j_2,\dots,j_m}$,

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n)(b_1 \otimes b_2 \otimes \cdots \otimes b_m) := \begin{cases} a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_m, & \text{if } i_n \neq j_1,\\ a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes (a_n b_1) \otimes b_2 \otimes \cdots \otimes b_m, & \text{if } i_n = j_1 \end{cases}$$

and then extend this definition to A by bilinearity. With this multiplication, simple tensors can be interpreted just as the multiplication of letters, so that, for example, we may simply write $a_1 a_2 \cdots a_n$ for $a_1 \otimes a_2 \otimes \cdots \otimes a_n$. This multiplication is associative and A becomes an algebra, which is denoted by $\bigsqcup_{i\in I} A_i$ and is called the **free product** or coproduct (see below). The algebra A contains each A_i as a direct summand, so that we can naturally interpret each A_i as a subalgebra of A.

The free product has a universality. Consider a family of algebras $(A_i)_{i\in I}$. An algebra A together with a family of homomorphisms $f_i \colon A_i \to A, i \in I$, is called a coproduct of $(A_i)_{i\in I}$ if for any family of homomorphisms g_i from A_i into an algebra B, $i \in I$, there exists a unique homomorphism $h \colon A \to B$ such that $h \circ f_i = g_i, i \in I$. A coproduct is unique up to isomorphisms. In fact, the free product $\bigsqcup_{i\in I} A_i$ together with the natural embeddings $\iota_i\colon A_i \to \bigsqcup_{i\in I} A_i$, satisfies the universality and hence is a coproduct.

2.2. **Monotone product of nc-probability spaces.** Given a family of nc-probability spaces $(A_i, \varphi_i)_{i\in I}$, where I is a totally ordered set, we set $A := \bigsqcup_{i\in I} A_i$. We aim to define a linear functional φ on A such that the subalgebras $(A_i)_{i\in I}$ are monotonically independent in (A, φ). We start from the case $I = \{1, 2\}$. The free product is then simpler:

$$A_1 \sqcup A_2 = \bigoplus_{n=1}^{\infty} \left[(\underbrace{A_1 \otimes A_2 \otimes A_1 \otimes \cdots}_{\text{length } n}) \oplus (\underbrace{A_2 \otimes A_1 \otimes A_2 \otimes \cdots}_{\text{length } n}) \right].$$

An advantage of the free product is that we can simply define φ to be the right hand side of (1.5), i.e., for any $i_1, i_2, ..., i_n \in \{1, 2\}$ with $i_1 \neq i_2, i_2 \neq i_3, ..., i_{n-1} \neq i_n$ and any $a_1a_2 \cdots a_n \in A_{i_1,i_2,...,i_n}$, we set

$$\varphi(a_1a_2 \cdots a_n) := \varphi_1 \left(\overrightarrow{\prod_{k\colon a_k\in A_1}} a_k \right) \prod_{k\colon a_k\in A_2} \varphi_2(a_k). \tag{2.1}$$

Since the right hand side of (2.1) is a multilinear functional on $A_{i_1} \times A_{i_2} \times \cdots \times A_{i_n}$, it makes sense as a definition by the universality of tensor product of vector spaces. We denote the above construction as

$$(A, \varphi) = (A_1, \varphi_1) \rhd (A_2, \varphi_2) = (A_1 \sqcup A_2, \varphi_1 \rhd \varphi_2)$$

and call it the **monotone product** of (A_1, φ_1) and (A_2, φ_2).

The monotone product has certain associativity. For three nc-probability spaces $(A_i, \varphi_i), i = 1, 2, 3$, there is a natural isomorphism

$$\Psi\colon (A_1 \sqcup A_2) \sqcup A_3 \simeq A_1 \sqcup (A_2 \sqcup A_3).$$

The isomorphism is defined by the natural rearrangement of the tensor components so that the resulting element belongs to the target space. For example, if $a_1a_2a_3a_4a_5 \in A_{2,3,2,1,3}$, then $a_1 \otimes a_2 \otimes (a_3a_4) \otimes a_5$ is an element of $(A_1 \sqcup A_2) \sqcup A_3$, where a_3a_4 stands for the multiplication in $A_1 \sqcup A_2$, while $\otimes$ is the multiplication in $(A_1 \sqcup A_2) \sqcup A_3$. Then

$$\Psi(a_1 \otimes a_2 \otimes (a_3a_4) \otimes a_5) := (a_1a_2a_3) \otimes a_4 \otimes a_5 \in A_1 \sqcup (A_2 \sqcup A_3).$$

Omitting parentheses and Ψ, we simply write $a_1a_2 \cdots a_n \in (A_1 \sqcup A_2) \sqcup A_3$ or $a_1a_2 \cdots a_n \in A_1 \sqcup (A_2 \sqcup A_3)$, which usually does not cause any confusion because the appropriate arrangement of parentheses is uniquely determined.

Proposition 2.1. *For three nc-probability spaces $(A_i, \varphi_i), i = 1, 2, 3$, we have*

$$(\varphi_1 \rhd \varphi_2) \rhd \varphi_3 = (\varphi_1 \rhd (\varphi_2 \rhd \varphi_3)) \circ \Psi.$$

Proof. Let $i_1, i_2, ..., i_n \in \{1, 2, 3\}$ with $i_1 \neq i_2, i_2 \neq i_3, ..., i_{n-1} \neq i_n$ and $a_1a_2 \cdots a_n \in A_{i_1,i_2,...,i_n}$. For a subset $J \subseteq \{1, 2, 3\}$, let $S_J \subseteq \{1, 2, \dots, n\}$ be defined by $S_J := \{p : i_p \in J\}$. Furthermore, we decompose $S_{2,3}$ into maximal intervals $T_1, T_2, ..., T_r$ of $\{1, 2, ..., n\}$. For example, if $(i_1, i_2, i_3, i_4, i_5, i_6, i_7) = (1, 3, 2, 3, 1, 3, 2)$, then $S_1 = \{1, 5\}$ and $S_{2,3} = \{2, 3, 4, 6, 7\}$, and $S_{2,3}$ is decomposed into $T_1 = \{2, 3, 4\}$ and $T_2 = \{6, 7\}$. By the definition of $\rhd$ we have

$$\begin{aligned}
&(\varphi_1 \rhd (\varphi_2 \rhd \varphi_3))(a_1a_2 \cdots a_n) \\
&= \varphi_1 \left(\overrightarrow{\prod_{p\in S_1}} a_p \right) (\varphi_2 \rhd \varphi_3) \left(\overrightarrow{\prod_{p\in T_1}} a_p \right) \cdots (\varphi_2 \rhd \varphi_3) \left(\overrightarrow{\prod_{p\in T_r}} a_p \right) \\
&= \varphi_1 \left(\overrightarrow{\prod_{p\in S_1}} a_p \right) \varphi_2 \left(\overrightarrow{\prod_{p\in T_1, i_p=2}} a_p \right) \cdots \varphi_2 \left(\overrightarrow{\prod_{p\in T_r, i_p=2}} a_p \right) \prod_{p\in S_3} \varphi_3(a_p),
\end{aligned}$$

where $\varphi_2 \left(\overrightarrow{\prod}_{p\in T_j, i_p=2} a_p \right)$ is set to be 1 if the product range for p is empty. On the other hand, we have

$$((\varphi_1 \rhd \varphi_2) \rhd \varphi_3)(a_1a_2 \cdots a_n) = (\varphi_1 \rhd \varphi_2) \left(\overrightarrow{\prod_{p\in S_{1,2}}} a_p \right) \prod_{p\in S_3} \varphi_3(a_p).$$

To compute the factor $(\varphi_1 \rhd \varphi_2) \left(\overrightarrow{\prod}_{p\in S_{1,2}} a_p \right)$, we decompose $S_{1,2}$ into S_1 and $S' := \{p \in S_{1,2} : i_p = 2\}$. Further, we decompose S' into maximal intervals of S' (not of $\{1, 2, ..., n\}$), which are exactly $T_1 \cap S_2, T_2 \cap S_2, ..., T_r \cap S_2$, so we are done. ∎

With associativity in hand, we generalize the definition of the monotone product to an arbitrary totally ordered finite set I. We may assume that $I = \{1, 2, ..., N\}$. Let $(A_i, \varphi_i)_{i\in I}$ be a family of nc-probability spaces. We can identify

$$\bigsqcup_{i=1}^{N} A_i \simeq (\cdots(((A_1 \sqcup A_2) \sqcup A_3) \sqcup \cdots) \sqcup A_N, \tag{2.2}$$

where the isomorphism is defined similarly to Ψ; it is just a suitable rearrangement of letters of words. On the right hand side of (2.2) we can define the linear function

$$(\cdots(((\varphi_1 \rhd \varphi_2) \rhd \varphi_3) \rhd \cdots) \rhd \varphi_N, \tag{2.3}$$

which induces a linear functional φ on $\bigsqcup_{i=1}^{N} A_i$ via the isomorphism. The associativity guarantees that the definition of φ does not change if we select another way of adding parentheses in (2.2). This definition of φ means that, when we compute $\varphi(a_1a_2\cdots a_n)$ for $a_1a_2...a_n \in A_{i_1,i_2,...,i_n}$, $i_1, i_2, ..., i_n \in [N]$ with $i_1 \neq i_2, i_2 \neq i_3, ..., i_{n-1} \neq i_n$, we first factor out

$$\prod_{p:\, i_p=N} \varphi_N(a_p),$$

and then repeat the same procedure for the rest $\varphi\left(\overrightarrow{\prod}_{p:\, 1\le i_p\le N-1} a_p\right)$ with N replaced with $N-1$, and so on until the factor $\varphi_1\left(\overrightarrow{\prod}_{p:\, i_p=1} a_p\right)$ appears.

Finally, we extend the definition of the monotone product to a possibly infinite totally ordered set I. For this purpose, it suffices to define φ on each direct summand $A_{i_1,i_2,...,i_n}$; then the definition can be extended by linearity to $A := \bigsqcup_{i\in I} A_i$. This is doable since $A_{i_1,i_2,...,i_n}$ can be regarded as a subspace of the free product $\bigsqcup_{j\in J} A_j$, where $J := \{i_1, i_2, ..., i_n\} \subseteq I$ is a finite totally ordered set. We denote this construction as

$$(A, \varphi) = \mathop{\rhd}_{i\in I}(A_i, \varphi_i) = \left(\bigsqcup_{i\in I} A_i, \mathop{\rhd}_{i\in I} \varphi_i\right)$$

and call it the **monotone product** of $(A_i, \varphi_i)_{i\in I}$. We also call φ the monotone product of $(\varphi_i)_{i\in I}$.

The associativity of the monotone product can be stated in a more general way as follows. First, for a family of algebras $(A_i)_{i\in I}$ and a disjoint decomposition $I = J \cup K$, we denote the natural isomorphism as

$$\Phi_{J,K} \colon \bigsqcup_{i\in I} A_i \simeq \left(\bigsqcup_{j\in J} A_j\right) \sqcup \left(\bigsqcup_{k\in K} A_k\right).$$

The definition of $\Phi_{J,K}$ is similar to Ψ and is omitted.

Proposition 2.2. *Suppose that a totally ordered set I decomposes as $I = J\cup K$, where J, K are nonempty disjoint subsets of I such that $j < k$ for all $j \in J, k \in K$. For any family of nc-probability spaces $(A_i, \varphi_i)_{i\in I}$ we have*

$$\mathop{\rhd}_{i\in I} \varphi_i = \left[\left(\mathop{\rhd}_{j\in J} \varphi_j\right) \rhd \left(\mathop{\rhd}_{k\in K} \varphi_k\right)\right] \circ \Phi_{J,K}.$$

Proof. This is a direct consequence of Proposition 2.1. More precisely, it suffices to consider the case of finite totally ordered set I because the definition of the infinite case is based on the finite case. Then the desired identity is just a rearrangement of parentheses, which can be justified by iterative use of Proposition 2.1. $\square$

In our definition (2.3) of the monotone product, we first factored out $\varphi_{i_p}(a_p)$ for all p for which i_p has the largest value among $i_1, i_2, ..., i_n$. Actually, we can factor out $\varphi(a_p)$'s when i_p is just a local maximum.

Proposition 2.3. *Let $(A_i, \varphi_i)_{i\in I}$ be a family of nc-probability spaces where I is a totally ordered set. Let (A, φ) be the monotone product of $(A_i, \varphi_i)_{i\in I}$. For any $i_1, i_2, ..., i_n \in I$ and $(a_1, a_2, ..., a_n) \in A_{i_1} \times A_{i_2} \times \cdots \times A_{i_n}$, we have*

$$\varphi(a_1a_2\cdots a_n) = \begin{cases} \varphi_{i_\ell}(a_\ell)\varphi(a_1\cdots a_{\ell-1}a_{\ell+1}\cdots a_n) & \text{if } 2 \le \ell \le n-1, i_{\ell-1} < i_\ell > i_{\ell+1}, \\ \varphi_{i_1}(a_1)\varphi(a_2a_3\cdots a_n) & \text{if } i_1 > i_2, \\ \varphi_{i_n}(a_n)\varphi(a_1a_2\cdots a_{n-1}) & \text{if } i_{n-1} < i_n. \end{cases} \tag{2.4}$$

Proof. We fix $1 \le \ell \le n$ such that $i_{\ell-1} < i_\ell > i_{\ell+1}$ (when $\ell = 1$ or n only one of the inequalities is considered) and set $m := i_\ell$. We decompose I into J, K, where $J := \{i \in I : i < m\}$ and $K := \{i \in I : i \ge m\}$. There is a natural isomorphism

$$A \simeq \left(\bigsqcup_{j\in J} A_j\right) \sqcup \left(\bigsqcup_{k\in K} A_k\right).$$

The linear functional φ on A induces a linear functional on the right hand side, which is exactly

$$\left(\mathop{\triangleright}_{j\in J}\varphi_j\right)\sqcup\left(\mathop{\triangleright}_{k\in K}\varphi_k\right)$$

by the associativity of the monotone product. This means that

$$\varphi(a_1a_2\cdots a_n)=\varphi\left(\overrightarrow{\prod_{p\colon i_p\in J}} a_p\right)\varphi\left(\overrightarrow{\prod_{p\in T_1}} a_p\right)\varphi\left(\overrightarrow{\prod_{p\in T_2}} a_p\right)\cdots\varphi\left(\overrightarrow{\prod_{p\in T_r}} a_p\right),$$

where T_j are the maximal intervals of $[n]$ such that $\min T_j>1$ implies $i_{\min T_j-1}\in J$ and $\max T_j<n$ implies $i_{\max T_j+1}\in J$. By the assumption $i_{\ell-1}<i_\ell>i_{\ell+1}$, some T_j is the singleton $\{\ell\}$. Factoring out this $\varphi(a_\ell)$ does not affect the other factorizations, so we get the conclusion. □

Comparing Proposition 2.3 and Definition 1.21, together with the fact that φ and φ_i coincide on A_i, yields the following.

Corollary 2.4. *Let $(A_i,\varphi_i)_{i\in I}$ be a family of nc-probability spaces with I a totally ordered set. Let (A,φ) be the monotone product of $(A_i,\varphi_i)_{i\in I}$. Then the family of subalgebras $(A_i)_{i\in I}$ is monotonically independent in (A,φ).*

The associativity of monotone product will be later used in the following form.

Corollary 2.5. *Let I be a totally ordered set and J,K be its nonempty disjoint subsets such that $I=J\cup K$ and $j<k$ for all $j\in J,k\in K$. Let (A,φ) be a nc-probability space and $(A_i)_{i\in I}$ be a family of monotonically independent subalgebras of A. Then the two subsets $\bigcup_{j\in J}A_j$ and $\bigcup_{k\in K}A_k$ are monotonically independent.*

Proof. We want to show that $B_1:=\langle A_j : j\in J\rangle$ and $B_2:=\langle A_k : k\in K\rangle$ are monotonically independent. For this, we refer to a universal space. Let $(\hat{A},\hat{\varphi})$ be the monotone product of $(A_i,\varphi|_{A_i})_{i\in I}$. By Proposition 2.2, the subalgebras $\bigsqcup_{j\in J}A_J$ and $\bigsqcup_{k\in K}A_i$ are monotonically independent in $(\hat{A},\hat{\varphi})$. With a slight abuse of notation, we have $\varphi|_{A_i}=\hat{\varphi}|_{A_i}$ for all $i\in I$. Since $(A_i)_{i\in I}$ is monotonically independent in both (A,φ) and $(\hat{A},\hat{\varphi})$, by Proposition 1.25, we have $\varphi(b_1b_2\cdots b_n)=\hat{\varphi}(b_1b_2\cdots b_n)$ for $b_p\in B_1\cup B_2$. Therefore,

$$\begin{aligned}\varphi(b_1b_2\cdots b_n)&=\hat{\varphi}(b_1b_2\cdots b_n)\\&=\hat{\varphi}\left(\overrightarrow{\prod_{p\colon b_p\in B_1}} b_p\right)\prod_{p\colon b_p\in B_2}\hat{\varphi}(b_p)=\varphi\left(\overrightarrow{\prod_{p\colon b_p\in B_1}} b_p\right)\prod_{p\colon b_p\in B_2}\varphi(b_p),\end{aligned}$$

showing that B_1,B_2 are monotonically independent. □

2.3. **Monotone product of $*$-probability spaces.** If $(A_i)_{i\in I}$ is a family of $*$-algebras, the free product $\bigsqcup_{i\in I}A_i$ also becomes a $*$-algebra with involution defined by

$$(a_1a_2\cdots a_n)^*:=a_n^*a_{n-1}^*\cdots a_1^*$$

and extended by antilinearity to the whole algebra. The following proposition shows that the monotone product preserves restricted states. Only in this section we say that a linear operator a on a pre-Hilbert space H is adjointable if there is a linear operator a^* on H such that $\langle a\xi,\eta\rangle=\langle\xi,a^*\eta\rangle$ for all $\xi,\eta\in H$. We introduce the notation

$$\mathbb{L}(H):=\{a\colon H\to H \mid \text{linear and adjointable}\},$$

which forms a unital $*$-algebra.

Proposition 2.6. *Let $(A_i,\varphi_i)_{i\in I}$ be a family of $*$-probability spaces where I is a totally ordered set. Then $\triangleright_{i\in I}(A_i,\varphi_i)$ is also a $*$-probability space.*

Proof. Let $(A,\varphi):=\triangleright_{i\in I}(A_i,\varphi_i)$ and $\tilde{\varphi}$ be the unital extension of $\triangleright_{i\in I}\varphi_i$ to $\tilde{A}:=\mathbb{C}\oplus A$. What has to be shown is the positivity $\tilde{\varphi}(a^*a)\ge 0$ for each $a\in\tilde{A}$. As a is a (finite) linear combination of elements of $\mathbb{C}$ and $A_{i_1,i_2,\dots,i_n}$, only finitely many A_i's are involved. Therefore, we can assume below that I is a finite set and $I=\{1,2,3,...,N\}$.

Let $\tilde{A}_i$ be the unitization of A_i and $\tilde{\varphi}_i\colon\tilde{A}_i\to\mathbb{C}$ be the unital extension of φ_i, which is positive. We take a triplet (π_i,H_i,ξ_i) consisting of a $*$-representation $\pi_i\colon A_i\to\mathbb{L}(H_i)$, a pre-Hilbert space H_i, and a unit vector $\xi_i\in H_i$ such that $\varphi_i(a)=\langle\xi_i,a\xi_i\rangle$ for all $a\in A_i$. Note that such a triplet exists by restricting the (algebraic) GNS-construction for $\tilde{A}_i$ onto A_i, see [86, Theorem 1.19] for the GNS-construction. Let $p_i\colon H_i\to\mathbb{C}\xi_i$ be the rank-one projection, $H:=H_1\otimes H_2\otimes\cdots\otimes H_N,\xi:=\xi_1\otimes\xi_2\otimes\cdots\otimes\xi_N$ as in Example 1.23, and $\lambda_i\colon\mathbb{L}(H_i)\to\mathbb{L}(H)$ be defined by

$$\lambda_i(x_i):=I_{H_1}\otimes\cdots\otimes I_{H_{i-1}}\otimes x_i\otimes p_{i+1}\otimes\cdots\otimes p_N,$$

which is a $*$-homomorphism. Then we define a $*$-representation $\tilde{\pi}\colon\tilde{A}\to\mathbb{L}(H)$ by

$$\begin{aligned}&\tilde{\pi}(\mathbf{1}_{\tilde{A}}):=I_H,\\&\tilde{\pi}(a_1a_2\cdots a_n):=\lambda_{i_1}(\pi_{i_1}(a_1))\lambda_{i_2}(\pi_{i_2}(a_2))\cdots\lambda_{i_n}(\pi_{i_n}(a)),\qquad a_1a_2\cdots a_n\in A_{i_1,i_2,\dots,i_n}.\end{aligned}$$

We show the formula

$$\tilde{\varphi}(a) = \langle \xi, \tilde{\pi}(a)\xi \rangle_H, \qquad a \in \tilde{A}. \tag{2.5}$$

This is obvious for $a = \mathbf{1}_{\tilde{A}}$ and for $a \in A_i$. For $a = a_1 a_2 \cdots a_n$, when computing $\varphi(a_1 a_2 \cdots a_n)$ we can use the monotone independence of $(A_i)_{i\in I}$ with respect to φ. On the other hand, when computing $\langle \xi, \pi(a_1 a_2 \cdots a_n)\xi\rangle = \langle \xi, b_1 b_2 \cdots b_n \xi\rangle$ where $b_k := \lambda_{i_k}(\pi_{i_k}(a_k))$, we can also use monotone independence of $B_i := \lambda_i(\mathbb{L}(H_i)) \subseteq \mathbb{L}(H)$ shown in Example 1.23. This fact and Proposition 1.25 yield (2.5) on A. Finally, formula (2.5) implies the positivity of $\tilde{\varphi}$ because

$$\tilde{\varphi}(a^* a) = \langle \xi, \tilde{\pi}(a^* a)\xi\rangle = \langle \xi, \tilde{\pi}(a)^*\tilde{\pi}(a)\xi\rangle = \langle \tilde{\pi}(a)\xi, \tilde{\pi}(a)\xi\rangle \geq 0, \qquad a \in \tilde{A}. \qquad \square$$

2.4. **Notes.** The associativity of monotone independence is addressed in [63] in the setting of $*$-probability spaces, in which the proof was based on the operator model in Example 1.23. In order to handle the monotone product of nc-probability spaces, we adopted a more combinatorial proof of Proposition 2.1. The proof of positivity in Proposition 2.6 is similar to the case of free product of states, see e.g. [143, Definition 1.5.4].

Given nc-probability spaces (A_1, φ_1) and (A_2, φ_2), there are other four kinds of definitions of a linear functional on the free product $A_1 \sqcup A_2$ that yield "good" independences [118, Definitions 2.2, 2.3]. One can check that they all satisfy the associativity.

(a) The antimonotone product

$$(\varphi_1 \lhd \varphi_2)(a_1 a_2 \cdots a_n) := \left[\prod_{k\colon a_k \in A_1} \varphi_1(a_k)\right] \varphi_2\left(\overrightarrow{\prod_{k\colon a_k\in A_2}} a_k\right),$$

which is just the flip of the monotone product and is essentially the same.

(b) The tensor product

$$(\varphi_1 \otimes \varphi_2)(a_1 a_2 \cdots a_n) := \varphi_1\left(\overrightarrow{\prod_{k\colon a_k\in A_1}} a_k\right) \varphi_2\left(\overrightarrow{\prod_{k\colon a_k\in A_2}} a_k\right).$$

(c) The Boolean product

$$(\varphi_1 \diamond \varphi_2)(a_1 a_2 \cdots a_n) := \prod_{k\colon a_k\in A_1} \varphi_1(a_k) \prod_{k\colon a_k \in A_2} \varphi_2(a_k).$$

(d) The last one is called the free product and its definition is of different flavour. First we consider the unitizations $\tilde{A}_i := \mathbb{C}\oplus A_i$ that naturally embed into $\tilde{A} := \mathbb{C}\oplus(A_1 \sqcup A_2)$. We define φ on $\tilde{A}$ by requiring that $\varphi(a_1 a_2 \cdots a_n) = 0$ whenever $(a_1, a_2, ..., a_n) \in \tilde{A}_{i_1} \times \tilde{A}_{i_2} \times \cdots \times \tilde{A}_{i_n}$, $i_1 \neq i_2, i_2 \neq i_3, ..., i_{n-1} \neq i_n$ and $\varphi(a_k) = 0$ for all $k \in [n]$. The free product $\varphi_1 * \varphi_2$ on $A_1 \sqcup A_2$ is defined as the restriction $\varphi|_{A_1 \sqcup A_2}$.

The universal constructions of independence on the free product algebra provide an appropriate framework for a classification program of independences. Speicher [140] and then Ben Ghorbal and Schürmann formulated a "good independence" as an associative product of linear functionals on the free product algebra with some conditions, and then classified them into tensor, free and Boolean [27]. Muraki dropped one assumption of Ben Ghorbal and Schürmann, and as a result, the classification list contained two more independences: monotone and antimonotone [118]. Muraki [119], Gerhold and Lachs [72] gave further results in this direction. As a closely related problem, Gerhold, Hasebe, Ulrich axiomatized a "good operator model" that contains the ones in Examples 1.14 and 1.17, and classified them [71].

3. Monotone cumulants

Cumulants are equivalents of moments and sometimes provide a clearcut description of random variables. In particular, (normalized) cumulants up to order four are called mean, variance, skewness and kurtosis, and are used in statistics. In probability theory, the characteristic function (or the Fourier transform) is often more powerful than cumulants because cumulants require that random variables have finite moments, while the characteristic function does not. However, in noncommutative probability theory, cumulants are quite useful because the theory substantially builds upon moments. In free probability theory, Voiculescu introduced single-variate free cumulants [144] and then Speicher defined multivariate free cumulants [139] that have discovered a wide range of applications so far.

In classical probability theory, (single-variate) cumulants are quantities that satisfy the following axioms.

(C1) There are universal polynomials $\tilde{P}_n(t_1, t_2, ..., t_{n-1}), n \geq 1$ with $\tilde{P}_1 := 0$ such that $C_n(X) = \mathbb{E}[X^n] + \tilde{P}_n(\mathbb{E}[X], \mathbb{E}[X^2], ..., \mathbb{E}[X^{n-1}])$ for all $n \geq 1$, $X \in L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})$ and any probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (Polynomiality)

(C2) $C_n(\lambda X) = \lambda^n C_n(X)$ for all $n \geq 1$ and $X \in L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})$. (Homogeneity)

(C3) If $X, Y \in L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})$ are independent then $C_n(X+Y) = C_n(X) + C_n(Y)$. (Additivity)

Remark 3.1. In (C1), by "universal" we emphasize that $\tilde{P}_n$ does not depend on X or the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Axiom (C1) is equivalent to the following reverted form of moments in terms of cumulants: there are universal polynomials $\tilde{Q}_n(t_1, t_2, ..., t_{n-1}), n \geq 1$ with $\tilde{Q}_1 := 0$ such that $\mathbb{E}[X^n] = C_n(X) + \tilde{Q}_n(C_1(X), C_2(X), ..., C_{n-1}(X))$ for all $n \geq 1$ and $X \in L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})$ and any probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We can give a construction of C_n as the coefficients of the logarithm of exponential moment generating function:

$$\log \mathbb{E}[e^{zX}] = \log\left(\sum_{n=0}^{\infty} \frac{\mathbb{E}[X^n]}{n!} z^n\right) = \sum_{n=1}^{\infty} \frac{C_n(X)}{n!} z^n.$$

Note that under the assumption $X \in L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})$, the above series might have convergence radius zero; then the above equalities can be interpreted as formal power series.

Our objective is to discover a monotone counterpart of cumulants, which we denote by κ_n. The most natural definition would be to replace the pair $(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ with a nc-probability space (A, φ), and replace the independence assumption in (C3) with monotone independence. However, the third condition would contradict the asymmetry of monotone independence. More precisely, suppose that x, y are monotonically independent real random variables in a unital C^*-probability space. Then $\kappa_n(x+y) = \kappa_n(x) + \kappa_n(y)$ does not depend on whether we assume x, y are monotonically independent or y, x are. However, monotonic independence of x, y implies $M_{x+y}(z) = M_x(M_y(z))$ and monotonic independence of y, x implies $M_{x+y}(z) = M_{y+x}(z) = M_y(M_x(z))$, and therefore, the distribution of $x+y$ is typically different if we switch the independence assumption.

Here we propose a weaker version of additivity, which we call extensivity because of its resemblance to the corresponding notion in thermodynamics.

Definition 3.2. A rule that associates with each nc-probability space (A, φ) and random variable $x \in A$ a sequence of complex numbers $\{\kappa_n(x)\}_{n \geq 1}$ is called **monotone cumulants** if

(M1) There are universal polynomials $P_n(t_1, t_2, ..., t_{n-1}), n \geq 1$ with $P_1 := 0$ such that $\kappa_n(x) = \varphi(x^n) + P_n(\varphi(x), \varphi(x^2), ..., \varphi(x^{n-1}))$ for all $n \geq 1$, $x \in A$ and any (A, φ). (Polynomiality)

(M2) $\kappa_n(\lambda x) = \lambda^n \kappa_n(x)$ for all $n \geq 1$, $\lambda \in \mathbb{C}, x \in A$ and (A, φ). (Homogeneity)

(M3) If $N \in \mathbb{N}$ and $x_1, x_2, ..., x_N$ are monotonically independent and identically distributed, then $\kappa_n(x_1 + x_2 + \cdots + x_N) = N\kappa_n(x_1)$. (Extensivity)

Remark 3.3. Similar to Remark 3.1, a recursive argument shows that condition (M1) is equivalent to that there are universal polynomials $Q_n(t_1, t_2, ..., t_{n-1}), n \geq 1$ with $Q_1 := 0$ such that

$$\varphi(x^n) = \kappa_n(x) + Q_n(\kappa_1(x), \kappa_2(x), ..., \kappa_{n-1}(x)) \tag{3.1}$$

for all $n \geq 1$, $x \in A$ and any (A, φ).

3.1. **Cumulants from moments of random walk.** We begin with showing the uniqueness of monotone cumulants, which also indicates how to show the existence. Note that the same reasoning below also applies to showing that classical cumulants $\{C_n\}_{n \geq 1}$ are unique. For that purpose an elementary lemma on polynomials is needed.

Lemma 3.4. *Let $P(N) = a_0 + a_1 N + a_2 N^2 + \cdots + a_k N^k$ and $Q(N) = b_0 + b_1 N + b_2 N^2 + \cdots + b_k N^k$ be two polynomial functions on $\mathbb{N}$ with complex coefficients a_i, b_i. If $P(N) = Q(N)$ for all $N \in \mathbb{N}$ then $a_i = b_i$ for all $0 \leq i \leq k$.*

Remark 3.5. This lemma allows us to naturally extend a polynomial $P(N)$ defined for $N \in \mathbb{N}$ to a polynomial $P(t)$ defined for $t \in \mathbb{R}$.

The result easily extends to polynomials in several variables. For the case of two variables, if $P(N, M) = \sum_{i,j=1}^{k} a_{i,j} N^i M^j$ and $Q(N, M) = \sum_{i,j=1}^{k} b_{i,j} N^i M^j$ are polynomials with complex coefficients $a_{i,j}, b_{i,j}$ and $P(N, M) = Q(N, M)$ for all $N, M \in \mathbb{N}$ then $a_{i,j} = b_{i,j}$ for all $0 \leq i, j \leq k$. The proof is just to fix one variable, say M, and apply the lemma for $P(\cdot, M)$ and $Q(\cdot, M)$, which yields $\sum_{j=1}^{k} a_{i,j} M^j = \sum_{j=1}^{k} b_{i,j} M^j$ for all $M \in \mathbb{N}$ and i. Then again applying the lemma gives the conclusion $a_{i,j} = b_{i,j}$.

Proof. In fact, a weaker assumption is enough; suppose $P(N) = Q(N)$ holds at distinct positive integers $N_1 < N_2 < \cdots < N_{k+1}$. Then, by setting $c_i := a_i - b_i$, we have

$$\begin{pmatrix} 1 & N_1 & N_1^2 & \cdots & N_1^k \\ 1 & N_2 & N_2^2 & \cdots & N_2^k \\ \vdots & & \ddots & \vdots & \\ 1 & N_{k+1} & N_{k+1}^2 & \cdots & N_{k+1}^k \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the coefficient matrix has nonzero determinant (called the Vandermonde determinant), the numbers c_i must be zero. □

Proposition 3.6. *Monotone cumulants are unique.*

Proof. Suppose that $(\kappa_n)_{n\geq 1}$ are monotone cumulants determined by universal polynomials $(P_n)_{n\geq 1}$. We can see that $P_n(t_1,t_2,...,t_{n-1})$ for $n\geq 2$ contains no linear terms or a constant term. Indeed, if we write $P_n(t_1,...,t_{n-1}) = \sum_{k_1k_2,...,k_{n-1}\geq 0} c_{k_1,k_2,...,k_{n-1}} t_1^{k_1}t_2^{k_2}\cdots t_{n-1}^{k_{n-1}}$, where $c_{k_1,k_2,...,k_{n-1}}$ are complex constants independent of (A,φ) and the tuple $(k_1,k_2,...,k_{n-1})$ runs over a finite subset of $\mathbb{N}_0^{n-1}$, then the homogeneity condition reads

$$\lambda^n\kappa_n(x) = \lambda^n\varphi(x^n) + \sum_{k_1k_2,...,k_{n-1}\geq 0} c_{k_1,k_2,...,k_{n-1}}\lambda^{k_1+2k_2+\cdots+(n-1)k_{n-1}}\varphi(x)^{k_1}\varphi(x^2)^{k_2}\cdots\varphi(x^{n-1})^{k_{n-1}}.$$

Since this holds for all $\lambda\in\mathbb{C}$ and all $x\in A$ and all (A,φ), comparing the coefficients of λ^p yields

$$\sum_{\substack{k_1k_2,...,k_{n-1}\geq 0\\ k_1+2k_2+\cdots+(n-1)k_{n-1}=p}} c_{k_1,k_2,...,k_{n-1}}\varphi(x)^{k_1}\varphi(x^2)^{k_2}\cdots\varphi(x^{n-1})^{k_{n-1}} = 0, \qquad p\neq n.$$

As the tuple $(s_1,s_2,...,s_{n-1}) := (\varphi(x),\varphi(x^2),...,\varphi(x^{n-1}))$ can take arbitrary vector in $\mathbb{C}^{n-1}$, we conclude $c_{k_1,k_2,...,k_{n-1}} = 0$ unless $k_1+2k_2+\cdots+(n-1)k_{n-1}=n$. In particular, the constant term and linear terms of P_n are all zero. This also implies that the constant and linear terms of Q_n in (3.1) are all zero.

Let us take monotonically independent and identically distributed random variables $x:=x_1,x_2,...,x_N$ in some nc-probability space. Then the extensivity condition yields

$$\varphi((x_1+x_2+\cdots+x_N)^n) = N\kappa_n(x) + Q_n(N\kappa_1(x),N\kappa_2(x),...,N\kappa_{n-1}(x)).$$

The right hand side is a polynomial in positive integers N and hence, by Lemma 3.4, their coefficients are uniquely determined. In particular, since the Q_n part has no linear term, $\kappa_n(x)$ is uniquely determined as the coefficient of N of $\varphi((x_1+x_2+\cdots+x_N)^n)$. ☐

The above proof also indicates how we can find monotone cumulants: $\kappa_n(x)$ should be the coefficient of N of *the nth moment of monotone random walk* $\varphi((x_1+x_2+\cdots+x_N)^n)$. In order for this definition to make sense, we need to show $\varphi((x_1+x_2+\cdots+x_N)^n)$ is a polynomial in N.

Proposition 3.7. *For each $n\in\mathbb{N}$ there is a universal polynomial $U_n(s,t_1,t_2,...,t_{n-1})$ with $U_1:=0$ such that $U_n(0,t_1,t_2,...,t_{n-1})=0$ and*

$$\varphi((x_1+x_2+\cdots+x_N)^n) = N\varphi(x^n) + U_n(N,\varphi(x),\varphi(x^2),...,\varphi(x^{n-1})), \qquad n,N\geq 1$$

for any monotonically independent and identically distributed random variables $x_1,x_2,...,x_N$ in any nc-probability space (A,φ).

Proof. The proof is based on induction on n. For $n=1$ we have

$$\varphi(x_1+x_2+\cdots+x_N) = N\varphi(x_1).$$

Suppose that the statement is the case up to $n-1$. We set $s_k := x_1+x_2+\cdots+x_k$ and $s_0:=0$. By Corollary 2.5, s_{N-1} and x_N are monotonically independent. According to the moment calculation (1.9), there exists a universal polynomial R_n of $2n-1$ variables such that

$$\begin{aligned}\varphi(s_N^n) &= \varphi((s_{N-1}+x_N)^n)\\ &= \varphi(s_{N-1}^n) + R_n(\varphi(s_{N-1}),\varphi(s_{N-1}^2),...,\varphi(s_{N-1}^{n-1}),\varphi(x_N),\varphi(x_N^2),...,\varphi(x_N^n))\\ &= \varphi(s_{N-1}^n) + R_n(\varphi(s_{N-1}),\varphi(s_{N-1}^2),...,\varphi(s_{N-1}^{n-1}),\varphi(x_1),\varphi(x_1^2),...,\varphi(x_1^n)).\end{aligned}$$

By the assumption of induction, $\varphi(s_N^n)-\varphi(s_{N-1}^n)$ is a polynomial in $N,\varphi(x),\varphi(x^2),...,\varphi(x^n)$, i.e., it is of the form

$$\varphi(s_N^n)-\varphi(s_{N-1}^n) = \sum_{k=0}^{d_n} V_{n,k}(\varphi(x),\varphi(x^2),...,\varphi(x^n))N^k.$$

Taking the sum over N we obtain

$$\varphi(s_N^n) = \sum_{k=0}^{d_n} V_{n,k}(\varphi(x),\varphi(x^2),...,\varphi(x^n))\sum_{M=1}^{N} M^k.$$

By Faulharbor's formula, $\sum_{M=1}^N M^k$ is a polynomial in N of degree $k+1$ without a constant term. ☐

Theorem 3.8. *Let (A,φ) be a nc-probability space and $x\in A$. We take monotonically independent random variables $x_1:=x,x_2,...,x_N$. Let $\kappa_n(x)$ be the coefficient of N of the polynomial $\varphi((x_1+x_2+\cdots+x_N)^n)$. Then $\kappa_n, n\in\mathbb{N}$ are monotone cumulants, i.e., conditions* (M1)–(M3) *hold. Moreover, the polynomial P_n $(n\geq 2)$ has no constant or linear terms.*

Remark 3.9. In the above definition of $\kappa_n(x)$, the existence of $x_1, x_2, ..., x_N$ is not discussed. In general, the existence is hopeless in the same algebra A. Since the value $\varphi((x_1+x_2+\cdots+x_N)^n)$ only depends on the moments of $x = x_1$, the other x_i's need not be in the same space A. We therefore take a larger space $(\hat{A}, \hat{\varphi})$ by setting $A_i := A, \varphi_i := \varphi$ and

$$(\hat{A}, \hat{\varphi}) := \mathop{\rhd}_{i\in\mathbb{N}} (A_i, \varphi_i).$$

Then we consider A as a subalgebra of $\hat{A}$ by identifying A with $A_1 \subseteq \hat{A}$. Also, there are other natural embeddings of A into $\hat{A}$ as $\iota_i \colon A \to A_i \subseteq \hat{A}$. For each $x \in A$ the random variables $x_i := \iota_i(x), i \in \mathbb{N}$ are by construction monotonically independent and have the same distribution as x. Then we can define the monotone cumulant $\kappa_n(x)$ to be the coefficient of N of the polynomial $\hat{\varphi}((x_1 + x_2 + \cdots + x_N)^n)$. This is a precise definition.

Proof of Theorem 3.8. Condition (M1) is clear from Proposition 3.7. Condition (M2) holds because $\varphi((\lambda x_1 + \lambda x_2 + \cdots + \lambda x_N)^n) = \lambda^n \varphi((x_1 + x_2 + \cdots + x_N)^n)$. In order to show condition (M3), we take iid sequence $(x_i)_{i=1}^{MN}$, and set $y_i := x_{N(i-1)+1} + x_{N(i-1)+2} + \cdots + x_{Ni}, i \in [M]$. By Corollary 2.5, the sequence $(y_i)_{i=1}^{M}$ is monotonically independent. Also, $(y_i)_{i=1}^{M}$ is identically distributed. For each $N \in \mathbb{N}$, the coefficient of M of

$$\varphi((y_1 + y_2 + \cdots + y_M)^n)$$

equals $\kappa_n(y_1)$, which is $\kappa_n(x_1 + x_2 + \cdots + x_N)$. On the other hand,

$$\varphi((y_1 + y_2 + \cdots + y_M)^n) = \varphi((x_1 + x_2 + \cdots + x_{MN})^n)$$

is a polynomial in MN whose coefficient of MN is $\kappa_n(x_1)$, and therefore the coefficient of M is $N\kappa_n(x_1)$. Combining the above arguments we conclude (M3). The last assertion on P_n is already proved in Proposition 3.6. $\square$

Example 3.10. From condition (M1), $\kappa_1(x) = \varphi(x)$. We compute monotone cumulants κ_2, κ_3 by finding the polynomials U_2, U_3 in Proposition 3.7. The method here is more straightforward than the proof of Proposition 3.7 in the sense that the polynomial R_n is not used.

Formula for U_3. It should be kept in mind that $\varphi(x_i^n) = \varphi(x^n)$ does not depend on i because $x = x_1, x_2, x_3, ...$ have an identical distribution. We first compute

$$\begin{aligned}\varphi(s_N^2) &= \sum_{i,j=1}^{N} \varphi(x_i x_j) = \sum_{i,j=1, i\neq j}^{N} \varphi(x_i x_j) + \sum_{i=1}^{N} \varphi(x_i^2) \\ &= \sum_{i,j=1, i\neq j}^{N} \varphi(x_i)\varphi(x_j) + \sum_{i=1}^{N} \varphi(x_i^2) = \sum_{i,j=1, i\neq j}^{N} \varphi(x)\varphi(x) + \sum_{i=1}^{N} \varphi(x^2) \\ &= N(N-1)\varphi(x)\varphi(x) + N\varphi(x^2),\end{aligned}$$

so that $U_2(N, t) = N(N-1)t^2$.

Formula for U_3. We begin with

$$\begin{aligned}\varphi(s_N^3) &= \sum_{i,j,k=1}^{N} \varphi(x_i x_j x_k) \\ &= \sum_{i,j,k \text{ distinct}} \varphi(x_i x_j x_k) + \sum_{i=j\neq k} \varphi(x_i^2 x_k) + \sum_{i=k\neq j} \varphi(x_i x_j x_i) + \sum_{i\neq k=j} \varphi(x_i x_j^2) + \sum_{i=j=k} \varphi(x_i^3) \\ &= \sum_{i,j,k \text{ distinct}} \varphi(x_i x_j x_k) + \sum_{i=j\neq k} \varphi(x_i^2)\varphi(x_k) + \sum_{i=k\neq j} \varphi(x_i x_j x_i) + \sum_{i\neq k=j} \varphi(x_i)\varphi(x_j^2) + \sum_{i=j=k} \varphi(x_i^3).\end{aligned}$$

In the above, $\varphi(x_i x_j x_k)$ for distinct integers i, j, k always factorizes into $\varphi(x_i)\varphi(x_j)\varphi(x_k)$; for example, if $i < j > k$ and $i \neq k$ then $\varphi(x_i x_j x_k) = \varphi(x_i x_k)\varphi(x_j) = \varphi(x_i)\varphi(x_j)\varphi(x_k)$ and if $i < j < k$ then $\varphi(x_i x_j x_k) = \varphi(x_i x_j)\varphi(x_k) = \varphi(x_i)\varphi(x_j)\varphi(x_k)$. On the other hand, the sum over $i = k \neq j$ is more delicate. In order to use monotone independence, we need to further specify the inequality between i and j. If $i < j$ then $\varphi(x_i x_j x_i) = \varphi(x_j)\varphi(x_i^2)$. If $i > j$ then $\varphi(x_i x_j x_i) = \varphi(x_i)\varphi(x_j x_i) = \varphi(x_i)\varphi(x_j)\varphi(x_i)$. Therefore,

$$\sum_{i=k\neq j} \varphi(x_i x_j x_i) = \sum_{i<j} \varphi(x_i x_j x_i) + \sum_{i>j} \varphi(x_i x_j x_i) = \sum_{i<j} \varphi(x_i^2)\varphi(x_j) + \sum_{i>j} \varphi(x_i)^2\varphi(x_j).$$

Overall, we arrive at

$$\begin{aligned}\varphi(s_N^3) &= N(N-1)(N-2)\varphi(x)^3 + N(N-1)\varphi(x^2)\varphi(x) + \frac{N(N-1)}{2}\varphi(x^2)\varphi(x) \\ &\quad + \frac{N(N-1)}{2}\varphi(x)^3 + N(N-1)\varphi(x^2)\varphi(x) + N\varphi(x^3) \\ &= N(N-1)\left(N-\frac{3}{2}\right)\varphi(x)^3 + \frac{5N(N-1)}{2}\varphi(x^2)\varphi(x) + N\varphi(x^3).\end{aligned}$$

This means

$$U_3(N,t_1,t_2) = N(N-1)\left(N-\frac{3}{2}\right)t_1^3 + \frac{5N(N-1)}{2}t_1t_2.$$

The above method can be generalized to any U_n, which provides another proof of Proposition 3.7.

Formulas for κ_2,κ_3. Finally, the monotone cumulants $\kappa_2(x)$ and $\kappa_3(x)$ are identified with the coefficients of N of $\varphi(s_N^2)$ and of $\varphi(s_N^3)$ respectively:

$$\kappa_2(x) = \varphi(x^2) - \varphi(x)^2, \tag{3.2}$$

$$\kappa_3(x) = \varphi(x^3) - \frac{5}{2}\varphi(x^2)\varphi(x) + \frac{3}{2}\varphi(x)^3. \tag{3.3}$$

A recursive formula for computing $\kappa_n(x)$ will be provided in Proposition 3.12. A combinatorial formula for $\varphi(x^n)$ in terms of $\kappa_\ell(x), 1 \le \ell \le n$ will be given in Theorem 3.23.

Remark 3.11. The nth monotone cumulant $\kappa_n(x)$ is determined by the moments of the random variable x up to order n. Therefore, for any probability measure μ having finite moments up to order n, we can define $\kappa_n(\mu) := \kappa_n(x)$ by taking a random variable x in a nc-probability space (A,φ) such that $\varphi(x^p) = \int_{\mathbb{R}} t^p\,\mu(dt), 1 \le p \le n$. We call $\kappa_n(\mu)$ the **nth monotone cumulant of μ**.

3.2. **Differential recursion for monotone cumulants.** To compute monotone cumulants, differential recursion is helpful.

Proposition 3.12. *Let x be a random variable and $x_1 := x, x_2, x_3, \ldots$ are monotonically independent in a nc-probability space (A,φ). Let $s_N := x_1 + x_2 + \cdots + x_N, s_0 := 0$. As we have seen above, for each $n \in \mathbb{N}$ the evaluation $\varphi(s_N^n)$ is a polynomial in N, so it can be extended to a polynomial in a real variable $t \in \mathbb{R}$, which we denote by $\mathbf{m}_n(t) \equiv \mathbf{m}_n(t;x)$. Then we have*

$$\mathbf{m}_n'(t) = \sum_{\ell=0}^{n-1}(\ell+1)\kappa_{n-\ell}(x)\mathbf{m}_\ell(t), \qquad n \ge 1; \qquad \mathbf{m}_0(t) \equiv 1, \tag{3.4}$$

$$\mathbf{m}_n(0) = 0, \qquad n \ge 1. \tag{3.5}$$

Proof. Note first that $\mathbf{m}_n(0) = 0$ comes from Proposition 3.7 showing that $\varphi(s_N^n)$ has no constant term on N. Let $s_M' := x_{N+1} + x_{N+2} + \cdots + x_{N+M}$. By Corollary 2.5, s_N and s_M' are monotonically independent. Note that s_M' also depends on N as an element of A; however its distribution only depends on the number M of the summands x_i's, so that we omit explicitly mentioning the dependence on N. In the obvious formula

$$\mathbf{m}_n(N+M) = \varphi((x_1 + x_2 + \cdots + x_{N+M})^n) = \varphi((s_N + s_M')^n),$$

the right-hand side is exactly the monotone convolution, so we can use the calculation in (1.9):

$$\varphi((s_N + s_M')^n) = \sum_{\ell=0}^{n} \sum_{\substack{k_0,k_1,\ldots,k_\ell \ge 0,\\ k_0+k_1+\cdots+k_\ell = n-\ell}} \varphi(s_N^\ell)\varphi(s_M^{k_0})\varphi(s_M^{k_1})\cdots\varphi(s_M^{k_\ell}). \tag{3.6}$$

Since each $\varphi(s_M^k)$ is a polynomial in M without a constant term, the contributions to the monomial M in the sum (3.6) only come from the tuples $(k_0,k_1,\ldots,k_\ell), 0 \le \ell \le n-1$ such that exactly one of k_i's is nonzero, so that we obtain

$$\begin{aligned}\varphi((s_N + s_M')^n) &= \sum_{\ell=0}^{n-1}(\ell+1)\varphi(s_N^\ell)\varphi(s_M^{n-\ell}) + R_1(M) \\ &= M\sum_{\ell=0}^{n-1}(\ell+1)\varphi(s_N^\ell)\kappa_{n-\ell}(x) + R_2(M),\end{aligned}$$

where $R_1(M)$ and $R_2(M)$ are polynomials in M without a constant or linear term. Since all the terms are polynomials, we can extend the variables N and M to real numbers t and s, so that

$$\mathbf{m}_n(t+s) = \sum_{\ell=0}^{n-1}(\ell+1)\kappa_{n-\ell}(x)\mathbf{m}_\ell(t) + R_2(s).$$

The desired formula follows by taking the derivative $\frac{d}{ds}\big|_{s=0}$. □

The differential recursion gives an efficient method for computing $\kappa_n(x)$.

Example 3.13. We set $n=1$ in (3.5) to obtain $\mathbf{m}_1'(t) = \kappa_1(x)$, which integrates to $\mathbf{m}_1(t) = \kappa_1(x)t$. Setting $t=1$, we obtain

$$\kappa_1(x) = \mathbf{m}_1(1) = \varphi(x),$$

which is already known. Formula (3.5) for $n=2$ reads

$$\mathbf{m}_2'(t) = \kappa_2(x)\mathbf{m}_0(t) + 2\kappa_1(x)\mathbf{m}_1(t) = \kappa_2(x) + 2t\kappa_1(x)^2,$$

which integrates to

$$\mathbf{m}_2(t) = \kappa_2(x)t + \kappa_1(x)^2 t^2.$$

Setting $t=1$ and using $\kappa_1(x) = \varphi(x)$ we obtain formula (3.2) for κ_2. In a similar manner we obtain formula (3.3) for κ_3. If n becomes larger and larger, this recursive method for computing κ_n seems more efficient than the one in Example 3.10.

Corollary 3.14. *As a polynomial in t, we have* $\deg(\mathbf{m}_n(t)) \le n$ *for all* $n \in \mathbb{N}\cup\{0\}$.

Proof. Integrating the differential recursion in Proposition 3.12 yields

$$\mathbf{m}_n(t) = \sum_{\ell=0}^{n-1}(\ell+1)\kappa_{n-\ell}(x)\int_0^t \mathbf{m}_\ell(s)\,ds, \qquad n \ge 1.$$

Starting from $\mathbf{m}_0(t) \equiv 1$, we can show the bound $\deg(\mathbf{m}_n(t)) \le n$ by induction. □

3.3. **Monotone central limit theorem.** In probability theory, a basic form of the central limit theorem says if $(X_i)_{i\ge1}$ is real-valued i.i.d. random variables such that $\mathbb{E}[X_i]=0$ and $\mathbb{E}[X_i^2] = \sigma^2 > 0$, then the distribution of

$$\frac{X_1 + X_2 + \cdots + X_N}{\sqrt{N\sigma^2}}$$

converges weakly to $\mathrm{N}(0,1)$ as $N\to\infty$. We consider a similar problem for monotonically independent random variables.

Theorem 3.15. *Let (A,φ) be a unital C^*-probability space and $(x_i)_{i\ge1}$ be a sequence of monotonically independent and identically distributed real random variables in A. Suppose that $\varphi(x_1)=0$ and $\varphi(x_1^2) = \sigma^2 > 0$. Then, for any $n\in\mathbb{N}$*

$$\lim_{N\to\infty}\varphi\left(\left(\frac{x_1+x_2+\cdots+x_N}{\sqrt{N\sigma^2}}\right)^n\right) = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{t^n}{\pi\sqrt{2-t^2}}dt.$$

In particular, the analytic distribution of $(x_1+x_2+\cdots+x_N)/\sqrt{N\sigma^2}$ converges weakly to the arcsine law with density $1/(\pi\sqrt{2-t^2})$, $-\sqrt{2}<t<\sqrt{2}$.

Proof. We first prove the convergence of monotone cumulants of $a_N := \frac{x_1+x_2+\cdots+x_N}{\sqrt{N\sigma^2}}$. By using conditions (M1) and (M2) we have

$$\kappa_n(a_N) = (N\sigma^2)^{-\frac{n}{2}}\kappa_n(x_1+x_2+\cdots+x_N) = (N\sigma^2)^{-\frac{n}{2}}N\kappa_n(x_1).$$

Recall here that $\kappa_1(x_1) = \varphi(x_1) = 0$ and $\kappa_2(x_1) = \varphi(x_1^2) - \varphi(x_1)^2 = \sigma^2$. Passing to the limit yields

$$\kappa_n := \lim_{N\to\infty}\kappa_n(a_N) = \begin{cases} 1, & \text{if } n=2, \\ 0, & \text{otherwise.}\end{cases}$$

By the polynomiality $\varphi(a_N^n) = \kappa_n(a_N) + Q_n(\kappa_1(a_N),\kappa_2(a_N),\dots,\kappa_{n-1}(a_N))$ and taking the limit, we obtain the convergence

$$\lim_{N\to\infty}\varphi(a_N^n) = \kappa_n + Q_n(\kappa_1,\kappa_2,\dots,\kappa_{n-1}), \qquad n\ge1.$$

Now, we come to use the differential recursion. Let $\mathbf{m}_n(t;a_N)$ be the polynomial constructed for $x:=a_N$ as in Proposition 3.12. From Example 3.13, we have $\mathbf{m}_1(t;a_N) = \varphi(a_N)t = 0$. Since $\mathbf{m}_n(t;a_N)$ is a polynomial in t and $\varphi(a_N^k)$, $1\le k\le n$, the limit

$$m_n(t) := \lim_{N\to\infty}\mathbf{m}_n(t;a_N)$$

exists. Since $\deg(\mathbf{m}_n(t;a_N)) \le n$ and the coefficient of each monomial t^k $(1 \le k \le n)$ of $\mathbf{m}_n(t;a_N)$ converges, the limit function $m_n(t)$ is also a polynomial with degree $\le n$ and the convergence is uniform on each finite interval of $\mathbb{R}$. By taking the limit in the integrated form of Proposition 3.12, we obtain

$$m_n(t) = \lim_{N\to\infty} \mathbf{m}_n(t;a_N) = \lim_{N\to\infty} \sum_{\ell=0}^{n-1} (\ell+1)\kappa_{n-\ell}(a_N) \int_0^t \mathbf{m}_\ell(s;a_N)\,ds = \int_0^t (n-1)m_{n-2}(s)\,ds.$$

Since $m_0(t) = 1$ and $m_1(t) = 0$ for all $t \in \mathbb{R}$, this can be easily solved by iterated integrals as

$$m_{2k}(t) = \frac{(2k-1)!!}{k!}t^k, \qquad m_{2k-1}(t) = 0, \qquad k \ge 1.$$

As $\mathbf{m}_n(1;a_N) = \varphi(a_N^n)$, we have thus obtained

$$\lim_{N\to\infty} \varphi(a_N^n) = m_n(1) = \begin{cases} \frac{(2k-1)!!}{k!}, & \text{if } n = 2k, k \in \mathbb{N} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{3.7}$$

This limit is exactly the moment sequence of the arcsine law, which has compact support and hence its moment sequence is determinate (Proposition A.3). The weak convergence is a consequence of Proposition A.6. ☐

3.4. **Poisson's law of small numbers.** The second limit theorem to be discussed is an analogue of Poisson's law of small numbers. In probability theory, the simplest formulation is as follows: for each $N \in \mathbb{N}$, suppose that $X_{N,1}, X_{N,2}, ..., X_{N,N}$ are independent random variables that have the identical distribution

$$\mathbb{P}[X_{N,k} = 0] = 1 - \frac{\lambda}{N}, \qquad \mathbb{P}[X_{N,k} = 1] = \frac{\lambda}{N}, \qquad 1 \le k \le N \tag{3.8}$$

for some fixed $\lambda > 0$. Then the distribution of $X_{N,1} + X_{N,2} + \cdots + X_{N,N}$ converges to the Poisson distribution with rate λ:

$$e^{-\lambda} \sum_{n=0}^\infty \frac{\lambda^n}{n!}\delta_n.$$

Theorem 3.16. *Let $\lambda > 0$. For each $N \in \mathbb{N}$, let $(x_{N,i})_{i=1}^N$ be monotonically independent and identically distributed real random variables in a unital C^*-probability space (A, φ) such that*

$$\mu_{x_{N,i}} = \left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1, \qquad 1 \le i \le N.$$

Then there exists a probability measure ρ_λ whose monotone cumulants are all equal to λ such that

$$\lim_{N\to\infty} \varphi((x_{N,1} + x_{N,2} + \cdots + x_{N,N})^n) = \int_{\mathbb{R}} t^n\,\rho_\lambda(dt), \qquad n \in \mathbb{N}.$$

Remark 3.17. We will study ρ_λ further in Example 5.22, where ρ_λ turns out to have compact support. Therefore, the analytic distribution of $x_{N,1} + x_{N,2} + \cdots + x_{N,N}$ weakly converges to ρ_λ. The measure ρ_λ will be called the **monotone Poisson distribution with parameter $\lambda > 0$.**

Proof of Theorem 3.16. Let $a_N := x_{N,1} + x_{N,2} + \cdots + x_{N,N}$. Since $(x_{N,i})_{i=1}^N$ is iid, we have

$$\kappa_n(a_N) = N\kappa_n(x_{N,1}).$$

Observe here that $\varphi(x_{N,1}^n) = \lambda/N$. From condition (M1), since P_n has no constant or linear terms,

$$N\kappa_n(x_{N,1}) = N\varphi(x_{N,1}^n) + NP_n(\varphi(x_{N,1}), \varphi(x_{N,1}^2), ..., \varphi(x_{N,1}^{n-1})) = \lambda + o(1).$$

Therefore we conclude that

$$\lim_{N\to\infty} \kappa_n(a_N) = \lambda.$$

This in turn implies

$$\lim_{N\to\infty} \varphi(a_N^n) = \lambda + Q_n(\lambda, \lambda, ..., \lambda). \tag{3.9}$$

Because for each N the sequence $\varphi(a_N^n), n = 0, 1, 2, ...$ is positive semi-definite, the limit sequence is also positive semi-definite. This guarantees that the limit (3.9) is a moment sequence of some probability measure on $\mathbb{R}$. ☐

3.5. **Monotone set partitions and monotone cumulants.** Cumulants are known to be intimately connected to set partitions. Let S be a finite set. A decomposition of S into nonempty disjoint subsets is called a **set partition** of S. A set partition is denoted as $\rho = \{B_1, B_2, \dots, B_k\}$, where B_i's are the nonempty disjoint subsets of S such that $S = B_1 \cup \cdots \cup B_k$. Each B_i is called a **block** of the set partition ρ. The number k of the blocks of ρ is denoted by $|\rho|$. Let $\mathcal{P}(S)$ stand for the set of all set partitions of S. For the special case $S = [n]$, we denote $\mathcal{P}(n) := \mathcal{P}([n])$. For example $\mathcal{P}(2)$ has two elements $\{\{1,2\}\}$ and $\{\{1\},\{2\}\}$ and $\mathcal{P}(3)$ has five elements $\{\{1,2,3\}\}, \{\{1,2\},\{3\}\}, \{\{1,3\},\{2\}\}, \{\{1\},\{2,3\}\}, \{\{1\},\{2\},\{3\}\}$.

Let $(\alpha_n)_{n\ge1}$ be a sequence of complex numbers. For each set partition $\rho = \{B_1, B_2, \dots, B_k\}$ of S, we define

$$\alpha_\rho := \alpha_{|B_1|}\alpha_{|B_2|}\cdots\alpha_{|B_k|}.$$

With this notation, the following formula holds.

Proposition 3.18. *For a random variable $X \in L^{\infty-}(\Omega, \mathcal{F}, P)$, we have*

$$\mathbb{E}[X^n] = \sum_{\rho\in\mathcal{P}(n)} C_\rho(X), \qquad n \in \mathbb{N}. \tag{3.10}$$

Proof. Although we do not need this formula later, the proof is sketched for the reader's convenience. We can see that, given a sequence of positive integers $(i_1, i_2, ..., i_k)$ with $i_1 + 2i_2 + \cdots + ki_k = n$, the number of $\rho \in \mathcal{P}(n)$ that has i_1 blocks of cardinality one, i_2 blocks of cardinality two, ..., i_k blocks of cardinality k, equals

$$\begin{aligned}&\binom{n}{i_1}\binom{n-i_1}{2i_2}\binom{n-i_1-2i_2}{3i_3}\cdots\binom{n-i_1-2i_2-\cdots-(k-1)i_{k-1}}{ki_k}\prod_{p=1}^k \frac{(pi_p)!}{(p!)^{i_p}(i_p!)}\\ &= \frac{n!}{i_1!i_2!\cdots i_k!(1!)^{i_1}(2!)^{i_2}\cdots(k!)^{i_k}},\end{aligned} \tag{3.11}$$

so that the coefficient of $C_1(X)^{i_1}C_2(X)^{i_2}\cdots C_k(X)^{i_k}$ in (3.10) is exactly (3.11). On the other hand, recall that the definition of cumulants is given by

$$\mathbb{E}[e^{zX}] = \sum_{n\ge0}\frac{\mathbb{E}[X^n]}{n!}z^n = \exp\left(\sum_{n\ge1}\frac{C_n(X)}{n!}z^n\right) = e^{C_1(X)z}e^{\frac{C_2(X)}{2!}z^2}e^{\frac{C_3(X)}{3!}z^3}\cdots \tag{3.12}$$

in the sense of formal power series. The coefficient of $C_1(X)^{i_1}C_2(X)^{i_2}\cdots C_k(X)^{i_k}$ in (3.12) is easily seen to be the number (3.11) divided by $n!$, as desired. ☐

Example 3.19. For $n = 1, 2, 3$, (3.10) reads

$$\mathbb{E}[X] = C_1(X), \tag{3.13}$$

$$\mathbb{E}[X^2] = C_2(X) + C_1(X)^2, \tag{3.14}$$

$$\mathbb{E}[X^3] = C_3(X) + 3C_2(X)C_1(X) + C_1(X)^3. \tag{3.15}$$

Our goal is to discover a similar formula for monotone cumulants. It turns out that an ordered set partition provides a suitable framework. An **ordered set partition** of a finite set S is a sequence $\pi = (B_1, B_2, ..., B_k)$, where $\{B_1, B_2, ..., B_k\}$ is a set partition, i.e., $B_1, B_2, ..., B_k$ are nonempty disjoint subsets of S such that S is the union of them. We can also consider that π is the set partition $\{B_1, B_2, ..., B_k\}$ equipped with the linear order on its blocks $B_1 \le B_2 \le \cdots \le B_k$. In this way, an equivalent definition is that an ordered set partition is a pair $\pi = (\rho, \le)$ of a set partition $\rho \in \mathcal{P}(S)$ and a linear (or total) order on ρ. We set the notations $|\pi| := |\rho|$ that is the length of the sequence π, and $\overline{\pi} := \rho$. Let $\mathcal{OP}(S)$ be the set of the ordered set partitions of S. With a slight abuse of notation, we will write $B_i \in \pi$, which more precisely means $B_i \in \overline{\pi}$.

Let T be a totally ordered finite set. A set partition $\rho \in \mathcal{P}(T)$ is said to have a crossing if there are two blocks $B_1, B_2 \in \rho$ and elements $a, b \in B_1$ and $b, c \in B_2$ such that $a < c < b < d$. A set partition that has no crossings is called a **noncrossing set partition**.

We consider a partial order on each $\rho \in \mathcal{P}(T)$ defined by a covering relation. For nonempty subsets $B_1, B_2 \subseteq T$, we say B_1 covers B_2, denoted as $B_1 \preceq B_2$, if $\min B_1 \le i \le \max B_1$ for all $i \in B_2$. On a set partition of T, the relation $\preceq$ becomes a partial order.

Definition 3.20. Let T be a totally ordered finite set. An ordered set partition $\pi = (\rho, \le)$ of T is called a **monotone set partition** if

(i) ρ is a noncrossing set partition,

(ii) if $B, B' \in \rho$ satisfies $B \preceq B'$ then $B \le B'$.

The set of monotone set partitions of T is denoted by $\mathcal{M}(T)$. For notational simplicity, we set $\mathcal{M}(n) := \mathcal{M}([n])$

The monotone set partitions $\mathcal{M}(T)$ can be generated from the following recursion.

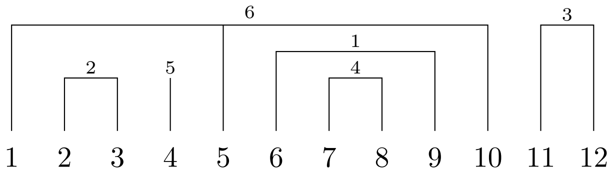

FIGURE 1. The diagram of $\pi = (B_1, B_2, \dots, B_6) \in \mathcal{OP}(12)$ where $B_1 = \{6, 9\}$, $B_2 = \{2, 3\}$, $B_3 = \{11, 12\}$, $B_4 = \{7, 8\}$, $B_5 = \{4\}$, and $B_6 = \{1, 5, 10\}$. This is not a monotone set partition because e.g. B_6 covers B_1. For example, the permuted one $\tilde{\pi} = (B_6, B_2, B_5, B_1, B_4, B_3)$ is a monotone set partition.

Proposition 3.21. *Let T be a totally ordered finite set. There exists a canonical bijection*

$$\beta\colon \mathcal{M}(T) \to \bigcup_I \mathcal{M}(T \setminus I),$$

where I runs over the set of nonempty intervals of T, the complement $T \setminus I$ is endowed with the linear order induced by T, and $\mathcal{M}(\emptyset) := \{\emptyset\}$. The bijection is given by $\beta\colon (\rho, \leq) \mapsto \rho \setminus \{B_{\max}\} \in \mathcal{M}(T \setminus B_{\max})$, where $B_{\max}$ is the largest block of ρ with respect to $\leq$.

Proof. Observe first that $B_{\max}$ is a nonempty interval of T since $B_{\max}$ does not cover any other element and ρ is noncrossing. It is straightforward that $\rho \setminus \{B_{\max}\}$ is a monotone set partition of $T \setminus B_{\max}$. Conversely, the mapping $\bigcup_I \mathcal{M}(T \setminus I) \ni (I, \pi') = (I, (\rho', \leq')) \mapsto (\rho' \cup \{I\}, \leq)$ where $\leq$ is the extension of $\leq'$ such that I is larger than any block of ρ', also gives a monotone set partition of T, and this is the inverse mapping. ☐

Proposition 3.22. *Let T be a totally ordered finite set. The cardinality of $\mathcal{M}(T)$ is $\frac{(|T|+1)!}{2}$.*

Proof. We may assume that $T = [n]$. Let $t_n := |\mathcal{M}(n)|, n \in \mathbb{N}$ and $t_0 := 1$. The previous bijection yields $t_n = \sum_I t_{n-|I|}$. For each $1 \leq k \leq n$, there are $n - k + 1$ intervals I such that $|I| = k$. This yields

$$t_n = \sum_{k=1}^{n} (n - k + 1) t_{n-k} = \sum_{p=0}^{n-1} (p + 1) t_p, \qquad n \geq 1; \qquad t_0 = 1.$$

Computing $t_n - t_{n-1}$ yields $t_n = (n+1)t_{n-1}$, so an induction argument shows the desired formula $t_n = (n+1)!/2$ for $n \in \mathbb{N}$. ☐

Given a sequence $(\alpha_n)_{n \geq 1}$ of complex numbers and $\pi = (B_1, B_2, \dots, B_k) \in \mathcal{OP}(S)$, we define

$$\alpha_\pi := \alpha_{|B_1|} \alpha_{|B_2|} \cdots \alpha_{|B_k|}.$$

Equivalently, we set $\alpha_\pi := \alpha_\rho$ for $\pi = (\rho, \leq)$.

Theorem 3.23. *On any nc-probability space (A, φ) and for any $x \in A$, we have*

$$\varphi(x^n) = \sum_{\pi \in \mathcal{M}(n)} \frac{1}{|\pi|!} \kappa_\pi(x), \qquad n \in \mathbb{N}. \tag{3.16}$$

Remark 3.24. For the classical cumulants $(C_n)_{n \geq 1}$, the moment-cumulant formula (3.10) can be written in the equivalent form

$$\mathbb{E}[X^n] = \sum_{\pi \in \mathcal{OP}(n)} \frac{1}{|\pi|!} C_\pi(X), \qquad n \in \mathbb{N}, \quad X \in L^{\infty-}, \tag{3.17}$$

since for each $\rho \in \mathcal{P}(n)$ there are $|\rho|!$ number of $\pi \in \mathcal{OP}(n)$ such that $\overline{\pi} = \rho$, corresponding to the permutations of the blocks. Comparing with formula (3.17) somehow justifies the naturality of the factor $\frac{1}{|\pi|!}$ in (3.16).

Proof of Theorem 3.23. Let $x_1 := x, x_2, x_3, \dots$ be monotonically independent, identically distributed sequence, possibly in a larger nc-probability space, which we still denote by (A, φ) for simplicity. Recall from Proposition 3.12 that $\mathbf{m}_n(t) = \mathbf{m}_n(t; x)$ is a polynomial in $t \in \mathbb{R}$ that coincides with $\varphi((x_1 + x_2 + \cdots + x_t)^n)$ at $t \in \mathbb{N}$. We prove a generalized formula

$$\mathbf{m}_n(t) = \sum_{\pi \in \mathcal{M}(n)} \frac{t^{|\pi|}}{|\pi|!} \kappa_\pi(x), \qquad n \in \mathbb{N} \tag{3.18}$$

which coincides with (3.16) for $t = 1$. We extend (3.18) to $n = 0$ by interpreting $\mathcal{M}(0) := \{\emptyset\}$, $|\emptyset| := 0$, $0! := 1$ and $\kappa_\emptyset(x) := 1$, so that the following calculations make sense.

Formula (3.18) is obviously the case for $n = 0$. Suppose that the formula holds up to $n - 1$. Then, using the differential recursion in Proposition 3.12, we proceed as

$$\begin{aligned}
\mathbf{m}_n(t) &= \sum_{\ell=0}^{n-1} (\ell+1)\kappa_{n-\ell}(x) \int_0^t \mathbf{m}_\ell(s)\, ds \\
&= \sum_{\ell=0}^{n-1} (\ell+1)\kappa_{n-\ell}(x) \sum_{\pi \in \mathcal{M}(\ell)} \frac{1}{|\pi|!}\kappa_\pi(x) \int_0^t s^{|\pi|}\, ds \\
&= \sum_{\ell=0}^{n-1} (\ell+1) \sum_{\pi \in \mathcal{M}(\ell)} \frac{t^{|\pi|+1}}{(|\pi|+1)!}\kappa_{n-\ell}(x)\kappa_\pi(x) \\
&= \sum_{p=1}^{n} (n-p+1) \sum_{\pi \in \mathcal{M}(n-p)} \frac{t^{|\pi|+1}}{(|\pi|+1)!}\kappa_p(x)\kappa_\pi(x). \qquad (3.19)
\end{aligned}$$

Since there are $n - p + 1$ intervals $I \subseteq [n]$ of size p, (3.19) can be written in the form

$$\sum_{\substack{\emptyset \neq I \subseteq [n] \\ \text{interval}}} \sum_{\pi \in \mathcal{M}([n]\setminus I)} \frac{t^{|\pi|+1}}{(|\pi|+1)!}\kappa_{|I|}(x)\kappa_\pi(x). \tag{3.20}$$

The last formula can be well described by the bijection β in Proposition 3.21: the ordered set partition $\sigma := \beta(I, \pi)$ runs over all elements of $\mathcal{M}(n)$ exactly once as (I, π) runs over the summation range of (3.20), and it holds that $|\pi| + 1 = |\sigma|$ and $\kappa_{|I|}(x)\kappa_\pi(x) = \kappa_\sigma(x)$. Therefore, the last expression (3.20) is exactly the desired (3.18). ☐

The monotone CLT says that the monotone cumulant sequence $(0, 1, 0, 0, 0, ...)$ corresponds to the moment sequence (3.7) of the arcsine law. This fact and Theorem 3.23 yield the cardinality of the set of monotone pair partitions

$$\mathcal{M}_2(2n) := \{\pi \in \mathcal{M}(2n) : \text{every block of } \pi \text{ has cardinality two}\}.$$

Corollary 3.25. *The cardinality of $\mathcal{M}_2(2n)$ is $(2n-1)!!$.*

The above proof of the moment-cumulant formula in Theorem 3.23 does not clarify well why the monotone set partitions appear. In fact, monotone set partitions have a more intrinsic meaning: they naturally appear when characterizing a "tensor-like" factorization of mixed moments.

Definition 3.26. Let I be a totally ordered set and $i_1, i_2, ..., i_n \in I$. Let $\ker(i_1, i_2, ..., i_n)$ be the ordered set partition of $[n]$, called the **kernel**, defined as follows: let $A_j := \{p \in [n] : i_p = j\}, j \in I$ and we collect all the nonempty sets $A_{j_1}, A_{j_2}, ..., A_{j_r}$, $j_1 < j_2 < \cdots < j_r$, and define $\ker(i_1, i_2, ..., i_n) := (A_{j_1}, A_{j_2}, ..., A_{j_r})$.

Example 3.27. Let $I = \mathbb{N}$. Then $\ker(3, 5, 2, 1, 5, 3, 5)$ is given by $(\{4\}, \{3\}, \{1, 6\}, \{2, 5, 7\})$.

Proposition 3.28. *Let I be a totally ordered set and $i_1, i_2, ..., i_n \in I$. Then $\ker(i_1, i_2, ..., i_n) \in \mathcal{M}(n)$ if and only if the factorization*

$$\varphi(a_1 a_2 \cdots a_n) = \prod_{B \in \ker(i_1, i_2, ..., i_n)} \varphi\left(\overrightarrow{\prod_{p \in B}} a_p\right) \tag{3.21}$$

holds for any random variables $a_1 \in A_{i_1}, a_2 \in A_{i_2}, \ldots, a_n \in A_{i_n}$ and any monotonically independent subalgebras $(A_i)_{i \in I}$ in any nc-probability space (A, φ).

Remark 3.29. The factorization (3.21) is exactly the formula that always holds irrespective of $\ker(i_1, ..., i_n)$ provided $(A_i)_{i \in I}$ were tensor independent. This proposition therefore characterizes the arrangements of random variables such that the factorization coincides with the case of tensor independence.

Proof. Let us check the statement through examples. In the following, $(A_i)_{i=1}^\infty$ are monotonically independent subalgebras in a nc-probability space (A, φ), $\mathbf{i} = (i_1, i_2, ..., i_n) \in \mathbb{N}^n$ and $a_1 \in A_{i_1}, ..., a_n \in A_{i_n}$.

Case 1: $\mathbf{i} = (i_1, i_2, ..., i_8) = (2, 4, 4, 4, 3, 2, 1, 1)$. Then the kernel

$$\pi_1 := \ker(\mathbf{i}) = (\{7, 8\}, \{1, 6\}, \{5\}, \{2, 3, 4\})$$

is a monotone set partition. We first focus on the largest block $\{2, 3, 4\}$ and obtain

$$\varphi(a_1 a_2 \cdots a_8) = \varphi(a_1(a_2 a_3 a_4)a_5 a_6 a_7 a_8) = \varphi(a_2 a_3 a_4)\varphi(a_1 a_5 a_6 a_7 a_8)$$

because $i_1 < i_2 = i_3 = i_4 > i_5$ and $a_2 a_3 a_4 \in A_{i_2}$. The remaining sequence $(i_1, i_5, i_6, i_7, i_8)$ associates the kernel ordered set partition $(\{7, 8\}, \{1, 6\}, \{5\})$, which is also a monotone set partition. Since now $i_1 < i_5 > i_6$ we have

$$\varphi(a_1 a_5 a_6 a_7 a_8) = \varphi(a_5)\varphi(a_1 a_6 a_7 a_8),$$

and finally we arrive at

$$\varphi(a_1a_2\cdots a_8) = \varphi(a_2a_3a_4)\varphi(a_5)\varphi(a_1a_6)\varphi(a_7a_8) = \prod_{B\in\pi_1}\varphi\left(\overrightarrow{\prod_{p\in B}} a_p\right).$$

In general, when the kernel is a monotone set partition, we can first factor out the expectation of elements corresponding to the largest block of the kernel, and then by Proposition 3.21 the remaining blocks still form a monotone set partition. Then we can repeat the same procedure to get the tensor-like factorization.

Case 2: $\mathbf{i} = (2,1,2,1)$. The associated kernel $\pi_2 := \ker(2,1,2,1) = (\{2,4\},\{1,3\})$ is not a monotone set partition because $\overline{\pi_2}$ has a crossing. By the definition of monotone independence we get

$$\varphi(a_1a_2a_3a_4) = \varphi(a_1)\varphi(a_2a_3a_4),$$

so that the block $\{1,3\}$ "splits" into the singletons $\{1\}$ and $\{3\}$. This shows the tensor-like factorization does not hold.

Case 3: $\mathbf{i} = (2,1,1,2)$. Then $\pi_3 := \ker(2,1,1,2) = (\{2,3\},\{1,4\})$. In this case $\overline{\pi_3}$ is noncrossing but the total order on $\overline{\pi_3}$ is not compatible with the covering relation, so that π_3 is not a monotone set partition. It holds that

$$\varphi(a_1a_2a_3a_4) = \varphi(a_1)\varphi(a_2a_3a_4),$$

so that the block $\{1,4\}$ again splits.

In general, as soon as $\ker(\mathbf{i})$ is not a monotone set partition, there always exists a block that splits, so that the tensor-like factorization fails. □

3.6. **Notes.** The monotone cumulants were defined by Hasebe and Saigo [83], and we basically followed this original paper with more detailed arguments. More general multivariate monotone cumulants are introduced in [82]. The proof of Theorem 3.23 followed [82]. The original definition of monotone cumulants was inspired by "umbral calculus" in combinatorics, in which "umbrae" correspond to i.i.d. copies of random variables, and a "dot operation" corresponds to the sum of i.i.d. random variables. The definition of monotone cumulants builds upon a Lie theoretic approach, which was already exploited by Voiculescu in the definition of free cumulants [144]. The Lie theoretic aspect of cumulants has been further pursued in the literature from Hopf-algebraic viewpoints, see e.g. [8, 57, 81, 106].

The proof of Theorem 3.15 (the monotone CLT) and Theorem 3.16 (the monotone Poisson's law of small numbers) more or less followed the lines of [83], being different from the proof of the original article [116]. Hora, Obata [86, Theorem 8.23] and Saigo [129] proved the monotone CLT allowing some non-identically distributed random variables. Wang analytically proved the monotone CLT by only assuming the finite second moment [147]. Arizmendi, Salazar and Wang provided a Berry-Esseen type result [14]. As for limit theorems other than the central one, Wang and Wendler showed a law of large numbers [148] using a martingale technique; Wang obtained a limit theorem of stable type [146]; Anshelevich and Williams established a rather general limit theorem converging to monotonically infinitely divisible distributions [6]; Franz, Hasebe and Schleißinger studied monotone convolutions of infinitesimal triangular arrays that allow non-identical probability measures [67]. These results can be seen as nontrivial limit theorems for iterated compositions of holomorphic self-maps and some of them have connections to ergodic theory.

The monotone set partitions first appeared in [117] in the form of Proposition 3.28. Lie theoretic approaches can make it clearer how Proposition 3.28 leads to the appearance of monotone set partitions in the moment-cumulant formula in Theorem 3.23, see [81, 106].

4. Cauchy transform

In this section we collect results on the Cauchy transform of probability measures and its relatives. Using these results we extend monotone convolutions to probability measures with unbounded support and analyze them in later sections.

4.1. **Measures.** Let X be a topological space and $\mathcal{B}(X)$ be the set of Borel subsets of X, i.e., $\mathcal{B}(X) \subseteq 2^X$ is the smallest σ-field that contains all open subsets of X. A **Borel measure** is a function $\mu\colon \mathcal{B}(X) \to [0,+\infty]$ such that

- $\mu(\emptyset) = 0$,
- $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty}\mu(B_n)$ whenever $B_1, B_2, B_3, \ldots$ are disjoint Borel subsets of X.

A Borel measure μ on X is called **finite** if the total mass $\mu(X)$ is finite. The set of the finite Borel measures on X is denoted by $\mathbf{M}_{\text{fin}}(X)$. We say that a Borel measure is **locally finite** if every point of X has an open neighborhood with finite mass. The **support** of a Borel measure μ is the smallest closed subset B of X such that $\mu(X \setminus B) = 0$, and the support is denoted as $\operatorname{supp}(\mu)$ if it exists. Note that the support of a Borel measure always exists if X is a separable metric space [56, Problem 3, Section 7.1]. Also, we say that a Borel measure μ is **supported** on B if $\mu(X \setminus B) = 0$. A **complex Borel measure** is a function of the form $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where μ_k are Borel measures. The domain of μ is $\{B \in \mathcal{B}(X) : \mu_k(B) < +\infty, k = 1, 2, 3, 4\}$.

Definition 4.1. Let X be a topological space. A sequence $(\tau_n)_{n\geq 1}$ in $\mathbf{M}_{\text{fin}}(X)$ is said to **weakly converge** to $\tau \in \mathbf{M}_{\text{fin}}(X)$ if for any bounded continuous function $f\colon X \to \mathbb{R}$ one has

$$\lim_{n\to\infty} \int_X f(x)\tau_n(dx) = \int_X f(x)\tau(dx).$$

There is a simple characterization of weak convergence, which is quite useful when combined with tightness.

Lemma 4.2. *Let X be a topological space. Let $\tau, \tau_n \in \mathbf{M}_{\text{fin}}(X)$ $(n \in \mathbb{N})$. Then the weak convergence $\tau = \lim_{n\to\infty} \tau_n$ holds if and only if any subsequence of $(\tau_n)_{n\geq 1}$ has a further subsequence that weakly converges to τ.*

Proof. The "only if" part is obvious. For the "if" part, suppose to the contrary that τ_n does not converge to τ; then there exist a bounded continuous function $f\colon X \to \mathbb{R}$, $\varepsilon > 0$ and a subsequence $(\tau_{n(j)})_{j\geq 1}$ such that $|\int f d\tau_{n(j)} - \int f d\tau| \geq \varepsilon$ for all j. This contradicts the assumption that $(\tau_{n(j)})$ has a further subsequence that converges to τ. □

Definition 4.3. Let X be a topological space and $\mathcal{M} \subseteq \mathbf{M}_{\text{fin}}(X)$.

(i) $\mathcal{M}$ is said to be **tight** if for any $\varepsilon > 0$ there exists a compact subset $K \subseteq X$ such that $\mu(X \setminus K) < \varepsilon$ for all $\mu \in \mathcal{M}$.

(ii) $\mathcal{M}$ is said to be **relatively compact** if any sequence in $\mathcal{M}$ has a further subsequence that is weakly convergent in $\mathbf{M}_{\text{fin}}(X)$.

Remark 4.4. The above definition of relative compactness coincides with the standard definition in topology theory when $\mathbf{M}_{\text{fin}}(X)$ is metrizable, which is the case if X is separable; see e.g. [35, Theorem 5, Appendix III].

Of course if X is a compact space, the whole set $\mathbf{M}_{\text{fin}}(X)$ is tight.

Theorem 4.5 (Prokhorov's theorem)**.** *Let X be a complete separable metric space and $\mathcal{M}$ be a subset of $\mathbf{M}_{\text{fin}}(X)$. Then $\mathcal{M}$ is relatively compact if and only if $\mathcal{M}$ is tight and $\{\tau(X) : \tau \in \mathcal{M}\} \subseteq [0, \infty)$ is bounded.*

Proof. For the case of probability measures, the reader is referred to [35, Theorems 6.1 and 6.2]. For finite Borel measures, dividing the measures by their total masses reduces the problem to probability measures. □

Remark 4.6. Actually we will use Theorem 4.5 only for $X = \mathbb{R}$ or circles in the plane. Then this theorem is a direct consequence of Helly's selection theorem, see e.g. [93, Theorem 13.33].

In the definition of weak convergence, we can take a smaller set of test functions provided that the total mass converges and the topological space is good enough. For the proof of the following result the reader is referred to [93, Theorem 13.16].

Proposition 4.7. *Let X be a locally compact, complete separable metric space and τ_n, τ $(n = 1, 2, 3, ...)$ be finite Borel measures on X such that $\lim_{n\to\infty} \tau_n(X) = \tau(X)$. Then τ_n converges weakly to τ if and only if*

$$\lim_{n\to\infty} \int_X f(x)\tau_n(dx) = \int_X f(x)\tau(dx)$$

holds for any continuous function f with compact support.

The definition of $\mathcal{B}(X)$ does not specify how to determine whether a given subset of X is a Borel subset or not. In measure theory and probability theory, one often sees such subsets, e.g. Lebesgue measurable subsets or elements of the direct product of σ-fields. When discussing such subsets, a standard technique is to consider a class of subsets rather than individual subsets. The following classes are helpful.

Definition 4.8. Let Ω be a set and $\mathcal{F}$ be a nonempty subset of 2^Ω.

(i) $\mathcal{F}$ is called a **π-system** if $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

(ii) $\mathcal{F}$ is called a **λ-system** if the following conditions are satisfied:

- $\Omega \in \mathcal{F}$;
- if $A, B \in \mathcal{F}$ and $A \subseteq B$ then $B \setminus A \in \mathcal{F}$;
- if $A_n \in \mathcal{F}$ $(n \in \mathbb{N})$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

(iii) $\mathcal{F}$ is called an **algebra** if the following conditions are satisfied:
- if $\Omega \in \mathcal{F}$;
- if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$;
- if $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$.

(iv) $\mathcal{F}$ is called a **monotone class** if the following conditions are satisfied:
- if $A_n \in \mathcal{F}$ $(n \in \mathbb{N})$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$;
- if $A_n \in \mathcal{F}$ $(n \in \mathbb{N})$ and $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ then $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

The following π-λ theorem and monotone class theorem are standard methods to prove statements concerning Borel subsets or other classes of subsets; the reader is referred to [93, Theorem 1.19] and [56, Theorem 4.4.2] for the proofs, respectively.

Theorem 4.9 (π-λ theorem)**.** *Let Ω be a set. Let $\mathcal{P}$ be a π-system and $\mathcal{L}$ be a λ-system of subsets of Ω such that $\mathcal{P} \subseteq \mathcal{L}$. Then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.*

Theorem 4.10 (Monotone class theorem)**.** *Let Ω be a set. Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be a monotone class of subsets of Ω such that $\mathcal{A} \subseteq \mathcal{M}$. Then $\sigma(\mathcal{A}) \subseteq \mathcal{M}$.*

A typical application of π-λ theorem is the following.

Proposition 4.11. *Let τ_1, τ_2 be locally finite Borel measures on $\mathbb{R}$ such that $\tau_1(I) = \tau_2(I)$ for all open intervals I of finite length. Then $\tau_1 = \tau_2$.*

Proof. The goal is to show $\tau_1(B) = \tau_2(B)$ for all $B \in \mathcal{B}(\mathbb{R})$. Since

$$\tau_i(B) = \lim_{N\to\infty} \tau_i(B \cap (-N, N)), \qquad i = 1, 2,$$

it suffices to show $\tau_1(B) = \tau_2(B)$ for bounded Borel subsets B. We therefore fix $N \in \mathbb{N}$. Let us consider the set $\mathcal{I} \subseteq 2^{(-N,N)}$ consisting of the empty set and the open subintervals of $(-N, N)$, and

$$\mathcal{L} := \{B \in \mathcal{B}((-N, N)) : \tau_1(B) = \tau_2(B)\}.$$

We can see that $\mathcal{I}$ is a π-system, $\mathcal{L}$ is a λ-system and, by assumption, $\mathcal{I}$ is contained in $\mathcal{L}$. By the π-λ theorem, $\sigma(\mathcal{I})$ is contained in $\mathcal{L}$, the former of which is known to be equal to $\mathcal{B}((-N, N))$. ☐

Remark 4.12. We can also use the monotone class theorem. Observe first that the above $\mathcal{L}$ is also a monotone class. Instead of $\mathcal{I}$ we consider the algebra $\mathcal{A} \subseteq 2^{(-N,N)}$ consisting of the empty set and finite disjoint unions of the intervals of the forms $(a_i, b_i]$ $(-N \le a_i < b_i \le N)$; note that $(a_i, N]$ is to be interpreted as (a_i, N). By taking limits we can see that each $(a_i, b_i]$ belongs to $\mathcal{L}$, and therefore $\mathcal{A} \subseteq \mathcal{L}$. By the monotone class theorem, $\mathcal{L}$ contains $\sigma(\mathcal{A}) = \mathcal{B}((-N, N))$.

Sometimes, finding an appropriate algebra is harder than finding a π-system, and therefore π-λ theorem is more useful. Later in Theorem 6.11 we also see the opposite situation where the the second condition of the λ-system is hard to check, and thus monotone class theorem is more useful.

4.2. **Holomorphic functions.** Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{C}^+$ the complex upper half-plane $\{z \in \mathbb{C} : \Im(z) > 0\}$. We consider $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ as a compact subset of the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Definition 4.13. Let X be a topological space. Let $f, f_n \colon X \to \mathbb{C}$ $(n \in \mathbb{N})$ be functions. We say that f_n **converges to f locally uniformly** if each point of X has a neighborhood on which f_n converges to f uniformly. We say that f_n **converges to f uniformly on compacta** if f_n converges to f uniformly on each compact subset of X. If X is an open subset of $\mathbb{C}$ then these two notions coincide.

The same idea of Lemma 4.2 works to the uniform convergence on compacta.

Lemma 4.14. *Let X be a topological space and let $f, f_n \colon X \to \mathbb{C}$ $(n \in \mathbb{N})$. Then f_n converges to f uniformly on compacta if and only if any subsequence of $(f_n)_{n\ge1}$ has a further subsequence that converges to f uniformly on compacta.*

When proving the convergence of a sequence of holomorphic functions, the following criterion is useful.

Theorem 4.15 (Vitali)**.** *Let $D \subseteq \mathbb{C}$ be a domain. Let $f_n \colon D \to \mathbb{C}$ $(n = 1, 2, 3, ...)$ be holomorphic functions that are uniformly bounded on each compact subset of D. Suppose that there is a sequence of distinct points $(z_k)_{k\ge1} \subseteq D$ such that $\lim_{k\to\infty} z_k \in D$ and $\lim_{n\to\infty} f_n(z_k)$ exists in $\mathbb{C}$ at all $k \in \mathbb{N}$. Then f_n converges to a holomorphic function locally uniformly on D.*

Proof. By Montel's theorem, (f_n) has a subsequence that converges locally uniformly to a holomorphic function f on D.

Let us apply Lemma 4.14. Take any subsequence $(f_{n(k)})_{k\geq 1}$. Again by Montel's theorem, it has a further subsequence $(f_{n(k(j))})_{j\geq 1}$ that converges to a holomorphic function g locally uniformly on D. By the assumption that $\lim_{n\to\infty} f_n(z_k) \in \mathbb{C}$ exists, we conclude that $f(z_k) = g(z_k)$ for all $k \in \mathbb{N}$. The identity theorem forces f and g to coincide. Therefore, $(f_{n(k(j))})_{j\geq 1}$ converges to f locally uniformly. Lemma 4.14 yields that the whole sequence $(f_n)_{n\geq 1}$ converges to f on D. □

Here we note a useful criterion that allows us to differentiate a holomorphic function under the integral sign. This fact will be used below without being mentioned.

Proposition 4.16. *Let $(T, \mathcal{F}, \mu)$ be a measure space. Let D be an open subset of $\mathbb{C}$. Let $f\colon D \times T \to \mathbb{C}$ be a function such that*

- *for a.e. $t \in T$ the function $f(\cdot, t)$ is holomorphic on D,*
- *for each $z \in D$ the function $f(z, \cdot)$ is μ-integrable,*
- *there is a μ-integrable function $g\colon T \to [0,\infty)$ such that $|f(z,t)| \leq g(t)$ for all $z \in D, t \in T$.*

Then the function $F(z) := \int_T f(z,t)\,\mu(dt)$ is holomorphic on D and $F^{(n)}(z) = \int_T \partial_z^n f(z,t)\,\mu(dt)$ for all $n \in \mathbb{N}$. Note that the assumptions above imply that $\partial_z^n f(z,\cdot)$ is μ-integrable for any $z \in D$ and $n \in \mathbb{N}_0$.

Proof. Let C be a circle $z_0 + re^{i\theta}, 0 \leq \theta \leq 2\pi$ sufficiently small so that C and its interior are contained in D. By Cauchy's integral formula we have

$$\partial_z f(z,t) = \frac{1}{2\pi i}\int_C \frac{f(w,t)}{(w-z)^2}\,dw.$$

This yields the estimate

$$|\partial_z f(z,t)| \leq \frac{1}{2\pi}\int_C \frac{g(t)}{|w-z|^2}\,|dw| \leq \frac{4g(t)}{r}, \qquad |z - z_0| < \frac{r}{2}. \tag{4.1}$$

Writing $f(z,t) = u(x,y,t) + iv(x,y,t)$ and $F(z) = U(x,y) + iV(x,y)$ with notation $z = x+iy$, the above estimate implies that the four functions $|\partial_x u|, |\partial_y u|, |\partial_x v|, |\partial_y v|$ are all bounded by $4g(t)/r$. Therefore, the usual criterion for the interchange of differentiation and integration yields that $U(x,y), V(x,y)$ are differentiable under the integral sign with respect to both x and y. Also we can check that U, V are C^1 functions by the dominated convergence theorem. This argument implies that the C^1 functions U, V satisfy the Cauchy-Riemann equations and hence F is holomorphic and the desired formula $F'(z) = \int_T \partial_z f(z,t)\,\mu(dt)$ holds.

For the higher-order derivatives, as we have established (4.1), the function $\partial_z f$ also satisfies the assumptions of the proposition, so that we can obtain the result for the second derivative. Repeating the above arguments we obtain the desired formula for higher-order derivatives. □

We introduce classes of holomorphic functions.

Definition 4.17. A holomorphic function $H\colon \mathbb{D} \to \{z \in \mathbb{C} : \Re(z) \geq 0\}$ is called a **Herglotz** function. A holomorphic function $N\colon \mathbb{C}^+ \to \mathbb{C}^+ \cup \mathbb{R}$ is called a **Nevanlinna function**.

Nevanlinna functions play central roles below. Upon fixing a conformal bijection $\psi\colon \mathbb{D} \to \mathbb{C}^+$, we have the bijection $N \mapsto -iN \circ \psi$ between the sets of Nevanlinna functions and Herglotz functions. Sometimes Herglotz functions make arguments clearer so we work with them instead.

Definition 4.18. For a function $f\colon \mathbb{C}^+ \to \mathbb{C}$ we say that f has a **nontangential limit** $\zeta \in \mathbb{C} \cup \{\infty\}$ at ∞ if for any $\gamma > 0$ we have

$$\lim_{\substack{z\in\nabla_\gamma \\ |z|\to\infty}} f(z) = \zeta, \tag{4.2}$$

where ∇_γ is the sector domain

$$\nabla_\gamma := \{z \in \mathbb{C}^+ : \gamma|\Re(z)| < \Im(z)\}.$$

The nontangential limit of f at ∞ is written as $\sphericalangle \lim_{z\to\infty} f(z)$ if it exists.

There is a conformal bijection of $\mathbb{C}^+$ that maps ∞ to $a \in \mathbb{R}$. This allows us to define a nontangential limit at a: f has a nontangential limit $\zeta \in \mathbb{C} \cup \{\infty\}$ at $a \in \mathbb{R}$ if for any $\gamma > 0$

$$\lim_{\substack{z\in a+\nabla_\gamma \\ z\to a}} f(z) = \zeta,$$

and we write $\zeta = \sphericalangle \lim_{z\to a} f(z)$.

In general, even if (4.2) exists for some $\gamma > 0$, the limit might not exist for smaller γ's. A remarkable fact is that such never happens for a large class of holomorphic functions: if (4.2) exists for some $\gamma > 0$, the limit exists for any $\gamma > 0$. This is a consequence of Lindelöf's theorem. Although this theorem is not essential below, we quote a version of Lindelöf's theorem for Nevanlinna functions as it helps to better understand some results. For the proof we refer the reader to [39, Theorem 1.5.7] or [44, Theorem 2.3]. A stronger version can be found in [44, Theorem 2.20].

Theorem 4.19 (Lindelöf)**.** *Let N be a Nevanlinna function. If there exists a continuous map $\gamma\colon [0,1) \to \mathbb{C}^+$ such that $\lim_{t\to 1}\gamma(t) = \infty$ and $\zeta := \lim_{t\to 1} N(\gamma(t)) \in \mathbb{C}^+ \cup \widehat{\mathbb{R}}$ exists, then the nontangential limit of N at ∞ exists and equals ζ.*

4.3. **Nevanlinna functions.** We collect various properties of Nevanlinna functions. We first demonstrate an integral formula for Nevanlinna functions. For this it is convenient to work with Herglotz functions first.

Lemma 4.20 (Poisson integral formula)**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function. For every $R \in (0,1)$ it holds that*

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re\left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right) f(Re^{i\phi})\,d\phi, \qquad |z| < R. \tag{4.3}$$

Proof. Let $z = re^{i\theta}$ with $0 \le r < R$. By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)}{w-z}\,dw. \tag{4.4}$$

Let $z^* := \frac{R^2}{r}e^{i\theta}$ called the reflection of z with respect to the circle $\{w : |w| = R\}$. Since $|z^*| > R$, Cauchy's integral theorem yields

$$0 = \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)}{w-z^*}\,dw. \tag{4.5}$$

Combining (4.4) and (4.5) gives

$$f(z) = \frac{1}{2\pi i}\int_{|w|=R} f(w)\left(\frac{1}{w-z} - \frac{1}{w-z^*}\right)dw.$$

With notation $w = Re^{i\phi}$ we obtain

$$\frac{1}{w-z} - \frac{1}{w-z^*} = \frac{R^2-r^2}{Re^{i\phi}(R^2 - 2rR\cos(\theta-\phi)+r^2)} = \frac{1}{Re^{i\phi}}\Re\left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right).$$

The conclusion follows by observing $dw = iRe^{i\phi}d\phi$. ☐

Proposition 4.21. *For a Herglotz function f there exist $b \in \mathbb{R}$ and a finite Borel measure σ on $\mathbb{T}$ such that*

$$f(z) = ib + \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,\sigma(d\zeta), \qquad z \in \mathbb{D}. \tag{4.6}$$

Proof. Let $0 < R < 1$ and g be a holomorphic function defined by

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Re^{i\phi}+z}{Re^{i\phi}-z}\Re[f(Re^{i\phi})]\,d\phi, \qquad |z| < R.$$

Since f has a representation in Lemma 4.20 we have $\Re[g(z)] = \Re[f(z)]$ for $|z| < R$. It is a well known consequence of the Cauchy–Riemann relations that a holomorphic function on a domain with a constant real part must be constant, which implies in our situation that $f(z) = g(z) + ib$ for some constant $b \in \mathbb{R}$. Therefore,

$$f(Rz) = ib + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\phi}+z}{e^{i\phi}-z}\Re[f(Re^{i\phi})]\,d\phi = ib + \int_{\mathbb{T}} \frac{w+z}{w-z}\,\sigma_R(dw), \qquad z \in \mathbb{D}, \tag{4.7}$$

where σ_R is the finite Borel measure on $\mathbb{T}$ defined by $\sigma_R(d\phi) := \frac{1}{2\pi}\Re[f(Re^{i\phi})]d\phi$. Selecting $z = 0$ in (4.7) we obtain $\sigma_R(\mathbb{T}) = \Re[f(0)]$, which implies that the family $\{\sigma_R(\mathbb{T}) \mid 0 < R < 1\} \subseteq [0,\infty)$ is bounded. Since $\mathbb{T}$ is compact, the family $\{\sigma_R \mid 0 < R < 1\}$ is tight. By Theorem 4.5, there exists a sequence $\{R_n\}_{n\ge 1}$ with $R_n \uparrow 1$ and a finite Borel measure σ such that $\sigma = \lim_{n\to\infty}\sigma_{R_n}$ weakly. Setting $R = R_n$ in (4.7) and letting $n \to \infty$ amounts to the desired (4.6). ☐

Remark 4.22. The number $b \in \mathbb{R}$ and the finite Borel measure σ are unique. We will prove this in the next theorem for the equivalent setting of Nevanlinna functions.

Theorem 4.23 (Nevanlinna formula)**.** *For a Nevanlinna function N, there exist $a \ge 0, b \in \mathbb{R}$ and a finite Borel measure τ on $\mathbb{R}$ such that*

$$N(z) = az - b + \int_{\mathbb{R}} \frac{1+tz}{t-z}\,\tau(dt) = -b + \int_{\widehat{\mathbb{R}}} \frac{1+tz}{t-z}\,\widehat{\tau}(dt), \tag{4.8}$$

where $\widehat{\tau}$ is the finite Borel measure on $\widehat{\mathbb{R}}$ defined by $\widehat{\tau}|_{\mathbb{R}} = \tau$ and $\widehat{\tau}(\{\infty\}) = a$, and $(1+tz)/(t-z)$ is set to be z at $t = \infty \in \widehat{\mathbb{R}}$. The triplet (a, b, τ) is uniquely determined as follows.

(i) $a = \sphericalangle \lim_{z\to\infty} \dfrac{N(z)}{z}$.

(ii) $b = -\Re[N(i)]$.

(iii) *The Borel measure $\rho(dt) = (1+t^2)\tau(dt)$ satisfies for each $-\infty < \alpha < \beta < \infty$*

$$\rho((\alpha,\beta)) + \frac{1}{2}(\rho(\{\alpha\}) + \rho(\{\beta\})) = \frac{1}{\pi}\lim_{y\to 0^+}\int_\alpha^\beta \Im[N(x+iy)]\,dx, \tag{4.9}$$

$$\rho(\{\alpha\}) = \lim_{y\to 0^+}(-iy)N(\alpha+iy), \tag{4.10}$$

*which will be referred to as the **Stieltjes inversion formula**.*

Remark 4.24. The pointwise limit (at each x) of $\Im[N(x+iy)]$ as $y \to 0^+$ will provide further information about ρ, e.g., about the Lebesgue decomposition of ρ, see [133, Appendix F].

Proof. The function

$$\psi(z) = \frac{-iz+1}{z-i}$$

is a homeomorphism from $\overline{\mathbb{D}}$ onto $\mathbb{C}^+ \cup \widehat{\mathbb{R}}$ with inverse

$$\psi^{-1}(z) = \frac{iz+1}{z+i}.$$

By Proposition 4.21 the function $-iN\circ\psi$ has a representation

$$-iN(\psi(w)) = ib + \int_{-\pi}^{\pi}\frac{e^{i\phi}+w}{e^{i\phi}-w}\,\sigma(d\phi) = ib + \int_{\widehat{\mathbb{R}}}\frac{\psi^{-1}(t)+w}{\psi^{-1}(t)-w}\,\widehat{\tau}(dt), \qquad w\in\mathbb{D}, \tag{4.11}$$

where $\widehat{\tau} = \sigma\circ\psi^{-1}|_{\widehat{\mathbb{R}}}$ is the push-forward measure. The variable $z = \psi(w)$ satisfies

$$\frac{\psi^{-1}(t)+\psi^{-1}(z)}{\psi^{-1}(t)-\psi^{-1}(z)} = -\frac{i(1+tz)}{t-z},$$

which transforms (4.11) into the desired (4.8).

(ii) is immediate.

(i) It suffices to show $\sphericalangle\lim_{z\to\infty} R(z)/z = 0$, where $R(z) := \int_{\mathbb{R}}\frac{1+tz}{t-z}\tau(dt)$. Let $t\in\mathbb{R}$, $\gamma>0$, $z = x+iy \in \nabla_\gamma$ with $y \geq 1$. Then

$$\left|\frac{1+tz}{z(t-z)}\right| = \left|\frac{1}{z(t-z)} + \frac{z}{t-z} + 1\right| \leq 1 + \left|\frac{z}{t-z}\right|\left(1+\frac{1}{|z|^2}\right) \leq 1 + 2\left|\frac{z}{t-z}\right|.$$

Since

$$\left|\frac{z}{z-t}\right| = \sqrt{\frac{x^2+y^2}{(x-t)^2+y^2}} \leq \sqrt{\frac{\gamma^{-2}y^2+y^2}{y^2}} = \sqrt{1+\gamma^{-2}}, \tag{4.12}$$

the function $\left|\frac{1+tz}{z(t-z)}\right|$ can be uniformly bounded by a constant independent of (t,z). By the dominated convergence theorem, $\lim_{z\to\infty, z\in\nabla_\gamma} R(z)/z = 0$.

(iii) First observe that

$$\frac{1+tz}{t-z} = \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)(1+t^2), \tag{4.13}$$

which is sometimes useful. Now this formula immediately implies

$$\begin{aligned}\Im[N(x+iy)] &= ay + \int_{\mathbb{R}}\frac{y}{(x-t)^2+y^2}\,\rho(dt)\\ &= ay + \underbrace{\int_{(\alpha-1,\beta+1)}\frac{y}{(x-t)^2+y^2}\,\rho(dt)}_{=:I_1(x,y)} + \underbrace{\int_{(-\infty,\alpha-1]\cup[\beta+1,\infty)}\frac{y}{(x-t)^2+y^2}\,\rho(dt)}_{=:I_2(x,y)}.\end{aligned}$$

Since I_2 is continuous on $[\alpha,\beta]\times[0,\infty)$ and vanishes on $y=0$, we have $\lim_{y\to 0^+}\int_{(\alpha,\beta)} I_2(x,y)\,dx = 0$. Concerning I_1, we use Tonelli's theorem to interchange the integrals

$$\begin{aligned}\lim_{y\to 0^+}\int_{(\alpha,\beta)} I_1(x,y)\,dx &= \lim_{y\to 0^+}\int_{(\alpha-1,\beta+1)}\rho(dt)\int_{(\alpha,\beta)}\frac{y}{(x-t)^2+y^2}\,dx \\ &= \lim_{y\to 0^+}\int_{(\alpha-1,\beta+1)}\left(\arctan\frac{\beta-t}{y}-\arctan\frac{\alpha-t}{y}\right)\rho(dt).\end{aligned}$$

Since

$$\lim_{y\to 0^+}\left(\arctan\frac{\beta-t}{y}-\arctan\frac{\alpha-t}{y}\right)=\begin{cases}0, & t\in(-\infty,\alpha)\cup(\beta,\infty),\\ \pi, & t\in(\alpha,\beta),\\ \frac{\pi}{2}, & t=\alpha,\beta,\end{cases}$$

the desired (4.9) follows by the dominated convergence theorem.

As for (4.10), due to the calculations

$$\begin{aligned}iyN(\alpha+iy) &= aiy(\alpha+iy)-iyb+iy\int_{\mathbb{R}}\left[\frac{1}{t-(\alpha+iy)}-\frac{t}{1+t^2}\right]\rho(dt)\\ &= o(1)-\rho(\{\alpha\})+\int_{\mathbb{R}\setminus\{\alpha\}} iy\left[\frac{(t-\alpha)+iy}{(t-\alpha)^2+y^2}-\frac{t}{1+t^2}\right]\rho(dt),\end{aligned}$$

it remains to show that the integral converges to zero as $y\to 0^+$. We split the integral region $\mathbb{R}\setminus\{\alpha\}$ into $J_1 := \{t\in\mathbb{R}: 0<|t-\alpha|<1\}$ and $J_2:=\{t\in\mathbb{R}:|t-\alpha|\ge 1\}$. The integral over J_1 tends to zero by the dominated convergence theorem because $\rho(J_1)<+\infty$ and for every $y\in(0,1)$, $t\in J_1$ we have

$$\begin{aligned}y\left|\frac{(t-\alpha)+iy}{(t-\alpha)^2+y^2}-\frac{t}{1+t^2}\right| &\le \frac{y|t-\alpha|+y^2}{(t-\alpha)^2+y^2}+\frac{|t|y}{1+t^2}\\ &\le \frac{\frac{1}{2}(y^2+(t-\alpha)^2)+y^2}{(t-\alpha)^2+y^2}+\frac{t^2+y^2}{2(1+t^2)}\le 2.\end{aligned}$$

The integral over J_2 also converges to zero by the dominated convergence because there is a constant $C>0$ such that for all $t\in J_2$ and $y\in(0,1)$

$$y\left|\frac{(t-\alpha)+iy}{(t-\alpha)^2+y^2}-\frac{t}{1+t^2}\right|\le\frac{C}{1+t^2}. \qquad \square$$

Finally, we verify the uniqueness of τ. Suppose that (a,b,τ') is another triplet. By the Stieltjes inversion formula, the measure $\rho'(dt):=(1+t^2)\tau'(dt)$ satisfies $\rho'(I)=\rho(I)$ for all open intervals I of finite length. Proposition 4.11 yields $\rho'=\rho$, and therefore, $\tau'=\tau$.

In many examples the Stieltjes inversion formula is used in the following form.

Corollary 4.25. *Let N be a Nevanlinna function with triplet (a,b,τ). If N extends to a continuous function $\tilde{N}\colon \mathbb{C}^+\cup[\alpha,\beta]\to\mathbb{C}^+\cup\mathbb{R}$ for some $-\infty<\alpha<\beta<\infty$, then the measure $\tau|_{[\alpha,\beta]}$ is Lebesgue absolutely continuous and its density is given by*

$$\frac{1}{\pi(1+t^2)}\Im[\tilde{N}(t)], \qquad t\in[\alpha,\beta].$$

Here we characterize the convergence of Nevanlinna functions.

Proposition 4.26. *Let $N_n, n=1,2,3,\dots$ be Nevanlinna functions with representations*

$$N_n(z) = -b_n+\int_{\widehat{\mathbb{R}}}\frac{1+tz}{t-z}\,\widehat{\tau}_n(dt).$$

The following statements are equivalent.

1. *N_n converges to a function N locally uniformly on $\mathbb{C}^+$.*
2. *There is a sequence of distinct points $(z_k)_{k\ge1}$ that converges to a point $z_\infty\in\mathbb{C}^+$, and $\lim_{n\to\infty}N_n(z_k)$ exists in $\mathbb{C}$ at all $k\in\mathbb{N}$.*
3. *the sequence $(b_n)_{n\ge1}$ converges to some $b\in\mathbb{R}$ and $(\widehat{\tau}_n)_{n\ge1}$ weakly converges to some finite Borel measure $\widehat{\tau}$ on $\widehat{\mathbb{R}}$.*

Moreover, if the above equivalent conditions hold, then the limit function N is the Nevanlinna function given by

$$N(z) = -b+\int_{\widehat{\mathbb{R}}}\frac{1+tz}{t-z}\,\widehat{\tau}(dt).$$

Proof. (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (3). By performing an affine transformation $z \mapsto pz+q$ ($p>0, q \in \mathbb{R}$), we may assume that $z_1 = i$. Since $N_n(i) = -b_n + i\widehat{\tau}_n(\widehat{\mathbb{R}})$, the sequence $(b_n)_{n\ge 1}$ converges to $b \in \mathbb{R}$ and the total mass $(\widehat{\tau}_n(\widehat{\mathbb{R}}))_{n\ge1}$ converges to a finite nonnegative number. By the compactness of $\widehat{\mathbb{R}}$ and by Theorem 4.5, the sequence $(\widehat{\tau}_n)_{n\ge1}$ has a weakly convergent subsequence $(\widehat{\tau}_{n'})$, whose limit is denoted by $\widehat{\tau}$. Let N be the Nevanlinna function determined by the pair $(b,\widehat{\tau})$. The definition of weak convergence implies that for each $z \in \mathbb{C}^+$

$$N_{n'}(z) = -b_{n'} + \int_{\widehat{\mathbb{R}}} \frac{1+tz}{t-z}\,\widehat{\tau}_{n'}(dt) \to -b + \int_{\widehat{\mathbb{R}}} \frac{1+tz}{t-z}\,\widehat{\tau}(dt) = N(z) \quad \text{as} \quad n' \to \infty. \tag{4.14}$$

We now take any subsequence $(\widehat{\tau}_{n(j)})_{j\ge1}$ of $(\widehat{\tau}_n)_{n\in\mathbb{N}}$. By the same reasoning as above, it has a further subsequence $(\widehat{\tau}_{n(j(k))})_{k\ge1}$ that converges to a finite Borel measure $\tilde{\tau}$. Denoting by $\tilde{N}$ the Nevanlinna function corresponding to $(b,\tilde{\tau})$, we obtain $N_{n(j(k))}(z) \to \tilde{N}(z)$ in the same way as (4.14). Therefore, for any $k \in \mathbb{N}$, we have $N(z_k) = \lim_{n\to\infty} N_n(z_k) = \tilde{N}(z_k)$. By the identity theorem we have $N = \tilde{N}$ on $\mathbb{C}^+$ and hence, by the uniqueness of the Nevanlinna formula, $\tilde{\tau} = \widehat{\tau}$. Lemma 4.2 implies the convergence of the whose sequence $\widehat{\tau}_n \to \widehat{\tau}$ as $n \to \infty$.

(3) $\Longrightarrow$ (1) The pointwise convergence $N_n(z) \to N(z)$ follows by the definition of weak convergence as $\widehat{\mathbb{R}} \ni t \mapsto \frac{1+tz}{t-z} \in \mathbb{C}^+$ is bounded and continuous. Moreover, for each compact subset $K \subseteq \mathbb{C}^+$, the function $(t,z) \mapsto (1+tz)/(t-z)$ is bounded on $\widehat{\mathbb{R}} \times K$, and so N_n is uniformly bounded on K. By Vitali's theorem (Theorem 4.15), the convergence $N_n \to N$ holds locally uniformly. ☐

Remark 4.27. Be aware that even if a sequence of Nevanlinna functions converges, the triplets in (4.8) might fail to converge to that of the limit function. Take for example the triplet $(0,0,\delta_n), n \in \mathbb{N}$. Then

$$N_n(z) := \frac{1+zn}{n-z} \to N(z) := z$$

while the limit function has triplet $(1,0,0)$. By contrast, the pair $(b,\widehat{\tau})$ works perfectly with respect to the convergence. In the above example, as finite Borel measures on $\widehat{\mathbb{R}}$, the convergence $\delta_n \to \delta_\infty$ holds.

Here is a technical lemma on Nevanlinna functions to be used in later sections.

Lemma 4.28. *Let I be an open interval of $\mathbb{R}$ and $N\colon I \times \mathbb{C}^+ \to \mathbb{C}^+ \cup \mathbb{R}$ be a function such that*

- *for each $z \in \mathbb{C}^+$, the map $t \mapsto N(t,z)$ is continuous,*
- *for each $t \in I$, the map $z \mapsto N(t,z)$ is a Nevanlinna function.*

Then $\frac{\partial^k N}{\partial z^k}$ is continuous on $I \times \mathbb{C}^+$ for every $k \in \mathbb{N}_0$.

Proof. For each $t \in I$ we have the formula

$$N(t,z) = a_t z - b_t + \int_{\mathbb{R}} \frac{1+xz}{x-z}\,\tau_t(dx).$$

Since $-b_t + i[a_t + \tau_t(\mathbb{R})] = \Re[N(t,i)]$ is a continuous function of $t \in I$, the functions $t \mapsto b_t, a_t, \tau_t(\mathbb{R}) \in \mathbb{R}$ are all bounded on a compact subinterval $J \subseteq I$. This implies that $N(t,z)$ is uniformly bounded on $J \times K$ for any compact $K \subseteq \mathbb{C}^+$.

We fix a point $(t,z) \in I \times \mathbb{C}^+$ and take a sequence $(t_n, z_n), n \in \mathbb{N}$, converging to (t,z). We choose a compact interval $J \subseteq I$ containing t, t_n and a compact subset $K \subseteq \mathbb{C}^+$ containing z, z_n, and further take a smooth simple closed curve C that surrounds K. By Cauchy's integral formula, we have

$$\frac{\partial^k N}{\partial z^k}(t_n, z_n) = \frac{1}{2\pi i k!} \int_C \frac{N(t_n,w)}{(w-z_n)^{k+1}}\,dw.$$

As $N(t_n,w)$ is uniformly bounded, the dominated convergence theorem allows us to conclude $\partial_z^k N(t_n,z_n) \to \partial_z^k N(t,z)$. ☐

4.4. **Cauchy transform and its relatives.** Let x be a real random variable in a unital C^*-probability space (A,φ). In Theorem 1.27 we encountered the shifted moment generating function $M_x(z)$ that can be written as

$$M_x(z) = \sum_{n=0}^{\infty} \varphi(x^n) z^{n+1} = \int_{\mathbb{R}} \frac{z}{1-zt}\,\mu_x(dt).$$

Replacing the variable z with $1/z$ gives a function called the Cauchy transform, which is widely used in noncommutative probability. In this section we also introduce some other related functions.

Definition 4.29. Let μ be a finite Borel measure on $\mathbb{R}$. The function

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\,\mu(dt), \qquad z \in \mathbb{C}^+$$

is called the **Cauchy transform** of μ; sometimes it is called the Stieltjes transform or Borel transform. The function $F_\mu(z) := 1/G_\mu(z)$ is called the **reciprocal Cauchy transform** of μ.

Proposition 4.30. *For a finite Borel measure μ on $\mathbb{R}$ and $-\infty < \alpha < \beta < \infty$, one has*

$$\mu((\alpha,\beta)) + \frac{1}{2}(\mu(\{\alpha\}) + \mu(\{\beta\})) = -\frac{1}{\pi}\lim_{y\to 0^+}\int_\alpha^\beta \Im[G_\mu(x+iy)]\,dx,$$
$$\mu(\{\alpha\}) = \lim_{y\to 0^+} iyG_\mu(\alpha+iy).$$

In particular, the map $\mu \mapsto G_\mu$ is injective.

Proof. The formula

$$-G_\mu(z) = -b + \int_{\mathbb{R}} \frac{1+tz}{t-z}\cdot\frac{\mu(dt)}{1+t^2}, \qquad b := -\int_{\mathbb{R}} \frac{t}{1+t^2}\,\mu(dt),$$

gives a Nevanlinna formula for $-G_\mu$ and allows us to apply Theorem 4.23(iii). The injectivity follows from the uniqueness part of the measure τ in Theorem 4.23. ☐

The following is a consequence of Corollary 4.25.

Corollary 4.31. *For a finite Borel measure μ on $\mathbb{R}$ and $-\infty < \alpha < \beta < \infty$, suppose that G_μ extends to a continuous function $\tilde{G}_\mu\colon \mathbb{C}^+ \cup [\alpha,\beta] \to (-\mathbb{C}^+)\cup\mathbb{R}$. Then $\mu|_{[\alpha,\beta]}$ is Lebesgue absolutely continuous and its density is given by $-\frac{1}{\pi}\Im[\tilde{G}_\mu(t)], t \in [\alpha,\beta]$.*

The Cauchy transform is characterized as follows.

Proposition 4.32. *Let $G\colon \mathbb{C}^+ \to (-\mathbb{C}^+)\cup\mathbb{R}$ be a holomorphic function. Then the following statements are equivalent.*

(1) *$G = G_\mu$ for some probability measure μ.*
(2) *$\lim_{y\to\infty} iyG(iy) = 1$.*
(3) *$\sphericalangle\lim_{z\to\infty} zG(z) = 1$.*

Proof. (1) $\Longrightarrow$ (3). By (4.12) and the dominated convergence theorem

$$zG_\mu(z) = \int_{\mathbb{R}} \frac{z}{z-t}\,\mu(dt) \to 1, \qquad z\to\infty, z \in \nabla_\gamma.$$

(3) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (1). From the Nevanlinna formula for $-G$ we have

$$G(z) = -az + b + \int_{\mathbb{R}} \frac{1+tz}{z-t}\,\tau(dt),$$

where $a \geq 0, b\in\mathbb{R}$ and a finite Borel measure τ on $\mathbb{R}$. Moreover,

$$\Re[iyG(iy)] = ay^2 + \int_{\mathbb{R}} \frac{y^2}{y^2+t^2}(1+t^2)\,\tau(dt).$$

From assumption (2) it must hold that $a = 0$, and from $\frac{y^2}{y^2+t^2} \uparrow 1$ $(y\uparrow\infty)$ the monotone convergence theorem yields

$$1 = \lim_{y\to\infty}\Re[iyG(iy)] = \int_{\mathbb{R}} (1+t^2)\,\tau(dt).$$

Therefore, the measure $\mu(dt) = (1+t^2)\tau(dt)$ is a probability measure on $\mathbb{R}$. The function G can be expressed in the form

$$G(z) = b + \int_{\mathbb{R}}\left(\frac{1}{z-t} + \frac{t}{1+t^2}\right)\mu(dt) = G_\mu(z) + b + \int_{\mathbb{R}}\frac{t}{1+t^2}\,\mu(dt),$$

which, together with assumption (2), implies $b + \int_{\mathbb{R}} \frac{t}{1+t^2}\,\mu(dt) = 0$ and hence $G = G_\mu$. ☐

Proposition 4.33. *Let $\mu, \mu_n, n \in \mathbb{N}$ be probability measures on $\mathbb{R}$. The following are equivalent.*

(1) *$\mu_n \to \mu$ weakly;*
(2) *G_{μ_n} converges to G_μ locally uniformly on $\mathbb{C}^+$;*

(3) *there exists a sequence of distinct points* $(z_k)_{k=1}^\infty \subset \mathbb{C}^+$ *such that* $\lim_{k\to\infty} z_k$ *exists in* $\mathbb{C}^+$, *and* $\lim_{n\to\infty} G_{\mu_n}(z_k) = G_\mu(z_k)$ *for every* $k \in \mathbb{N}$.

Proof. (2) $\iff$ (3) follows from Proposition 4.26 (1) and (2).

(1) $\implies$ (3) is obvious from the definition of weak convergence.

(2) $\implies$ (1). We extend μ_n to a probability measure $\widehat{\mu}_n$ on $\widehat{\mathbb{R}}$ by setting $\widehat{\mu}_n(\{\infty\}) := 0$. Since $\{\widehat{\mu}_n(\widehat{\mathbb{R}}) = 1\}_{n\ge1}$ is bounded, by Theorem 4.5, there is a subsequence $(\widehat{\mu}_{n(j)})_{j\ge1}$ that converges weakly to a probability measure $\widehat{\mu}$ on $\widehat{\mathbb{R}}$. Since $\mathbb{R} \ni t \mapsto 1/(z-t)$ can be regarded as a bounded continuous function on $\widehat{\mathbb{R}}$ vanishing at infinity, we have

$$G_{\mu_n}(z) = \int_{\widehat{\mathbb{R}}} \frac{1}{z-t}\,\widehat{\mu}_n(dt) \to \int_{\widehat{\mathbb{R}}} \frac{1}{z-t}\,\widehat{\mu}(dt) = \int_{\mathbb{R}} \frac{1}{z-t}\,\widehat{\mu}(dt) = G_{\widehat{\mu}|_{\mathbb{R}}}(z), \qquad z \in \mathbb{C}^+.$$

By assumption (2), the limit function $G_{\widehat{\mu}|_{\mathbb{R}}}$ must be $G_\mu(z)$. Since the Cauchy transform determines the underlying finite Borel measure on $\mathbb{R}$ uniquely (see Proposition 4.30), we have $\widehat{\mu}|_{\mathbb{R}} = \mu$. Recalling that μ is a probability measure on $\mathbb{R}$ and $\widehat{\mu}$ is a probability measure on $\widehat{\mathbb{R}}$, we must have $\widehat{\mu}(\{\infty\}) = 0$. The weak convergence $\widehat{\mu}_{n(j)} \to \widehat{\mu}$ implies

$$\int_{\mathbb{R}} f(x)\mu_{n(j)}(dx) \to \int_{\mathbb{R}} f(x)\,\mu(dx) \tag{4.15}$$

for any continuous function f on $\mathbb{R}$ with compact support since such an f can be regarded as a continuous function on $\widehat{\mathbb{R}}$. By Proposition 4.7, the convergence (4.15) implies that $\mu_{n(j)}$ weakly converges to μ on $\mathbb{R}$. By Lemma 4.2, the weak convergence of the whole sequence $\mu_n \to \mu$ holds as one can apply the above arguments to any subsequence of (μ_n) instead of (μ_n) itself. ☐

Proposition 4.34. *Let N be a Nevanlinna function. The following are equivalent.*

(1) $N = F_\mu$ *for some probability measure* μ *on* $\mathbb{R}$.

(2) $\displaystyle\lim_{y\to\infty} \frac{N(iy)}{iy} = 1.$

(3) $\displaystyle\sphericalangle\lim_{z\to\infty} \frac{N(z)}{z} = 1.$

(4) *There are* $b \in \mathbb{R}$ *and a finite Borel measure* τ *on* $\mathbb{R}$ *such that*

$$N(z) = z - b + \int_{\mathbb{R}} \frac{1+tz}{t-z}\tau(dt). \tag{4.16}$$

The probability measure μ *is unique.*

Proof. The equivalence (1) $\iff$ (2) $\iff$ (3) follows from Proposition 4.32 applied to $G := 1/N$. Theorem 4.23 implies the equivalence (3) $\iff$ (4). The uniqueness is a consequence of the injectivity of $\mu \mapsto G_\mu$ addressed in Proposition 4.30. ☐

Remark 4.35. For a probability measure μ the following inequality holds:

$$\Im[F_\mu(z)] \ge \Im(z), \qquad z \in \mathbb{C}^+, \tag{4.17}$$

and the equality holds at some $z \in \mathbb{C}^+$ if and only if μ is a delta measure. The latter statement holds because the equality holds if and only if $\tau = 0$.

A uniform version of Proposition 4.34(2) or (3) for a family of probability measures characterizes the tightness.

Proposition 4.36. *Let $\mathcal{P}$ be a family of probability measures on $\mathbb{R}$. Then the following are equivalent.*

(1) $\mathcal{P}$ *is tight.*

(2) $\displaystyle\sphericalangle\lim_{z\to\infty}\sup_{\mu\in\mathcal{P}}\left|\frac{F_\mu(z)}{z} - 1\right| = 0.$

(3) $\displaystyle\lim_{y\to\infty}\sup_{\mu\in\mathcal{P}}\left|\frac{F_\mu(iy)}{iy} - 1\right| = 0.$

Proof. Observe first that (2) and (3) are respectively equivalent to:

(2′) $\displaystyle\sphericalangle\lim_{z\to\infty}\sup_{\mu\in\mathcal{P}}|zG_\mu(z) - 1| = 0,$

(3′) $\displaystyle\lim_{y\to\infty}\sup_{\mu\in\mathcal{P}}|iyG_\mu(iy) - 1| = 0.$

$(2') \implies (3')$ is obvious.

$(3') \implies (1)$ follows from the estimates for $y > 0$:

$$-\Re[iyG_\mu(iy) - 1] = \int_{\mathbb{R}} \frac{t^2}{t^2+y^2}\,\mu(dt) \geq \int_{|t|>y} \frac{t^2}{t^2+y^2}\,\mu(dt) \geq \frac{1}{2}\mu(\mathbb{R}\setminus[-y,y]).$$

$(1) \implies (2')$. We fix $\gamma > 0$ for the nontangential domain. By the tightness assumption, for every $\varepsilon > 0$ there is $T > 0$ such that

$$\mu(\mathbb{R}\setminus[-T,T]) \leq \frac{\varepsilon}{1+\sqrt{1+\gamma^{-2}}}, \qquad \mu \in \mathcal{P}.$$

Combining the above inequality and (4.12), for all $\mu \in \mathcal{P}$ and $z \in \nabla_\gamma$ we have

$$\begin{aligned} |zG_\mu(z) - 1| &= \left| \int_{\mathbb{R}} \left(\frac{z}{z-t} - 1 \right) \mu(dt) \right| \\ &\leq \int_{[-T,T]} \left| \frac{t}{z-t} \right| \mu(dt) + (\sqrt{1+\gamma^{-2}}+1)\mu(\mathbb{R}\setminus[-T,T]) \\ &\leq \int_{[-T,T]} \frac{|t|}{\Im(z)} \mu(dt) + \varepsilon \leq \frac{T}{\Im(z)} + \varepsilon. \end{aligned}$$

Therefore, we obtain $\displaystyle\sup_{\mu\in\mathcal{P}, z\in\nabla_\gamma, \Im(z)>T/\varepsilon} |zG_\mu(z) - 1| < 2\varepsilon.$ ☐

Finally we introduce and characterize transforms useful to study multiplicative monotone convolution.

Definition 4.37. Let μ be a probability measure on $\mathbb{R}$. The function

$$\psi_\mu(z) := \int_{\mathbb{R}} \frac{zt}{1-zt}\,\mu(dt), \qquad z \in \mathbb{C}^+$$

is called the **ψ-transform** of μ (or the moment generating function of μ) and

$$\eta_\mu(z) := \frac{\psi_\mu(z)}{1+\psi_\mu(z)}, \qquad z \in \mathbb{C}^+$$

is called the **η-transform** of μ.

We can check by straightforward calculations that $\psi_\mu(z) = zG_\mu(1/z) - 1$, where G_μ is defined on the lower half-plane $-\mathbb{C}^+$ by the same formula in Definition 4.29, i.e., $G_\mu(z) := \overline{G_\mu(\overline{z})}$, and also

$$\eta_\mu(z) = 1 - zF_\mu\left(\frac{1}{z}\right), \tag{4.18}$$

where $F_\mu := 1/G_\mu$ on $-\mathbb{C}^+$.

Proposition 4.38. *For any probability measure μ on $\mathbb{R}$ such that $\mu \neq \delta_0$, the following hold.*

(i) *η_μ is a holomorphic map from $\mathbb{C}^+$ into $\mathbb{C}\setminus[0,+\infty)$.*

(ii) *$\arg z \leq \arg \eta_\mu(z) \leq \arg z + \pi$, i.e., $\eta_\mu(z)/z \in \mathbb{C}^+ \cup \mathbb{R}$, for all $z \in \mathbb{C}^+$.*

(iii) *$\sphericalangle \lim_{z\to 0} \eta_\mu(z) = 0$.*

Conversely, if a holomorphic map $\eta\colon \mathbb{C}^+ \to \mathbb{C}\setminus[0,+\infty)$ satisfies $\arg z \leq \arg \eta(z) \leq \arg z + \pi$ and $\sphericalangle \lim_{z\to 0} \eta(z) = 0$, then there is a unique probability measure μ on $\mathbb{R}$ such that $\mu \neq \delta_0$ and $\eta = \eta_\mu$.

Proof. The holomorphicity is immediate from (4.18). Inequality (4.17) implies $1/z - F_\mu(1/z) \in (\mathbb{C}^+ \cup \mathbb{R}) \setminus \{0\}$, which in turn yields $\eta_\mu(\mathbb{C}^+) \subseteq \mathbb{C}\setminus[0,+\infty)$ and condition (ii). Condition (iii) is equivalent to Proposition 4.34 (3) for $N = F_\mu$.

For the last converse statement, we can see that the function $N(z) := z[1 - \overline{\eta(1/\overline{z})}]$ is a Nevanlinna function and satisfies $\lim_{y\to+\infty} N(iy)/(iy) = 1$. Therefore, by Proposition 4.34 there is a unique probability measure μ on $\mathbb{R}$ such that $N(z) = F_\mu(z)$. Since η is assumed not to take 0, the value $N(z)$ never equals z and so $\mu \neq \delta_0$. We thus obtain

$$\eta_\mu(z) = 1 - \overline{zN\left(\frac{1}{\overline{z}}\right)} = \eta(z). \qquad \square$$

4.5. **Support and moments of probability measures.** From the Cauchy transform one can extract information about support and moments of the underlying probability measures. As a general symbol, for a finite Borel measure τ on $\mathbb{R}$ and $n \in \mathbb{N}_0$, if $\int_\mathbb{R} |t|^n \,\tau(dt) < +\infty$ we set

$$m_n(\tau) := \int_\mathbb{R} t^n \,\tau(dt)$$

and call it the n**th moment of** τ. Note that Hölder's inequality implies

$$\int_\mathbb{R} |x|^k \tau(dx) \le \tau(\mathbb{R})^{1-\frac{k}{\ell}} \left[\int_\mathbb{R} |x|^\ell \tau(dx)\right]^{\frac{k}{\ell}}, \qquad 0 < k < \ell < +\infty, \tag{4.19}$$

so if the absolute ℓth moment is finite, then the lower order moments all exist.

Proposition 4.39. *Let μ be a probability measure on $\mathbb{R}$, C a closed subset of $\mathbb{R}$, and $-\infty < \alpha < \beta < \infty$.*

(i) *$\mu(C) = 1$ holds if and only if G_μ has an analytic continuation $\tilde{G}_\mu$ on $\mathbb{C} \setminus C$ such that $\tilde{G}_\mu(\overline{z}) = \overline{\tilde{G}_\mu(z)}$.*

(ii) *If $\mu([\alpha, \beta]) = 1$ then $\tilde{G}_\mu$ on $\mathbb{C} \setminus [\alpha, \beta]$ considered above has the Laurent series expansion*

$$\tilde{G}_\mu(z) = \sum_{n \ge 0} \frac{m_n(\mu)}{z^{n+1}}, \qquad z \in \mathbb{C},\ |z| > \max\{|\alpha|, |\beta|\}.$$

(iii) *If $\mu([\alpha, \beta]) = 1$ then F_μ has an analytic continuation $\tilde{F}_\mu$ on $\mathbb{C} \setminus [\alpha, \beta]$ such that $\tilde{F}_\mu(\overline{z}) = \overline{\tilde{F}_\mu(z)}$ and there exist real numbers $b_1, b_2, \dots$ such that*

$$\tilde{F}_\mu(z) = z - \sum_{n \ge 1} \frac{b_n}{z^{n-1}}, \qquad z \in \mathbb{C},\ |z| > \max\{|\alpha|, |\beta|\}.$$

(iv) *Suppose that F_μ has an analytic continuation $\tilde{F}_\mu$ on $\mathbb{C} \setminus [\alpha, \beta]$ such that $\tilde{F}_\mu(\overline{z}) = \overline{\tilde{F}_\mu(z)}$. Then* $\operatorname{supp}(\mu) \cap (\beta, +\infty)$ *contains at most one point. An atom exists in $(\beta, +\infty)$ if and only if $\tilde{F}_\nu$ has a zero on $(\beta, +\infty)$, in which case μ has an atom at the zero of $\tilde{F}$. Similarly,* $\operatorname{supp}(\mu) \cap (-\infty, \alpha)$ *contains at most one point and an atom exists in $(-\infty, \alpha)$ if and only if $\tilde{F}_\nu$ has a zero on $(-\infty, \alpha)$.*

Proof. (i) Suppose that $\mu(C) = 1$. Then the integral formula

$$\tilde{G}_\mu(z) = \int_C \frac{1}{z - t}\, \mu(dt) \tag{4.20}$$

gives an analytic continuation of G_μ to $\mathbb{C} \setminus C$ with $\tilde{G}_\mu(\overline{z}) = \overline{\tilde{G}_\mu(z)}$. Conversely, if G_μ has an analytic continuation $\tilde{G}_\mu$ on $\mathbb{C} \setminus C$ with $\tilde{G}_\mu(\overline{z}) = \overline{\tilde{G}_\mu(z)}$, then the latter condition implies $\tilde{G}_\mu(x)$ takes real values for $x \in \mathbb{R} \setminus C$. For each interval $[a, b] \subseteq \mathbb{R} \setminus C$, by the Stieltjes inversion formula in Corollary 4.31 we have $\mu([a, b]) = 0$. This implies $\mu(\mathbb{R} \setminus C) = 0$.

(ii) The Laurent series expansion can be obtained from formula (4.20) and the fact that the series

$$\frac{1}{z - t} = \sum_{n \ge 0} \frac{t^n}{z^{n+1}}$$

converges uniformly over $t \in [\alpha, \beta]$ and $z \in \{w \in \mathbb{C} : |w| \ge \varepsilon + \max\{|\alpha|, |\beta|\}\}$ for each $\varepsilon > 0$.

(iii) Since $\tilde{G}_\mu$ in (4.20) has no zeros in $\mathbb{C} \setminus [\alpha, \beta]$, its reciprocal $\tilde{F}_\mu := \tilde{G}_\mu$ is the desired analytic continuation. For the series expansion, we take a Nevanlinna formula for F_μ

$$F_\mu(z) = z - b + \int_\mathbb{R} \frac{1 + tz}{t - z}\, \tau(dt), \qquad z \in \mathbb{C}^+. \tag{4.21}$$

Since the extension $\tilde{F}_\mu$ takes real values on $\mathbb{R} \setminus [\alpha, \beta]$, the Stieltjes inversion (see Corollary 4.25) implies τ is supported on $[\alpha, \beta]$. Hence, the analytic continuation is given by

$$\tilde{F}_\mu(z) = z - b + \int_{[\alpha,\beta]} \frac{1 + tz}{t - z}\, \tau(dt), \qquad z \in \mathbb{C} \setminus [\alpha, \beta]. \tag{4.22}$$

The remaining argument is similar to (ii) thanks to formula (4.13); the coefficients b_n are given by $b_1 = b + \int_{[\alpha,\beta]} t\, \tau(dt)$ and $b_n = \int_{[\alpha,\beta]} t^{n-2}(1 + t^2)\, \tau(dt), n \ge 2$.

(iv) As discussed in (iii), the assumption implies that F_μ has an analytic extension $\tilde{F}_\mu$ of the form (4.22). One can see that

$$\tilde{F}_\mu'(x) = 1 + \int_{[\alpha,\beta]} \frac{1 + t^2}{(t - x)^2}\, \tau(dt) \ge 1, \qquad x \in \mathbb{R} \setminus [\alpha, \beta], \tag{4.23}$$

so that $\tilde{F}_\mu$ is strictly increasing on $\mathbb{R}\setminus[\alpha,\beta]$ and $\lim_{x\to\pm\infty}\tilde{F}_\nu(x)=\pm\infty$. In particular, $\tilde{F}_\nu$ has at most one zero in each interval $(-\infty,\alpha)$ and $(\beta,+\infty)$. If $\tilde{F}_\nu$ has a zero $\gamma\in(\beta,+\infty)$ then $\tilde{G}_\mu:=1/\tilde{F}_\mu$ is analytic in $(\beta,+\infty)\setminus\{\gamma\}$ taking real values there, and

$$q:=\lim_{y\to 0} iyG_\mu(\gamma+iy)=\lim_{y\to 0}\frac{iy}{\tilde{F}_\mu(\gamma+iy)-\tilde{F}_\mu(\gamma)}=\frac{1}{\tilde{F}'_\mu(\gamma)}\in(0,1].$$

Therefore, μ assigns the mass q to the point γ and $\mu((\beta,+\infty)\setminus\{\gamma\})=0$. By contrast, if $\tilde{F}_\mu$ has no zeros on $(\beta,+\infty)$ then $\tilde{G}_\mu=1/\tilde{F}_\mu$ is analytic in $\mathbb{C}\setminus(-\infty,\beta]$ taking real values, so that μ is supported on $(-\infty,\beta]$. A similar analysis is valid on the interval $(-\infty,\alpha)$. ☐

A particular class is the set of probability measures on the nonnegative real line. This class admits characterizations by means of the reciprocal Cauchy transform and η-transform.

Proposition 4.40. *Let μ be a probability measure on $\mathbb{R}$. Let (b,τ) be the pair appearing for $N=F_\mu$ in (4.16). The following are equivalent.*

(1) *μ is supported on $[0,+\infty)$.*

(2) *F_μ has an analytic continuation $\tilde{F}_\mu$ defined on $\mathbb{C}\setminus[0,+\infty)$ such that $\tilde{F}_\mu(\overline{z})=\overline{\tilde{F}_\mu(z)}$ and $\lim_{x\uparrow 0}\tilde{F}_\mu(x)\in(-\infty,0]$.*

(3) *τ is supported on $(0,+\infty)$ and $\int_0^\infty t^{-1}\,\tau(dt)\le b$.*

Proof. (1) $\Longrightarrow$ (2). The existence of analytic continuation $\tilde{F}_\mu$ can be proved in a similar way to the proof of Proposition 4.39 (iii); it is given by $\tilde{F}_\mu=1/\tilde{G}_\mu$, where

$$\tilde{G}_\mu(z)=\int_{[0,+\infty)}\frac{1}{z-t}\,\mu(dt),\qquad z\in\mathbb{C}\setminus[0,+\infty).$$

Obviously, $\tilde{G}_\mu(x)<0$ for all $x<0$, so that $\tilde{F}_\mu(x)<0$. Since $\tilde{F}_\mu$ is (strictly) increasing on $(-\infty,0)$ we obtain $\tilde{F}_\mu(0-)\in(-\infty,0]$.

(2) $\Longrightarrow$ (3). Let (b,τ) be the pair in (4.16) for $N=F_\mu$. By the Stieltjes inversion formula, τ is supported on $[0,+\infty)$. Then the Nevanlinna formula (4.16) naturally gives the expression for $\tilde{F}_\mu$. We write

$$\tilde{F}_\mu(z)=z-b+\underbrace{\int_{[0,1)}\frac{1+zt}{t-z}\,\tau(dt)}_{=:I_1(z)}+\underbrace{\int_{[1,+\infty)}\frac{1+zt}{t-z}\,\tau(dt)}_{=:I_2(z)},\qquad z\in\mathbb{C}\setminus[0,+\infty). \tag{4.24}$$

Since the function $(-1,0)\ni x\mapsto(1+xt)/(t-x)$ is increasing and positive for all $t\in[0,1)$, we can use the monotone convergence theorem to conclude that $I_1(x)\uparrow\int_{[0,1)}t^{-1}\,\tau(dt)$ as $x\to 0^-$. On the other hand, for I_2 we can use the dominated convergence theorem to deduce $I_2(x)\to\int_{[1,+\infty)}t^{-1}\,\tau(dt)$ as $x\to 0^-$ because $|(1+tx)/(t-x)|\le(1+t)/t$ for all $t\ge 1$ and $-1<x<0$. By the assumption $\tilde{F}_\mu(0-)\le 0$, we must have $\int_{[0,1)}t^{-1}\,\tau(dt)<+\infty$; in particular, $\tau(\{0\})=0$. The condition $\tilde{F}_\mu(0-)\le 0$ now reads $\int_0^\infty t^{-1}\,\tau(dt)\le b$.

(3) $\Longrightarrow$ (1). Repeating the above arguments we have that $\tilde{F}_\mu$ in (4.24) satisfies $\tilde{F}_\mu(0-)=-b+\int_0^\infty t^{-1}\,\tau(dt)\le 0$. Since $\tilde{F}_\mu$ is increasing on $(-\infty,0)$, it is negative and so the holomorphic function $\tilde{G}_\mu:=1/\tilde{F}_\mu$ on $\mathbb{C}\setminus[0,+\infty)$ satisfies the condition of Proposition 4.39 (i). ☐

The η-transform of a probability measure on $[0,+\infty)$ analytically extends to $\mathbb{C}\setminus[0,+\infty)$ due to (4.18). The following proposition gives a useful characterization of η-transforms.

Proposition 4.41. *Let $\eta\colon\mathbb{C}\setminus[0,+\infty)\to\mathbb{C}$ be a holomorphic function. There is a probability measure $\mu\ne\delta_0$ on $[0,+\infty)$ such that $\eta=\eta_\mu$ on $\mathbb{C}^+$ if and only if*

(a) *$\eta(\overline{z})=\overline{\eta(z)}$ on $\mathbb{C}\setminus[0,+\infty)$,*

(b) *η is a self-map of $\mathbb{C}\setminus[0,+\infty)$ and $\arg z\le\arg\eta_\mu(z)<\pi$ on $\mathbb{C}^+$,*

(c) *For any $\theta\in(0,\pi)$ we have*

$$\lim_{\substack{z\to 0\\ \theta<\arg z<2\pi-\theta}}\eta(z)=0.$$

Moreover, if the above conditions hold then η has the following formula

$$\eta(z)=b'z+\int_{(0,+\infty)}\frac{z}{1-tz}\cdot\frac{1+t^2}{t}\,\tau(dt),\qquad z\in\mathbb{C}\setminus[0,+\infty), \tag{4.25}$$

where $b'\ge 0$ and τ is a finite Borel measure on $(0,\infty)$ such that $\int_0^\infty t^{-1}\,\tau(dt)<+\infty$.

Proof. Suppose that $\eta|_{\mathbb{C}^+} = \eta_\mu$ for some probability measure $\mu \neq \delta_0$ on $[0, +\infty)$. From Proposition 4.40 (2) and (4.18), η is given by

$$\eta(z) = 1 - z\tilde{F}_\mu\left(\frac{1}{z}\right).$$

This implies condition (a). Before proving condition (b) we first establish (4.25): combining formula (4.24) and Proposition 4.40 (3) we have

$$\eta(z) = \left(b - \int_0^\infty \frac{1}{t}\,\tau(dt)\right) z + \int_0^\infty \left(\frac{z(z+t)}{1-zt} + \frac{z}{t}\right) \tau(dt),$$

which is formula (4.25) with $b' := b - \int_0^\infty t^{-1}\,\tau(dt)$. As one of b' and τ is nonzero, formula (4.25) obviously yields that $\eta(x) < 0$ for all $x < 0$. In addition, as $z/(1 - tz) \in \mathbb{C}^+$ for all $z \in \mathbb{C}^+$ and $t > 0$, we have $\eta(z) \in \mathbb{C}^+$. Combining the fact $\eta(z) \in \mathbb{C}^+$ and the inequality $\arg z \le \eta(z) \le \arg z + \pi$ known in Proposition 4.38 we conclude condition (b). Condition (c) is equivalent to

$$\lim_{\substack{|z|\to\infty \\ \theta<\arg z<2\pi-\theta}} \frac{\tilde{F}_\mu(z)}{z} = 1, \tag{4.26}$$

which can be shown as in Theorem 4.23 (i). More precisely, the bound (4.12) actually holds for all $z \in \mathbb{C}$ with $\theta < \arg z < 2\pi - \theta$ and $t \ge 0$ with $\theta := \arctan \gamma$, because $|z/(z-t)| \le 1$ for $\Re(z) < 0$. The remaining proof is the same as Theorem 4.23 (i).

For the converse, suppose that η satisfies conditions (a)–(c). Let $F(z) := z[1 - \eta(1/z)], z \in \mathbb{C} \setminus [0, +\infty)$, which is expected to be the reciprocal Cauchy transform of the desired μ. Condition (a) implies F is holomorphic with $F(\overline{z}) = \overline{F(z)}$. Condition (b) implies $F|_{\mathbb{C}^+}$ is a Nevanlinna function. Condition (c) implies $F(z)/z \to 1$ as $z \to \infty, z \in \nabla_\gamma$ for any $\gamma > 0$. Therefore, by Proposition 4.34 there is a probability measure μ on $\mathbb{R}$ such that $F = F_\mu$ on $\mathbb{C}^+$. Moreover, condition (b) implies η takes negative values on $(-\infty, 0)$, so that F takes also negative values there. By Proposition 4.40, μ is supported on $[0, +\infty)$. $\square$

Remark 4.42. The limit of a function $f\colon \mathbb{C} \setminus [0, +\infty) \to \mathbb{C}$ as $z \to 0$ satisfying $\arg z \in (\theta, 2\pi - \theta)$ could be called the nontangential limit of f at 0. This is because the domain $\mathbb{C} \setminus [0, +\infty)$ is conformally equivalent to $\mathbb{C}^+$ by the mapping $z \mapsto \sqrt{z}$, and then the domain $\{z : \arg z \in (\theta, 2\pi - \theta)\}$ is mapped exactly onto the sector $\nabla_{\tan(\theta/2)}$.

Accordingly, there is an alternative proof of (4.26) based on Lindelöf's theorem: observing that the function $\tilde{F}_\mu(z)/z$ maps $\mathbb{C} \setminus [0, +\infty)$ into $\mathbb{C} \setminus (-\infty, 0]$ that is also conformally equivalent to $\mathbb{C}^+$ (see Proposition 4.40 (2)), one can use Lindelöf's theorem to extend the known nontangential limit from the upper half-plane in Theorem 4.23 (i) to (4.26).

We turn our attention to the existence of finite moments.

Proposition 4.43. *Let μ be a probability measure on $\mathbb{R}$ and let $n \in \mathbb{N}$. Let τ be the finite Borel measure in* (4.16) *for $N = F_\mu$. Then the following conditions are equivalent.*

(1) $\displaystyle\int_{\mathbb{R}} t^{2n}\,\mu(dt) < +\infty.$

(2) *There exist $a_1, a_2, ..., a_{2n} \in \mathbb{R}$ such that*

$$G_\mu(z) = \frac{1}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \cdots + \frac{a_{2n}}{z^{2n+1}} + o(|z|^{-(2n+1)}) \tag{4.27}$$

for $z = iy$ as $y \to +\infty$.

(3) $\displaystyle\int_{\mathbb{R}} t^{2n}\,\tau(dt) < +\infty.$

(4) *There exist $b_1, b_2, \cdots, b_{2n} \in \mathbb{R}$ such that*

$$F_\mu(z) = z - b_1 - \frac{b_2}{z} - \cdots - \frac{b_{2n}}{z^{2n-1}} + o(|z|^{-(2n-1)}) \tag{4.28}$$

for $z = iy$ as $y \to +\infty$.

If the above equivalent conditions are satisfied then the expansions (4.27) *and* (4.28) *hold as $z \to \infty$ nontangentially, i.e., the remainder term $o(|z|^{-(2n+1)})$ in* (4.27) *is a function $r(z)$ that satisfies $\sphericalangle \lim_{z\to\infty} z^{2n+1} r(z) = 0$,* *and similar for* (4.28). *Moreover, it holds that*

$$a_\ell = m_\ell(\mu), \qquad 1 \le \ell \le 2n, \tag{4.29}$$

$$b_\ell = \int_{\mathbb{R}} t^{\ell-2}(1+t^2)\tau(dt), \qquad 2 \le \ell \le 2n, \tag{4.30}$$

$$b_1 = a_1 = m_1(\mu), \tag{4.31}$$

$$b_2 = a_2 - a_1^2 = \mathrm{Var}(\mu). \tag{4.32}$$

Proof. (1) $\Longrightarrow$ (2). The assumption implies that $\int_{\mathbb{R}} |t|^\ell \mu(dt) < \infty$ for $1 \le \ell \le 2n$. Observe that the identity

$$\frac{1}{z-t} = \sum_{\ell=0}^{2n} \frac{t^\ell}{z^{\ell+1}} + \frac{t^{2n+1}}{z^{2n+1}(z-t)} \tag{4.33}$$

holds, which is integrated into

$$G_\mu(z) = \sum_{\ell=0}^{2n} \frac{m_\ell(\mu)}{z^{\ell+1}} + \underbrace{\frac{1}{z^{2n+1}} \int_{\mathbb{R}} \frac{t^{2n+1}}{z-t}\, \mu(dt)}_{=:R_n(z)}. \tag{4.34}$$

By the dominated convergence theorem we can show $R_n(iy) = o(y^{-2n-1})$ as $y \to +\infty$.

(2) $\Longrightarrow$ (1). We only consider the case $n=2$ which should well clarify how the general n can be dealt with. Keeping in mind that $a_1/(iy)^2$ is real, we observe that

$$y^3 \Im\left[G_\mu(z) - \frac{1}{z} - \frac{a_1}{z^2}\right] = y^3 \Im\left[G_\mu(z) - \frac{1}{z}\right] = \int_{\mathbb{R}} \frac{y^2t^2}{y^2+t^2}\, \mu(dt), \qquad z = iy.$$

By the assumption, the left-hand side above is bounded as $y \to \infty$. By the monotone convergence theorem we get

$$\int_{\mathbb{R}} t^2\, \mu(dt) < +\infty.$$

This implies that $\int |t|\mu(dt)$ is also finite; see (4.19). By the established implication (1) $\Longrightarrow$ (2) for $n=1$, we have

$$G_\mu(z) = \frac{1}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + o(|z|^{-3}).$$

Since the asymptotic expansion is unique, we have $a_1 = m_1(\mu)$ and $a_2 = m_2(\mu)$. Next, we take the imaginary part of the expansion $G_\mu(z) = \frac{1}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \frac{a_3}{z^4} + \frac{a_4}{z^5} + o(|z|^{-5})$. Integrating (4.33) with $n=1$ yields

$$y^5 \Im\left[G_\mu(z) - \frac{1}{z} - \frac{m_1(\mu)}{z^3} - \frac{m_2(\mu)}{z^3} - \frac{a_3}{z^4}\right] = -\int_{\mathbb{R}} \frac{y^2t^4}{y^2+t^2}\, \mu(dt), \qquad z = iy.$$

This must be bounded as $y \to +\infty$, and hence, by the monotone convergence theorem,

$$\int_{\mathbb{R}} t^4\, \mu(dt) < +\infty$$

as desired. We have already verified (4.29) in the course of the proof above.

(3) $\Longleftrightarrow$ (4). The proof is very similar to the equivalence of (1) and (2). In the course of the proof, formula (4.30) naturally appears. Note that formula (4.13) is helpful.

(2) $\Longleftrightarrow$ (4) is an easy consequence of the relation $G_\mu(z) = 1/F_\mu(z)$ and the geometric series expansion $1/(1-\zeta) = \zeta + \zeta^2 + \cdots$. For example in case $n=1$, assuming $G_\mu(z) = \frac{1}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + o(z^{-3})$ we have

$$\begin{aligned}
F_\mu(z) &= \frac{1}{\frac{1}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + o(z^{-3})} = \frac{z}{1 + \frac{m_1(\mu)}{z} + \frac{m_2(\mu)}{z^2} + o(z^{-2})} \\
&= z\left[1 - \left(\frac{m_1(\mu)}{z} + \frac{m_2(\mu)}{z^2} + o(z^{-2})\right) + \left(\frac{m_1(\mu)}{z} + o(z^{-1})\right)^2 + o(z^{-2})\right] \\
&= z - m_1(\mu) - \frac{m_2(\mu) - m_1(\mu)^2}{z} + o(z^{-1}), \qquad z = iy,\ y \to +\infty.
\end{aligned}$$

This also verifies (4.31) and (4.32).

(4.27) *and* (4.28) *as the nontangential limits.* As for (4.27), assuming (1) and fixing $\gamma > 0$, let us prove $z^{2n+1}R_n(z) \to 0$ as $z \to \infty, z \in \nabla_\gamma$. It suffices to find a bound of the form $1/|z-t| \le C/(1+|t|)$ with a constant C independent of $z \in \nabla_\gamma \cap \{\Im z > 1\}$. For $t \in [-1,1]$ we can simply find an upper bound

$1/|z-t| \le 1/\Im(z) \le 1 \le 2/(1+|t|)$. For $|t| \ge 1$ we consider $z = x+iy \in \nabla_\gamma$ with $y > 1$, and proceed as

$$\begin{aligned} |z-t|^2 &= (x-t)^2 + y^2 \ge x^2 - 2xt + t^2 + \gamma^2 x^2 \\ &= (1+\gamma^2)\left(x - \frac{t}{1+\gamma^2}\right)^2 + \frac{1}{1+\gamma^{-2}}t^2 \ge \frac{1}{1+\gamma^{-2}}t^2, \end{aligned}$$

so that

$$\frac{1}{|z-t|} \le \frac{\sqrt{1+\gamma^{-2}}}{|t|} \le \frac{2\sqrt{1+\gamma^{-2}}}{1+|t|}.$$

Of course (4.28) can be similarly proved. □

Remark 4.44. (a) The coefficients b_n are called the Boolean cumulants of μ that are central notions in "Boolean probability theory", another type of noncommutative probability. For this reason the minus signs are put in formula (4.28). The interested reader is referred to [141].

(b) The assumption that the coefficients $a_1, a_2, ...$ are real is crucial. Indeed, for the Cauchy distribution (4.36) the Cauchy transform (4.35) has a convergent series expansion

$$G_\mu(z) = \sum_{n\ge 0} \frac{(a-ib)^n}{z^{n+1}}, \qquad z \in \mathbb{C}^+, |z| > \sqrt{a^2+b^2}.$$

However, the second moment is infinity.

(c) The last part of the proof $\sphericalangle \lim_{z\to\infty} z^{2n+1}R_n(z) = 0$ has an alternative proof based on Lindelöf's theorem. For this we decompose $z^{2n+1}R_n = S_n^+ - S_n^-$, where

$$S_n^+(z) = \int_{[0,\infty)} \frac{t^{2n+1}}{z-t}\mu(dt), \qquad S_n^-(z) = \int_{(-\infty,0]} \frac{-t^{2n+1}}{z-t}\mu(dt).$$

Because $-S_n^\pm(z)$ are holomorphic mappings from $\mathbb{C}^+$ into $\mathbb{C}^+ \cup \mathbb{R}$ and $S_n^\pm(iy) \to 0$ as already proved, Lindelöf's theorem implies $\sphericalangle \lim_{z\to\infty} S_n^\pm(z) = 0$.

4.6. **Methods to compute Cauchy transforms.** In some situations, we first obtain an explicit formula for the Cauchy transform before knowing the underlying probability measure, e.g. when we solve a functional equation or a differential equation satisfied by the Cauchy transform. In such a situation, we can compute the measure by the Stieltjes inversion formula. The next two examples naturally appear from the study of monotone convolution semigroups and infinitely divisible distributions, see Example 5.18.

Example 4.45. Let us consider the function

$$G(z) = \frac{1}{z-a+ib} \tag{4.35}$$

where $a \in \mathbb{R}, b > 0$ are constants. Since G is a holomorphic function from $\mathbb{C}^+$ into $-\mathbb{C}^+$, and $\lim_{z\to\infty} zG(z) = 1$, $G = G_\mu$ for some probability measure μ on $\mathbb{R}$. Moreover, G extends to a continuous function $G\colon \mathbb{C}^+ \cup \mathbb{R} \to \mathbb{C}^+$, so that by Corollary 4.31, μ is Lebesgue absolutely continuous on $\mathbb{R}$, and

$$\frac{d\mu}{dt} = -\frac{1}{\pi}\Im\left[\frac{1}{t-a+ib}\right] = \frac{b}{\pi[(t-a)^2+b^2]}. \tag{4.36}$$

This is the Cauchy distribution. Thus, we have verified that the Cauchy transform of the Cauchy distribution μ is given by (4.35).

Example 4.46. Let $r > 0$. The function

$$G(z) := \frac{1}{\sqrt{z^2 - 2r}}$$

where $\sqrt{w}$ is defined on $\mathbb{C} \setminus [0,\infty)$ by $\sqrt{w} = |w|^{1/2} e^{(i/2)\arg w}, 0 < \arg w < 2\pi$. Then G is a holomorphic function from $\mathbb{C}^+$ into $-\mathbb{C}^+$. One can check that $iyG(iy) \to 1$, so that G is the Cauchy transform of a probability measure μ. The function G extends to a continuous function on $\mathbb{C}^+ \cup (\mathbb{R} \setminus \{\pm\sqrt{2r}\})$, so that by Corollary 4.31, the underlying probability measure μ is Lebesgue absolutely continuous on $\mathbb{R} \setminus \{\pm\sqrt{2r}\}$ and the density is given by

$$-\frac{1}{\pi}\lim_{\varepsilon\to 0^+} \Im[G(t+i\varepsilon)] = \frac{1}{\pi\sqrt{2r-t^2}}\chi_{(-\sqrt{2r},\sqrt{2r})}(t). \tag{4.37}$$

Since $G(z)$ diverges as $z \to \pm\sqrt{2r}$, one has to check whether μ has an atom at these points or not. Because we have an estimate $|G(z)| \le C|z \mp \sqrt{2r}|^{-1/2}$ as $z \to \pm\sqrt{2r}$, we have $\mu(\{\pm\sqrt{2r}\}) = \lim_{\varepsilon\to 0^+} i\varepsilon G(\pm\sqrt{2r}+i\varepsilon) = 0$. Hence, μ is Lebesgue absolutely continuous on $\mathbb{R}$, supported on $(-\sqrt{2r}, \sqrt{2r})$ and $d\mu/dt$ is given by (4.37). Another way to check the absence of atoms is to show that (4.37) has total mass 1 by directly computing the integral.

The other direction, computing the Cauchy transform of a given probability measure, is usually harder. For example if we do not know formula (4.35) but want to compute G_μ for the Cauchy distribution μ, we need to calculate the integral

$$\int_{\mathbb{R}} \frac{1}{z-t} \cdot \frac{b}{\pi[(t-a)^2+b^2]}\, dt.$$

In this case, the residue theorem allows us to perform the calculations.

When μ has compact support and the moments are explicit, we can sometimes find a closed formula for

$$G_\mu(z) = \sum_{n=0}^\infty \frac{1}{z^{n+1}} \int_{\mathbb{R}} t^n \,\mu(dt)$$

for large $|z|$, and then perform the analytic continuation to $\mathbb{C}^+$. As an example, one can calculate the Cauchy transform of the semicircle distribution of mean 0 and variance $r>0$

$$\int_{-2\sqrt{r}}^{2\sqrt{r}} \frac{1}{z-t} \cdot \frac{\sqrt{4r-t^2}}{2\pi r}\, dt = \frac{z-\sqrt{z^2-4r}}{2r}, \qquad r>0,\ z\in\mathbb{C}^+, \tag{4.38}$$

see [120, Lemma 2.21].

The pushforward of a symmetric distribution around the origin by the map $t\mapsto t^2$ can be calculated from the original Cauchy transform.

Proposition 4.47. *Let ν be a probability measure on $\mathbb{R}$ that is symmetric around 0, i.e., $\nu(B)=\nu(-B)$ holds for all $B\in\mathcal{B}(\mathbb{R})$. Let μ be the pushforward of ν by the map $t\mapsto t^2$. Then we have $z\tilde{G}_\mu(z^2) = G_\nu(z)$ on $\mathbb{C}^+$, where $\tilde{G}_\mu$ is the analytic continuation of G_μ to $\mathbb{C}\setminus[0,+\infty)$ given in Proposition 4.39.*

Proof. The desired formula follows from the straightforward calculations

$$\tilde{G}_\mu(z^2) = \int_{\mathbb{R}} \frac{1}{z^2-t^2}\nu(dt) = \frac{1}{2z}\int_{\mathbb{R}} \left(\frac{1}{z-t}+\frac{1}{z+t}\right)\nu(dt) = \frac{1}{z}G_\nu(z). \qquad \square$$

Example 4.48. Let μ be the Marchenko–Pastur distribution

$$\mu(dt) = \frac{1}{2\pi r}\sqrt{\frac{4r-t}{t}}\,\chi_{(0,4r)}(t)\,dt, \qquad r>0,$$

which is the pushforward of the semicircle distribution $\nu(dt) = \frac{\sqrt{4r-t^2}}{2\pi r}dt$ by the map $t\mapsto t^2$. Using (4.38) and Proposition 4.47 one obtains

$$G_\mu(z) = \frac{1}{\sqrt{z}} \cdot \frac{\sqrt{z}-\sqrt{z-4r}}{2r} = \frac{z-\sqrt{z^2-4rz}}{2rz}, \qquad z\in\mathbb{C}^+.$$

Yet another way is to establish a differential equation for the Cauchy transform by integration by parts.

Example 4.49. Let us consider $G(z) := G_{\mathrm{N}(0,1)}(z) = \int_{\mathbb{R}} \frac{1}{z-t}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt$. By integration by parts we have

$$\begin{aligned} G'(z) &= -\int_{\mathbb{R}} \frac{1}{(z-t)^2}\cdot\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt = \int_{\mathbb{R}} \frac{1}{z-t}\cdot\frac{1}{\sqrt{2\pi}}(e^{-t^2/2})'\,dt \\ &= \int_{\mathbb{R}} \frac{1}{z-t}\cdot\frac{1}{\sqrt{2\pi}}(z-t-z)e^{-t^2/2}\,dt = 1-zG(z). \end{aligned}$$

First we solve the homogeneous equation $H'(z) = -zH(z)$, which has a general solution

$$H(z) = C_1 e^{-z^2/2}.$$

Next we replace the constant C_1 with a function and set $G(z) := f(z)e^{-z^2/2}$. The equation $G'(z) = 1-zG(z)$ yields

$$f'(z) = e^{z^2/2},$$

so that

$$f(z) = C_2 + \int_{L_z} e^{w^2/2}\,dw,$$

where C_2 is a constant and the line integral is performed over the half-line $L_z := \{z+iy : -\infty<y<0\}$ started at ∞ and terminated at z. To determine the constant C_2, let us consider

$$G(iy) = e^{y^2/2}\left[C_2 + i\int_{-\infty}^{y} e^{-t^2/2}\,dt\right].$$

Since G is a Cauchy transform we must have $G(iy)\to 0$ as $y\to+\infty$. This forces $C_2 = -\sqrt{2\pi}i$, so that we obtain

$$G_{\mathrm{N}(0,1)}(z) = e^{-z^2/2}\left[-\sqrt{2\pi}i + \int_{L_z} e^{w^2/2}\,dw\right], \qquad z\in\mathbb{C}^+.$$

4.7. **Notes.** The proof of the Nevanlinna formula in Theorem 4.23 is based on the expositions by Akhiezer and Glazman [4, Section 59] and Bhatia [33, Chapter V.4]. The characterization of weak convergence in Proposition 4.33 is an extension of Maassen's result [105, Theorem 2.5]. The part of Proposition 4.36 that a tight family of probability measures satisfies the convergence $\sphericalangle \lim_{z\to\infty} F_\mu(z)/z = 1$ uniformly over μ was proved by Bercovici and Voiculescu in the remark following [31, Proposition 5.1].

The characterization of η-transform in Proposition 4.38 is adopted from [9, Proposition 3.2] and [10, Proposition 2.4]. The characterization of probability measures on $[0,+\infty)$ in terms of η_μ in Proposition 4.41 was given by Belinschi and Bercovici [19, Proposition 2.2] for the purpose of studying multiplicative free convolution. The characterization of probability measures on $[0,+\infty)$ in terms of F_μ in Proposition 4.40 is adopted from [76, Proposition 2.5]. Concerning the characterization of finite even moments, the equivalence (1) $\iff$ (2) in Proposition 4.43 is due to [3, Theorem 3.2.1]. Some of the results in this section can also be found in the book of Mingo and Speicher [108].

5. Analysis of monotone convolution

Let x, y be monotonically independent real random variables in a unital C^*-probability space. The reciprocal Cauchy transform of μ_x and the shifted moment generating function of x are connected as

$$F_{\mu_x}(z) = 1/M_x(1/z).$$

Accordingly, Theorem 1.27 can be written in the form $F_{\mu_{x+y}}(z) = F_{\mu_x}(F_{\mu_y}(z))$. From the complex-analytic perspective, we can extend the additive monotone convolution to any probability measures on $\mathbb{R}$.

Theorem 5.1. *Let μ, ν be probability measures on $\mathbb{R}$. Then there exists a unique probability measure λ on $\mathbb{R}$ such that*

$$F_\lambda(z) = F_\mu(F_\nu(z)), \qquad z \in \mathbb{C}^+.$$

The measure λ is denoted by $\mu \rhd \nu$ and is called the ***(additive) monotone convolution*** *of μ and ν.*

Proof. Note that $\Im[F_\nu(z)] \geq \Im z > 0$ and in particular $F_\nu(\mathbb{C}^+) \subseteq \mathbb{C}^+$. The function $N(z) := F_\mu(F_\nu(z))$ is therefore a Nevanlinna function. According to Proposition 4.34, it suffices to prove $\lim_{y\to\infty} N(iy)/(iy) = 1$. Proposition 4.34 guarantees that $F_\nu(iy) = iy(1+o(1))$ as $y \to \infty$. From this one can see that for any fixed $\gamma > 0$ there exists $y_0 > 0$ such that $F_\nu(iy) \in \nabla_\gamma$ for all $y > y_0$. Therefore, by Proposition 4.34 (3), it holds that $\frac{F_\mu(F_\nu(iy))}{F_\nu(iy)} \to 1$ and hence

$$\frac{N(iy)}{iy} = \frac{F_\mu(F_\nu(iy))}{F_\nu(iy)} \cdot \frac{F_\nu(iy)}{iy} \to 1. \qquad \square$$

Proposition 5.2. *Let μ, μ_n, ν, ν_n ($n \in \mathbb{N}$) be probability measures on $\mathbb{R}$ such that $\mu_n \to \mu$ and $\nu_n \to \nu$ weakly. Then $\mu_n \rhd \nu_n \to \mu \rhd \nu$ weakly.*

Proof. By Proposition 4.33, it suffices to show the pointwise convergence $F_{\mu_n}(F_{\nu_n}(z)) \to F_\mu(F_\nu(z))$ on $\mathbb{C}^+$. For this we fix $z \in \mathbb{C}^+$, set $w_n := F_{\nu_n}(z), w := F_\nu(z)$ and begin with

$$|F_{\mu_n}(w_n) - F_\mu(w)| \leq |F_{\mu_n}(w_n) - F_\mu(w_n)| + |F_\mu(w_n) - F_\mu(w)|.$$

The second term clearly converges to 0 because $w_n \to w$. The first term also converges to zero since the convergence $F_{\mu_n} \to F_\mu$ is locally uniform. $\square$

Example 5.3. The measure $\mu \rhd \delta_a$ is the translation of μ by a. Indeed, the fact $F_{\delta_a}(z) = 1/G_{\delta_a}(z) = z - a$ yields

$$G_{\mu\rhd\delta_a}(z) = G_\mu(F_{\delta_a}(z)) = G_\mu(z-a) = \int_{\mathbb{R}} \frac{1}{z-(t+a)}\,\mu(dt), \tag{5.1}$$

showing that $\mu \rhd \delta_a$ is the pushforward of μ by the map $t \mapsto t + a$.

On the other hand, the measure $\delta_a \rhd \mu$ is not a translation of μ for generic $a \in \mathbb{R}$ and μ. For example, if μ is the arcsine law $1/(\pi\sqrt{2r-t^2})\chi_{(-\sqrt{2r},\sqrt{2r})}(t)\,dt$ in Example 4.46, then $F_\mu(z) = \sqrt{z^2-2r}$ and so $F(z) := F_{\delta_a\rhd\mu}(z) = \sqrt{z^2-2r} - a$. Observe first that F has an analytic extension to $\mathbb{C} \setminus [-\sqrt{2r}, \sqrt{2r}]$, which we denote by the same symbol F. If $a > 0$, then F has a zero at $x = \sqrt{a^2+2r}$, while F has no zero on $(-\infty, -\sqrt{2r})$ as $F(-x) = -\sqrt{x^2-2r} - a < 0$ for $x > \sqrt{2r}$. In view of Proposition 4.39 (iv) and its proof, $\delta_a \rhd \mu$ has an atom at $\sqrt{a^2+2r}$ and its weight is $1/F'(\sqrt{a^2+2r}) = a/\sqrt{a^2+2r}$. By the Stieltjes inversion formula $\delta_a \rhd \mu$ has a Lebesgue absolutely continuous part on $[-\sqrt{2r}, \sqrt{2r}]$ with density

$$\frac{-1}{\pi}\Im\frac{1}{\sqrt{(t+i0)^2-2r}-a} = \frac{-1}{\pi}\Im\frac{1}{i\sqrt{2r-t^2}-a} = \frac{\sqrt{2r-t^2}}{\pi(a^2+2r-t^2)}.$$

By symmetry, a similar result holds for $a < 0$, and consequently, we obtain

$$\delta_a \rhd \mu = \frac{|a|}{\sqrt{a^2+2r}}\delta_{\mathrm{sign}(a)\sqrt{a^2+2r}} + \frac{\sqrt{2r-t^2}}{\pi(a^2+2r-t^2)}\chi_{(-\sqrt{2r},\sqrt{2r})}(t)\,dt, \qquad a \in \mathbb{R},\ r > 0.$$

Example 5.4. Let μ_r be the Marchenko–Pastur distribution with scale parameter $r > 0$ in Example 4.48. For $a \in \mathbb{R}$ we have

$$F_{\delta_a \rhd \mu_r}(z) = \frac{z - 2a + \sqrt{(z-2r)^2 - 4r^2}}{2}.$$

In particular, $\delta_{2r} \rhd \mu_r$ is the semicircle distribution with mean $2r$ and variance r^2:

$$(\delta_{2r} \rhd \mu_r)(dt) = \frac{1}{2\pi r^2}\sqrt{4r^2 - (t-2r)^2}\,\chi_{[0,4r]}(t)\,dt;$$

see (4.38) and (5.1).

In a similar vein, we can define multiplicative monotone convolution by extending the formula $\eta_{\sqrt{x}y\sqrt{x}}(z) = \eta_x(\eta_y(z))$ in Theorem 1.29.

Theorem 5.5. *Let μ, ν be probability measures on $[0, +\infty)$ and on $\mathbb{R}$, respectively. Suppose that $\nu \neq \delta_0$. Let $\tilde{\eta}_\mu$ denote the analytic continuation of η_μ to $\mathbb{C} \setminus [0, +\infty)$ as given in Proposition 4.41. Then there exists a unique probability measure λ on $\mathbb{R}$ such that*

$$\eta_\lambda(z) = \tilde{\eta}_\mu(\eta_\nu(z)), \qquad z \in \mathbb{C}^+.$$

The measure λ is denoted by $\mu \circlearrowright \nu$ and is called the ***multiplicative monotone convolution*** *of μ and ν. We also define $\mu \circlearrowright \delta_0 := \delta_0$.*

Proof. If $\mu = \delta_0$ then $\eta_\lambda = 0$ and so $\lambda = \delta_0$. We therefore assume $\mu \neq \delta_0$. Note that the property $\eta_\nu(\mathbb{C}^+) \subseteq \mathbb{C} \setminus [0, +\infty)$ in Proposition 4.38 implies that the composition $\tilde{\eta}_\mu \circ \eta_\nu$ is well defined and is holomorphic on $\mathbb{C}^+$. Moreover, if $\arg \eta_\nu(z) \in [\arg z, \pi]$ then, by Proposition 4.41, $\pi \geq \arg \tilde{\eta}_\mu(\eta_\nu(z)) \geq \arg \eta_\nu(z) \geq \arg z$. If $\arg \eta_\nu(z) \in (\pi, \arg z + \pi]$ then by the symmetry $\tilde{\eta}_\mu(\overline{z}) = \overline{\tilde{\eta}_\mu(z)}$ we deduce that $\pi \leq \arg \tilde{\eta}_\mu(\eta_\nu(z)) \leq \arg \eta_\nu(z) \leq \arg z + \pi$. In any case the inequality $\arg z \leq \arg \eta_\lambda(z) \leq \arg z + \pi$ holds on $\mathbb{C}^+$, and so condition (ii) in Proposition 4.38 holds. It remains to check condition (iii) in Proposition 4.38. As $z \to 0, z \in \nabla_\gamma$, $\eta_\nu(z)$ converges to 0, and moreover, the inequality $\arg z \leq \arg \eta_\nu(z) \leq \arg z + \pi$ implies $\arg \eta_\nu(z) \in (\theta, 2\pi - \theta)$ for some $\theta \in (0, \pi)$. Therefore, using Proposition 4.40 (c) yields $\tilde{\eta}_\mu(\eta_\nu(z)) \to 0$.

The uniqueness of λ is a consequence of the uniqueness result in Proposition 4.34 and the relation $\eta_\lambda(z) = 1 - zF_\lambda(1/z)$ noted in (4.18). ☐

Although $\eta_{\delta_0}(z) \equiv 0$ is not contained in the domain of $\tilde{\eta}_\mu$, the above exceptional definition $\mu \circlearrowright \delta_0 := \delta_0$ is natural because multiplicative monotone convolution comes from the distribution of $\sqrt{x}y\sqrt{x}$ and $\nu = \delta_0$ corresponds to $y = 0$. This definition can also be justified from the perspective of continuity.

Proposition 5.6. *Let μ, μ_n $(n \in \mathbb{N})$ be probability measures on $[0, +\infty)$ and ν, ν_n $(n \in \mathbb{N})$ be probability measures on $\mathbb{R}$ such that $\mu_n \to \mu$ and $\nu_n \to \nu$ weakly. Then $\mu_n \circlearrowright \nu_n \to \mu \circlearrowright \nu$ weakly.*

Proof. Observe first that the weak convergence of measures is equivalent to the locally uniform convergence of η-transforms due to (4.18) and Proposition 4.33.

In case $\nu \neq \delta_0$, the proof is the same as Proposition 5.2 because $\eta_\nu(z)$ belongs to $\mathbb{C} \setminus [0, +\infty)$ that is the domain of $\tilde{\eta}_{\mu_n}$ and $\tilde{\eta}_\mu$.

The case $\nu = \delta_0$ needs to be handled separately. In this case, we first extend Proposition 4.36 as follows: for a tight family $\mathcal{P}$ of probability measures on $[0, +\infty)$, it holds that

$$\lim_{\substack{z \to \infty \\ \theta < \arg z < 2\pi - \theta}} \sup_{\mu' \in \mathcal{P}} \left| \frac{\tilde{F}_{\mu'}(z)}{z} - 1 \right| = 0 \tag{5.2}$$

for each $\theta \in (0, \pi)$. The proof is almost the same; the required modification is that inequality (4.12) holds for all z with $\arg z \in (\theta, 2\pi - \theta)$ and all $t \geq 0$ where $\theta := \arctan \gamma$ and that $|t/(z-t)|$ can be bounded by $t/|z|$ for $\Re(z) < 0$ instead of $t/\Im(z)$.

In our situation, since the weakly convergent family $\{\mu_n : n \in \mathbb{N}\}$ is tight, (5.2) yields

$$\lim_{\substack{z \to 0 \\ \theta < \arg z < 2\pi - \theta}} \sup_{n \in \mathbb{N}} |\tilde{\eta}_{\mu_n}(z)| = 0.$$

This estimate and the fact that $\eta_{\nu_n}(z) \to 0$ and $\arg z \leq \arg \eta_{\nu_n}(z) \leq \arg z + \pi$ (if $\nu_n \neq \delta_0$) imply that $\tilde{\eta}_{\mu_n}(\eta_{\nu_n}(z)) \to 0$ as $n \to \infty$ for each $z \in \mathbb{C}^+$. This further implies the weak convergence $\mu_n \circlearrowright \nu_n \to \delta_0$ by Proposition 4.33. ☐

Example 5.7. In a similar way to Example 5.3, $\mu \circlearrowright \delta_b$ is the dilation of μ by $b \in \mathbb{R}$, i.e., the pushforward of the measure μ by the map $x \mapsto bx$. On the other hand, $\delta_a \circlearrowright \nu$ is different from the dilation for generic $a > 0$ and ν.

In the rest of this section and also in Sections 6 and 7, we focus on additive monotone convolution and simply call it monotone convolution. In many cases similar results can be obtained for multiplicative monotone convolution but are omitted. We come back to multiplicative monotone convolution in Section 8 to explore the eigenvalues of perturbed random matrices.

5.1. **Support and moments for monotone convolution.** We study support and moments of monotone (additive) convolution of probability measures.

Definition 5.8. Let S, T be topological spaces. A **Borel kernel** from S to T is a function $\tau\colon S \times \mathcal{B}(T) \to [0, +\infty]$ such that

- $B \mapsto \tau(s, B)$ is a Borel measure for every $s \in S$,
- $s \mapsto \tau(s, B)$ is measurable for every $B \in \mathcal{B}(T)$.

If each $\tau(s, \cdot)$ is a probability measure on T, we call τ a **probability kernel** (from S to T). If $S = T$ we simply say τ is a Borel kernel on S.

The following fact is well known in the theory of Markov processes.

Lemma 5.9. *Let R, S, T be topological spaces. Let ρ be a Borel measure on S, σ be a Borel kernel from R to S and τ be a Borel kernel from S to T. Then the* ***compositions***

$$(\rho\tau)(B) := \int_S \rho(ds)\tau(s, B), \qquad B \in \mathcal{B}(T), \tag{5.3}$$

$$(\sigma\tau)(r, B) := \int_S \sigma(r, ds)\tau(s, B), \qquad r \in R,\ B \in \mathcal{B}(T) \tag{5.4}$$

define a Borel measure on T and a Borel kernel from R to T, respectively. Moreover, for every measurable function $f\colon S \to [0, +\infty]$,

$$\int_T f(t)\,(\rho\tau)(dt) = \int_S \left[\int_T f(t)\tau(s, dt)\right] \rho(ds), \tag{5.5}$$

$$\int_T f(t)\,(\sigma\tau)(r, dt) = \int_S \left[\int_T f(t)\tau(s, dt)\right] \sigma(r, ds), \qquad r \in R. \tag{5.6}$$

Remark 5.10. In mathematics, it is more common to write an integrand in front of a measure; however, when discussing composition, the notation such as (5.3) and (5.4) is more convenient; see also Section 7.2.

Proof. Since $\sigma\tau$ is more general, we only discuss $\sigma\tau$. The σ-additivity of $\sigma\tau$ for the second component follows from the monotone convergence theorem. For the measurability $r \mapsto (\sigma\tau)(r, B)$, we return to the definition of integration and take a sequence of nonnegative simple functions $f_n(s) \uparrow \tau(s, B)$ of the form $f_n(s) = \sum_{i=1}^{m_n} a_{n,i}\chi_{A_{n,i}}(s)$. Then the function $r \mapsto \int_S f_n(s)\,\sigma(r, ds) = \sum_{i=1}^{m_n} a_{n,i}\sigma(r, A_{n,i})$ is measurable. Therefore its limit $\int_S \sigma(r, ds)\tau(s, B)$ is also a measurable function of r.

Formula (5.6) is obvious for linear combinations of indicator functions, and then extends to general f's by the monotone convergence theorem. □

Lemma 5.11. *Let S be a topological space. Let $(\tau_s)_{s\in S}$ be a family of finite Borel measures on $\mathbb{R}$. Then $\tau(s, B) := \tau_s(B)$ is a Borel kernel from S to $\mathbb{R}$ if and only if $s \mapsto G_{\tau_s}(z)$ is measurable for every $z \in \mathbb{C}^+$.*

Proof. If τ is a Borel kernel, then the definition of integral (i.e., approximating the integrand $1/(z - x)$ by simple functions) implies the measurability of $s \mapsto G_{\tau_s}(z)$. Conversely, if $s \mapsto G_{\tau_s}(z)$ is measurable then the Stieltjes inversion formula (Proposition 4.30) implies that $s \mapsto \tau_s(I)$ is measurable for each finite open interval I. By taking the limit, the same holds for infinite intervals I. Since each $\tau(s, \cdot)$ is a finite measure, the set

$$\mathcal{L} := \{B \in \mathcal{B}(\mathbb{R}) : s \mapsto \tau(s, B) \text{ is measurable}\}$$

is a λ-system and contains the π-system $\mathcal{I}$ of all open intervals of $\mathbb{R}$. By the π-λ theorem (Theorem 4.9), $\mathcal{L}$ contains $\sigma(\mathcal{I})$ that coincides with $\mathcal{B}(\mathbb{R})$. Therefore, $s \mapsto \tau(s, B)$ is measurable for any $B \in \mathcal{B}(\mathbb{R})$. □

Proposition 5.12. *Let μ, ν be probability measures on $\mathbb{R}$. For $x \in \mathbb{R}$ we define*

$$\tilde{\nu}(x, dy) := (\delta_x \rhd \nu)(dy).$$

Then $\tilde{\nu}(\cdot, \cdot)$ is a probability kernel on $\mathbb{R}$ and the identity $\mu \rhd \nu = \mu\tilde{\nu}$ holds, i.e.,

$$(\mu \rhd \nu)(B) = \int_{\mathbb{R}} \mu(dx)\tilde{\nu}(x, B), \qquad B \in \mathcal{B}(\mathbb{R}). \tag{5.7}$$

Proof. Since $x \mapsto G_{\tilde{\nu}(x,\cdot)}(z) = G_{\delta_x}(F_\nu(z)) = \frac{1}{F_\nu(z)-x}$ is measurable in x, Lemma 5.11 ensures that $\tilde{\nu}(\cdot, \cdot)$ is a probability kernel. Denote by λ the right-hand side of (5.7), which is a probability measure. For $z \in \mathbb{C}^+$, we apply (5.5) to $f(y) := 1/(z - y)$ (by decomposing f as $f = g_+ - g_- + i(h_+ - h_-)$) to obtain

$$\begin{aligned}
G_\lambda(z) &= \int_{\mathbb{R}} \frac{1}{z - y}\,\lambda(dy) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{1}{z - y}\,\tilde{\nu}(x, dy)\right] \mu(dx) \\
&= \int_{\mathbb{R}} \frac{1}{F_{\tilde{\nu}(x,\cdot)}(z)}\,\mu(dx) = \int_{\mathbb{R}} \frac{1}{F_\nu(z) - x}\,\mu(dx) \\
&= G_\mu(F_\nu(z)),
\end{aligned}$$

so that $F_\lambda = F_\mu \circ F_\nu$. $\square$

The probability kernel in Proposition 5.12 is helpful for studying monotone convolution, in particular to see how a property of $\mu \rhd \nu$ is inherited to μ and ν. Below we study the support and moments of monotone convolution. Another remarkable aspect is a connection of this probability kernel to a certain Markov process, which will be explored in Section 7.2.

Proposition 5.13. *Let μ, ν be probability measures on $\mathbb{R}$. Then $\mu \rhd \nu$ has compact support if and only if both μ, ν have compact support. Moreover, if $\mu \rhd \nu$ is supported on $[-R, R]$ then F_ν extends analytically to $\mathbb{C} \setminus [-R, R]$ with $F_\nu(\overline{z}) = \overline{F_\nu(z)}$.*

Proof. If μ and ν have compact support, then there are monotonically independent real random variables x and y in a unital C^*-probability space such that the distributions of x and y are μ and ν, respectively. Since the distribution of $x + y$ coincides with $\mu \rhd \nu$, it has compact support.

Suppose that $\mu \rhd \nu$ has compact support. Let $\tilde{\nu}$ be the probability kernel defined in Proposition 5.12. There exists some $B = \{x \in \mathbb{R} : |x| > R\}$ such that

$$0 = (\mu \rhd \nu)(B) = \int_{\mathbb{R}} \mu(dx)\tilde{\nu}(x, B),$$

and so $\tilde{\nu}(x, B) = 0$ for μ-a.e. $x \in \mathbb{R}$. Pick such an $x \in \mathbb{R}$ that $\tilde{\nu}(x, B) = 0$. Then $F_{\tilde{\nu}(x,\cdot)}$ has an analytic continuation to $\mathbb{C} \setminus [-R, R]$ such that $F_{\tilde{\nu}(x,\cdot)}(\overline{z}) = \overline{F_{\tilde{\nu}(x,\cdot)}(z)}$, and the same holds for $F_\nu(z) = F_{\tilde{\nu}(x,\cdot)}(z) + x$. By Proposition 4.39, ν is supported on some compact interval.

To show that μ is compactly supported, it suffices by symmetry to show that the support of μ is bounded from above. We use the expansion in Proposition 4.39 (with the same symbol F_μ for simplicity)

$$F_\nu(z) = z - \sum_{n=1}^{\infty} \frac{b_n}{z^{n-1}}, \qquad |z| > R,$$

which implies $F_\nu(z) = z + O(1)$ as $z \to \infty$. Hence, there is $c > 0$ such that for all $y \in \mathbb{R}$ with $y \geq R + 1$ one has $y - c < F_\nu(y) < y + c$. Since $F_\nu'(y) \geq 1$ on $(R, +\infty)$ (see (4.23)), this implies that for each x with $x \geq R + c + 1$ the function $F_{\tilde{\nu}(x,\cdot)}(y) = F_\nu(y) - x$ has a unique zero y_x on $(x - c, x + c)$. This means that $\tilde{\nu}(x, \cdot)$ has an atom at y_x, so that $\tilde{\nu}(x, (x - c, x + c)) > 0$. Since for $x \geq R + c + 1$ we have $(x - c, x + c) \subseteq B$, so that $\tilde{\nu}(x, B) > 0$. However, $\tilde{\nu}(x, B) = 0$ for μ-a.e. x, i.e., there is an $S \in \mathcal{B}(\mathbb{R})$ such that $\mu(S) = 1$ and $\tilde{\nu}(x, B) = 0$ for all $x \in S$. Therefore, $S \cap (R + c + 1, +\infty) = \emptyset$, showing that the support of μ is bounded from above. $\square$

We extend the combinatorial formula (1.9) for the moments of the sum $x + y$ to the monotone convolution of probability measures.

Proposition 5.14. *Let μ and ν be probability measures on $\mathbb{R}$ and let $n \in \mathbb{N}$. Then the $2n$th moment $\int_{\mathbb{R}} t^{2n}\,(\mu \rhd \nu)(dt)$ is finite if and only if both $\int_{\mathbb{R}} t^{2n}\,\mu(dt)$ and $\int_{\mathbb{R}} t^{2n}\,\nu(dt)$ are finite. Moreover, if all these $2n$th moments are finite, then we have, for all $1 \leq p \leq 2n$,*

$$m_p(\mu \rhd \nu) = \sum_{\ell=0}^{p} \sum_{\substack{k_0, k_1, \dots, k_\ell \geq 0, \\ k_0 + k_1 + \cdots + k_\ell = p - \ell}} m_\ell(\mu) m_{k_0}(\nu) m_{k_1}(\nu) \cdots m_{k_\ell}(\nu); \tag{5.8}$$

in particular,

$$m_1(\mu \rhd \nu) = m_1(\mu) + m_1(\nu), \tag{5.9}$$

$$\mathrm{Var}(\mu \rhd \nu) = \mathrm{Var}(\mu) + \mathrm{Var}(\nu). \tag{5.10}$$

Proof. First we assume that $\int_{\mathbb{R}} t^{2n}\,(\mu \rhd \nu)(dt)$ is finite. By Proposition 5.12 and Lemma 5.9, we obtain, with notation $\tilde{\nu}(x, \cdot) = (\delta_x \rhd \nu)(\cdot)$,

$$\int_{\mathbb{R}} t^{2n}\,(\mu \rhd \nu)(dt) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} t^{2n}\,\tilde{\nu}(x, dt) \right] \mu(dx),$$

which implies $\int_{\mathbb{R}} t^{2n}\,\tilde{\nu}(x, dt) < +\infty$ for μ-a.e. x. We choose such an x. According to Proposition 4.43, $F_{\tilde{\nu}(x,\cdot)}(z)$ has an expansion of the form (4.28), and the same applies to $F_\nu(z) = F_{\tilde{\nu}(x,\cdot)}(z) + x$. Thus we obtain $\int_{\mathbb{R}} t^{2n}\,\nu(dt) < +\infty$. For the finiteness of $\int_{\mathbb{R}} t^{2n}\,\mu(dt)$ we use an asymptotic expansion of the inverse function of F_ν. Because we do not use this part later and the proof is rather long, the proof is postponed to Appendix.

Next we assume that $\int_{\mathbb{R}} t^{2n}\,\mu(dt)$ and $\int_{\mathbb{R}} t^{2n}\,\nu(dt)$ are finite. As discussed in the proof of Theorem 5.1, for any fixed $\gamma > 0$ we have $F_\nu(iy) \in \nabla_\gamma$ for all sufficiently large y. By (4.28) on the domain ∇_γ, we obtain

$$F_\mu(F_\nu(iy)) = F_\nu(iy) - b_1 - b_2 G_\nu(iy) - \cdots - b_{2n} G_\nu(iy)^{2n-1} + o(|F_\nu(iy)|^{-(2n-1)}). \tag{5.11}$$

The remainder term above can be written in the form $o(y^{-(2n-1)})$ because

$$(iy)^{2n-1}o(|F_\nu(iy)|^{-(2n-1)}) = \left(\frac{y}{F_\nu(iy)}\right)^{2n-1} \cdot F_\nu(iy)^{2n-1}o(|F_\nu(iy)|^{-(2n-1)}) \to 0$$

as $y \to \infty$. Expanding $G_\nu(iy)$ and $F_\nu(iy)$ in the forms (4.27) and (4.28) respectively, substituting them into (5.11) and recollecting the terms shows that for some reals $c_1, \cdots, c_{2n}$

$$F_\mu(F_\nu(z)) = z - c_1 - \frac{c_2}{z} - \cdots - \frac{c_{2n}}{z^{2n-1}} + o(|z|^{-(2n-1)})$$

for $z = iy$ as $y \to +\infty$. Again Proposition 4.43 guarantees that the $2n$th moment of $\mu \rhd \nu$ is finite.

Finally, formula (5.8) is obtained by expanding the right hand side of $G_{\mu\rhd\nu}(z) = G_\mu(1/G_\nu(z))$, which is just tracing the calculations (1.10)–(1.14) backwards, where the infinite sum is to be replaced with truncated finite sum with remainder terms and $\varphi(x^\ell), \varphi(y^k)$ are to be replaced with $m_\ell(\mu), m_k(\nu)$, respectively. □

5.2. **Convolution semigroups.** In probability theory, time-homogeneous random walk is the sum of independent, identically distributed random variables $(X_n)_{n\geq 1}$:

$$S_n := X_1 + X_2 + \cdots + X_n; \qquad S_0 := 0.$$

The distribution μ_n of S_n is the n-fold convolution $\mu * \mu * \cdots * \mu$, where μ is the distribution of X_1. Obviously, we have $\mu_m * \mu_n = \mu_{m+n}$ and $\mu_0 = \delta_0$. A continuous-time analogue of random walk is called a Lévy process, which is characterized by a convolution semigroup $(\mu_t)_{t\geq 0}$, i.e., $\mu_t, t \geq 0$, are probability measures on $\mathbb{R}$ such that $\mu_{s+t} = \mu_s * \mu_t,\ s,t \geq 0$, $\mu_0 = \delta_0$ and $t \mapsto \mu_t$ is weakly continuous.

We consider a monotone analogue of Lévy processes. The process itself will be explored later in Section 7. Here we investigate the distributional properties of monotone convolution semigroups with complex-analytic methods.

Definition 5.15. A family $(\mu_t)_{t\geq 0}$ of probability measures on $\mathbb{R}$, indexed by nonnegative reals t, is called a **monotone convolution semigroup** if

(i) $t \mapsto \mu_t$ is weakly continuous, i.e., for every bounded continuous function f on $\mathbb{R}$, the function $t \mapsto \int_{\mathbb{R}} f(x)\mu_t(dx)$ is continuous.

(ii) $\mu_{s+t} = \mu_s \rhd \mu_t$ for all $s,t \geq 0$,

(iii) $\mu_0 = \delta_0$, where δ_0 is the delta measure at 0.

Theorem 5.16. *Let $(\mu_t)_{t\geq 0}$ be a monotone convolution semigroup. Let F_t be the reciprocal Cauchy transform of μ_t. Then the limit*

$$A(z) := \lim_{t\to 0^+} \frac{F_t(z) - z}{t} \tag{5.12}$$

exists locally uniformly on $\mathbb{C}^+$ and it holds that

(i) *A is a Nevanlinna function and $\sphericalangle \lim_{z\to\infty} A(z)/z = 0$,*

(ii) $\dfrac{d}{dt}F_t(z) = A(F_t(z)), \quad t \geq 0,\ z \in \mathbb{C}^+.$

Conversely, given a function A satisfying (i) *above then equation* (ii) *has a unique solution $(F_t)_{t\geq 0}$, which consists of the reciprocal Cauchy transforms of a monotone convolution semigroup $(\mu_t)_{t\geq 0}$.*

*The function A is called the **infinitesimal generator** of (F_t) and also of (μ_t).*

Proof. According to Proposition 4.33, the weak continuity $t \mapsto \mu_t$ ensures the continuity $t \mapsto F_t$ with respect to the locally uniform convergence. The existence of the limit (5.12) is part of the Berkson–Porta's work (see the original paper [32, Theorem 1.1] or [39, Theorem 10.1.4]). The inequality $\Im[F_{\mu_t}(z)] \geq \Im z$, which follows by (4.16), implies A is a Nevanlinna function. Taking the derivative $F_{s+t}(z) = F_s(F_t(z))$ with respect to s at $s = 0$ we get the differential equation (ii). It remains to show $\sphericalangle \lim_{z\to\infty} A(z)/z = 0$. To begin with, integrating equation (ii) yields

$$F_t(z) = z + \int_0^t A(F_s(z))\,ds, \qquad z \in \mathbb{C}^+. \tag{5.13}$$

Let $a := \sphericalangle \lim_{z\to\infty} A(z)/z$. Since $s \mapsto \mu_s$ is weakly continuous, for a fixed $t > 0$, the family $\{\mu_s : 0 \leq s \leq t\}$ is tight. By Proposition 4.36, $\sup_{s\in[0,t]} |F_s(iy) - iy| = o(y)$ as $y \to \infty$. This in particular implies that for any $\gamma > 0$, there is $y_0 > 0$ such that $F_s(iy) \in \nabla_\gamma$ for all $y > y_0$ and $s \in [0,t]$. Therefore,

$$\sup_{s\in[0,t]} \left|\frac{A(F_s(iy))}{iy} - a\right| = \sup_{s\in[0,t]} \left|\frac{A(F_s(iy))}{F_s(iy)} \cdot \frac{F_s(iy)}{iy} - a\right| \to 0 \quad \text{as} \quad y \to \infty.$$

Dividing (5.13) by $z = iy$ and using the obtained uniform convergence, we obtain

$$\frac{F_t(iy)}{iy} = 1 + \int_0^t \frac{A(F_s(iy))}{iy}\,ds \to 1 + ta \quad \text{as} \quad y \to \infty.$$

Since $\frac{F_t(iy)}{iy} \to 1$, a must be zero.

Conversely, if A is a function satisfying (i), then [32, Section 2] guarantees that the equation in (ii) has a unique solution $(F_t)_{t\geq 0}$ consisting of holomorphic self-maps of $\mathbb{C}^+$ such that $F_{s+t} = F_s \circ F_t$. To show F_t is the reciprocal Cauchy transform of a probability measure, it suffices to show $F_t(iy)/(iy) \to 1$. Let us use the PDE $\frac{\partial}{\partial t}F_t(z) = A(z)\frac{\partial}{\partial z}F_t(z)$ that arises by taking the derivative $d/ds|_{s=0}$ of the relation $F_{t+s}(z) = F_t(F_s(z))$. The integrated form reads

$$\frac{F_t(z)}{z} = 1 + \frac{A(z)}{z}\int_0^t \frac{\partial F_s}{\partial z}(z)\, ds. \tag{5.14}$$

By the assumption on A we know that $A(iy)/(iy) \to 0$ as $y \to +\infty$. To estimate the integral part, let us write the Nevanlinna formula

$$F_s(z) = a_s z - b_s + \int_{\mathbb{R}} \frac{1+xz}{x-z}\, \tau_s(dx).$$

Using the continuous function $f(s) := \Im[F_s(i)] = a_s + \tau_s(\mathbb{R})$ we obtain the bound

$$\begin{aligned}\left|\frac{\partial F_s}{\partial z}(iy)\right| &= \left|a_s + \int_{\mathbb{R}} \frac{1+x^2}{(x-iy)^2}\, \tau_s(dx)\right| \\ &\leq a_s + \int_{\mathbb{R}} \frac{1+x^2}{x^2+y^2}\, \tau_s(dx) \leq a_s + \tau_s(\mathbb{R}) = f(s), \qquad s \geq 0,\ y \geq 1.\end{aligned} \tag{5.15}$$

Combining this estimate and (5.14) yields

$$\left|\frac{F_t(iy)}{iy} - 1\right| \leq \left|\frac{A(iy)}{iy}\right| \int_0^t f(s)\, ds \to 0 \quad \text{as} \quad y \to +\infty,$$

thereby $F_t = F_{\mu_t}$ for some probability measure μ_t on $\mathbb{R}$. The continuity of $t \mapsto F_t(z)$, the semigroup relation $F_{s+t} = F_s \circ F_t$, and the initial condition $F_0 = \mathrm{id}$ imply that $(\mu_t)_{t\geq 0}$ is a monotone convolution semigroup. □

Remark 5.17. When proving $a = 0$ in the first part of the above proof, we could also use formula (5.14) instead of (5.13).

The Nevanlinna formula for $A(z)$ in Theorem 5.16 is of the form

$$A(z) = -\gamma + \int_{\mathbb{R}} \frac{1+xz}{x-z}\sigma(dx), \tag{5.16}$$

where $\gamma \in \mathbb{R}$ and σ is a finite Borel measure on $\mathbb{R}$. In this case we write $A(z) = A^{(\gamma,\sigma)}(z)$. This integral formula is referred to as the monotone Lévy–Khintchine representation of $(\mu_t)_{t\geq 0}$.

Example 5.18. In the following cases of A, we can explicitly solve the complex ODE in Theorem 5.16(ii). The function w^β below is defined on $\mathbb{C} \setminus [0, +\infty)$ as $|w|^\beta e^{i\beta \arg w}$ in such a way that $\arg w \in (0, 2\pi)$.

(a) Let $A(z) = A^{a,0}(z) = -a$ with $a \in \mathbb{R}$. Then $F_t(z) = z - at$ and $\mu_t = \delta_{at}$.

(b) Let $A(z) = -a + ib$ with $a \in \mathbb{R}$ and $b > 0$. Then $F_t(z) = z - (a - ib)t$ and

$$\mu_t(dx) = \frac{1}{\pi} \cdot \frac{bt}{(x-at)^2 + (bt)^2}\, dx, \qquad x \in \mathbb{R},\ t > 0,$$

see Example 4.45. Note that the measure σ for A can be computed from the Stieltjes inversion (see Theorem 4.23) and we obtain $\sigma(dx) = b[\pi(1+x^2)]^{-1}\, dx$.

(c) Let $A(z) = A^{0,r\delta_0}(z) = -\frac{r}{z}$ with $r > 0$. Then $F_t(z) = \sqrt{z^2 - 2rt}$. The measure μ_t is the arcsine law

$$\mu_t(dx) = \frac{1}{\pi\sqrt{2rt - x^2}}\, dx, \qquad |x| < \sqrt{2rt},$$

see Example 4.46.

(d) Let $A(z) = e^{i\rho\alpha\pi} z^{1-\alpha}$, where $\alpha \in (0, 2]$ and $\rho \in [0, 1] \cap [1 - 1/\alpha, 1/\alpha]$. Then F_t is given by $F_t(z) = (z^\alpha + te^{i\rho\alpha\pi})^{1/\alpha}$. The corresponding distribution μ_t is called a monotone stable distribution. This monotone convolution semigroup is characterized by the condition $D_c(\mu_t) = \mu_{c^\alpha t}$ for all $c, t > 0$, where $D_c(\nu)$ is the push-forward of ν by the mapping $x \mapsto cx$. See [84] and references therein for further information.

In what follows, we characterize the monotone convolution semigroups having compact support and finite moments of even orders. We also clarify the connection between convolution semigroups and monotone cumulants defined in Remark 3.11.

Proposition 5.19. *Let $(\mu_t)_{t\geq 0}$ be a monotone convolution semigroup with infinitesimal generator $A^{(\gamma,\sigma)}$. The following are equivalent:*

(1) *μ_t has compact support at every $t \geq 0$;*

(2) *μ_t has compact support at some $t > 0$;*

(3) *σ has compact support.*

If the above conditions are fulfilled then there are constants $C_1, C_2 > 0$ such that μ_t is supported on $[-C_1 - C_2 t, C_1 + C_2 t]$ for all $t \geq 0$.

Proof. (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (3). Suppose that μ_{t_0} is supported on a compact interval $[-R, R]$ for some $t_0 > 0$ and $R > 0$. For each $t \in (0, t_0)$, μ_t is also compactly supported according to Proposition 5.13 applied to $\mu_{t_0} = \mu_{t_0 - t} \rhd \mu_t$. Moreover, the same proposition shows $F_t := F_{\mu_t}$ has an analytic continuation to $\mathbb{C} \setminus [-R, R]$ with $F_t(\overline{z}) = \overline{F_t(z)}$, and the same holds for the Nevanlinna function $A_t(z) := [F_t(z) - z]/t, t > 0$. This implies a Nevanlinna formula

$$A_t(z) = -\gamma_t + \int_{[-R,R]} \frac{1 + xz}{x - z}\, \sigma_t(dx).$$

According to Theorem 5.16, the function $A_t(z)$ converges to $A(z)$ on $\mathbb{C}^+$ as $t \to 0^+$, so that $\Im[A_t(i)] = \sigma_t([-R, R])$ converges to a finite nonnegative number. In particular, the family $\{\sigma_t([-R, R]) : 0 < t < t_0\}$ is bounded. This implies that $\{A_t : t \in (0, t_0)\}$ is uniformly bounded on each compact subset of $\mathbb{C} \setminus [-R, R]$. By Vitali's theorem (Theorem 4.15), A_t converges to a holomorphic function $\tilde{A}$ locally uniformly on $\mathbb{C} \setminus [-R, R]$ as $t \to 0^+$. The function $\tilde{A}$ is an analytic continuation of $A^{(\gamma,\sigma)}$ to $\mathbb{C} \setminus [-R, R]$, taking real values on $\mathbb{R} \setminus [-R, R]$. By the Stieltjes inversion formula, we have $\sigma(\mathbb{R} \setminus [-R, R]) = 0$.

(3) $\Longrightarrow$ (1). Suppose that σ is supported on $[-R, R]$. Let $\rho(dx) := (1 + x^2)\sigma(dx)$. Then we can write

$$A(z) = A^{(\gamma,\sigma)}(z) = -a + \int_{[-R,R]} \frac{1}{x - z}\, \rho(dx), \tag{5.17}$$

where $a := \gamma + \int_{\mathbb{R}} x\, \sigma(dx)$. Denote by $\tilde{A}$ the analytic continuation of A to $\mathbb{C} \setminus [-R, R]$ given by the right hand side of (5.17).

The idea is to solve the differential equation $\frac{d}{dt}\tilde{F}_t(z) = \tilde{A}(\tilde{F}_t(z)), \tilde{F}_0(z) = z$ by the usual Picard iteration method, and show that the solution is holomorphic on $\mathbb{C} \setminus [-R_t, R_t]$ for some $R_t > 0$ with $\tilde{F}_t(\overline{z}) = \overline{\tilde{F}_t(z)}$. Beware that the existence of unique solution $(\tilde{F}_t)_{t \geq 0}$ is already known for $z \in \mathbb{C} \setminus \mathbb{R}$ in Theorem 5.16, but for $z \in \mathbb{R} \setminus [-R, R]$ a solution does not globally exists in general, as $\tilde{F}_t(z)$ might hit $+R$ or $-R$ in finite time.

Observe first that there exists $C > 0$ such that

$$|\tilde{A}(z)| \leq C \quad \text{for all} \quad z \in \mathbb{C}, |z| > R + 1. \tag{5.18}$$

Let $F_t^n(z), n = 0, 1, 2, ...$, be recursively defined by

$$F_t^n(z) = z + \int_0^t \tilde{A}(F_s^{n-1}(z))\, ds, \qquad F_t^0(z) := z.$$

All F_t^n are well defined holomorphic functions on $|z| > R+Ct+2$. To see this, it suffices to show that $|F_t^n(z)| \geq R+2$ for all $n \in \mathbb{N}, t \geq 0$ and $|z| \geq R + Ct + 2$. Indeed, supposing the claim is the case for $F_t^{n-1}(z)$, we have for all $t \geq 0$ and $|z| \geq R + Ct + 2$

$$|F_t^n(z)| \geq |z| - \int_0^t |\tilde{A}(F_s^{n-1}(z))|\, ds \geq R + Ct + 2 - Ct = R + 2$$

as desired.

We can then easily show that $|F_t^n(z)| \leq |z| + Ct$. Since $(F_t^n(z))_{n \geq 1}$ is known to converge whenever $z \in \mathbb{C} \setminus \mathbb{R}$ (one can also show this directly by Picard's iteration; a more general setting is treated in Theorem 6.11(i)), by Vitali's theorem, for each fixed $t \geq 0$, the functions F_t^n converge locally uniformly to a holomorphic function $\tilde{F}_t$ on $|z| > R + Ct + 2$ with $\tilde{F}_t(\overline{z}) = \overline{\tilde{F}_t(z)}$. By the dominated convergence theorem, we obtain

$$\tilde{F}_t(z) = z + \int_0^t \tilde{A}(\tilde{F}_s(z))\, ds, \qquad \tilde{F}_0(z) = z.$$

Therefore, we have constructed an analytic continuation $\tilde{F}_t$ of F_t on the domain $\mathbb{C} \setminus [-R - Ct - 2, R + Ct + 2]$. By Proposition 4.39, every μ_t has compact support.

Moreover, the proof of (3) $\Longrightarrow$ (1) shows that μ_t is supported on $[-R - Ct - 2, R + Ct + 2]$ because $\tilde{F}_t(z)$ satisfies $|\tilde{F}_t(z)| \geq R + 2$ for all $|z| \geq R + Ct + 2$ and hence $\tilde{F}_t$ has no zeros on $\mathbb{R} \setminus [-R - Ct - 2, R + Ct + 2]$; see Proposition 4.39 (iv). ☐

Remark 5.20. The above proof (2) $\Longrightarrow$ (3) actually shows that if μ_t is supported on $[-R_t, R_t]$ $(t > 0)$ then σ is supported on $\bigcap_{t>0}[-R_t, R_t]$.

Proposition 5.21. *Let $(\mu_t)_{t\geq 0}$ be a monotone convolution semigroup with infinitesimal generator $A^{(\gamma,\sigma)}$. If σ is supported on a compact interval $[-R,R]$, then A has a convergent series expansion*

$$A(z) = -\sum_{n=1}^{\infty} \frac{\alpha_n}{z^{n-1}}, \qquad |z| > R, \tag{5.19}$$

and the nth monotone cumulant of μ_t coincides with $t\alpha_n$ for all $t \geq 0$ and $n \in \mathbb{N}$.

Proof. Observe that

$$A(z) = -\gamma + \int_{\mathbb{R}} \left(\frac{1+x^2}{x-z} - t\right) \sigma(dx) = -a - \sum_{n\geq 0} \frac{1}{z^{n+1}} \int_{\mathbb{R}} x^n (1+x^2)\sigma(dx),$$

where $a := \gamma + \int_{\mathbb{R}} x\, \sigma(dx)$, so that the sequence $(\alpha_n)_{n\geq 1}$ in (5.19) is given by

$$\alpha_1 := a; \qquad \alpha_n := \int_{\mathbb{R}} x^{n-2}(1+x^2)\sigma(dx), \quad n \geq 2.$$

Taking the derivative of $F_{t+s}(z) = F_t(F_s(z))$ with respect to s at 0 yields the first order PDE

$$\frac{\partial}{\partial t} F_t(z) = A(z) \frac{\partial}{\partial z} F_t(z),$$

which is equivalent to $\frac{\partial}{\partial t} G_t(z) = A(z) \frac{\partial}{\partial z} G_t(z)$ and so its integrated form

$$G_t(z) = \frac{1}{z} + A(z) \int_0^t \frac{\partial}{\partial z} G_s(z)\, ds, \qquad t \geq 0. \tag{5.20}$$

As a result of Lemma 4.28, the map $(s,z) \mapsto \frac{\partial}{\partial z} G_s(z)$ is continuous, so that we may interchange the integral and $\frac{\partial}{\partial z}$ in (5.20). Let $m_n(t)$ be the nth moment of μ_t. In view of Proposition 5.19, for each $T > 0$, the series

$$G_t(z) = \sum_{n\geq 0} \frac{m_n(t)}{z^{n+1}}$$

converges uniformly on $[0,T] \times \{z : |z| > C_1 + C_2 T + 1\}$. This shows that $t \mapsto m_n(t) = \frac{1}{2\pi i} \int_{|z|=R} z^n G_t(z)\, dz$ is continuous on $[0,T]$, where $R := C_1 + C_2 T + 1$. The above arguments allow us to perform series expansions of functions in (5.20) to obtain

$$\sum_{n\geq 0} \frac{m_n(t)}{z^{n+1}} = \frac{1}{z} + \left(\sum_{n=1}^{\infty} \frac{-\alpha_n}{z^{n-1}}\right) \left(\sum_{n=0}^{\infty} \frac{-(n+1)}{z^{n+2}} \int_0^t m_n(s)\, ds\right).$$

Comparing the coefficient of $\frac{1}{z^{n+1}}$ and by Cauchy's product formula, we obtain $m_0(t) = 1$, $m_n(0) = 0$ for $n \in \mathbb{N}$ and

$$m_n'(t) = \sum_{\ell=0}^{n-1} (\ell+1)\alpha_{n-\ell} m_\ell(t), \qquad n \geq 1, t \geq 0. \tag{5.21}$$

This is exactly the recursion for $\mathbf{m}_n(t)$ in Proposition 3.12, so that $m_n(t) = \mathbf{m}_n(t)$ and α_n is the nth monotone cumulant of μ_1.

The above argument around (5.21) works for the rescaled monotone convolution semigroup $\tilde{\mu}_t := \mu_{st}, t \geq 0$, where $s > 0$ is fixed. Then $\tilde{m}_n(t) := m_n(st)$ is the nth moment of $\tilde{\mu}_t$ and the recursion (5.21) implies

$$\frac{d}{dt} \tilde{m}_n(t) = \sum_{\ell=0}^{n-1} (\ell+1) s\alpha_{n-\ell} \tilde{m}_\ell(t), \qquad n \geq 1, t \geq 0.$$

showing that $(s\alpha_n)_{n\geq 1}$ is the sequence of monotone cumulants of $\tilde{\mu}_1 = \mu_s$. ☐

Example 5.22 (Monotone Poisson distribution)**.** Let $A(z) = -1 + \frac{1}{1-z} = -\sum_{n\geq 1} \frac{1}{z^{n-1}}$. From Propositions 5.19 and 5.21, the corresponding monotone convolution semigroup $(\mu_t)_{t\geq 0}$ consists of compactly supported measures and $\kappa_n(\mu_t) = t$. Therefore μ_t has the same moment sequence as ρ_t in Theorem 3.16. Since μ_t has compact support, we conclude $\mu_t = \rho_t$ by Proposition A.3.

By the standard technique to solve the ODE, the solution $(F_t)_{t\geq 0}$ to the ODE in Theorem 5.16 (ii) is given by the implicit formula $h(F_t(z)) = h(z) + t$, where

$$h(z) = C + \int_i^z \frac{dw}{A(w)} = \log z - z,$$

where $\log z$ is the principal branch and the arbitrary constant C is suitably selected. Since $h'(z) = 1/A(z)$ has negative imaginary part on $\mathbb{C}^+$, by Lemma B.1 h is injective. We determine the range $h(\mathbb{C}^+)$. This is basically determined by $h(\mathbb{R})$ (see discussions below). The function h on $(0,1]$ is strictly increasing with range $(-\infty,-1]$ and h on $[1,+\infty)$ is strictly decreasing with the same range $(-\infty,-1]$. The function h on $(-\infty,0)$ is injective with

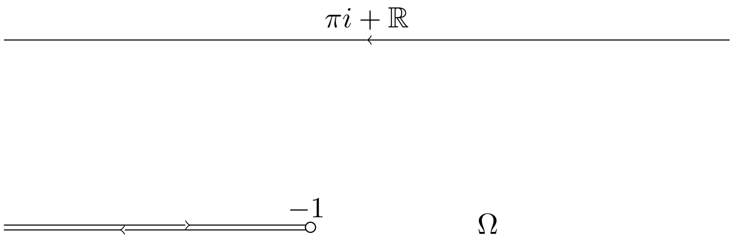

FIGURE 2. The range $h(\mathbb{C}^+)$

range $i\pi + \mathbb{R}$. This shows that $h(\mathbb{C}^+)$ is the region $\Omega := \{z \in \mathbb{C} : \Im(z) < \pi\} \setminus (-\infty, -1]$ (see Figure 2). Indeed, considering the behavior $h(w) = -w + o(w)$ as $w \to \infty$, every point $z \in \Omega$ has rotation number 1 with respect to the closed simple curve

$$h(w), \quad w \in [-R, -1/R] \cup \{(1/R)e^{-i\theta} : -\pi \le \theta \le 0\} \cup [1/R, R] \cup \{Re^{i\theta} : 0 \le \theta \le \pi\}$$

for sufficiently large $R > 1$, so that $z \in h(\mathbb{C}^+)$ by the argument principle. We can easily see from $\Im \log z < \pi$ that $h(\mathbb{C}^+) \subseteq \{z \in \mathbb{C} : \Im(z) < \pi\}$. We can also see that any $x \in (-\infty, -1]$ does not belong to $h(\mathbb{C}^+)$; indeed, if $z = re^{i\theta} \in \mathbb{C}^+$ and $h(z) \in \mathbb{R}$ then the condition $\Im[h(z)] = 0$ implies $r = \theta/\sin\theta$. Then the function

$$f(\theta) := \Re[h(re^{i\theta})] = \log\frac{\theta}{\sin\theta} - \frac{\theta\cos\theta}{\sin\theta}$$

has the positive derivative $f'(\theta) = [(\theta - \sin\theta\cos\theta)^2 + \sin^4\theta]/[\theta\sin^2\theta]$, so that $f(\theta) > f(+0) = -1$.

Note that Ω is invariant under the positive shifts $z \mapsto z + t$, which is actually a direct consequence of $h(F_t(z)) = h(z) + t$. Then the formula $F_t(z) = h^{-1}(h(z) + t)$ is well defined. Since h is analytic and injective on $(-\infty, 0)$, the function F_t extends to $(-\infty, 0)$, which is analytic and takes values in $(-\infty, 0)$. This implies that μ_t is supported on $[0, +\infty)$. This fact can also be deduced from Theorem 3.16 because the weak convergence limit of measures on $[0, +\infty)$ is supported on $[0, +\infty)$ as well.

Since h^{-1} has singularity at -1, let β_t be the unique solution $x > 1$ to $h(x) + t = -1$ and α_t the unique solution $0 < x < 1$ to $h(x) + t = -1$. Then F_t has an analytic extension to $(0, \alpha_t) \cup (\beta_t, +\infty)$ taking values in $(0, +\infty) \setminus \{1\}$. Therefore, μ_t is supported on $\{0\} \cup [\alpha_t, \beta_t]$. We can actually show that F_t extends to a continuous function on $\mathbb{C}^+ \cup \mathbb{R}$. For $x \in (\alpha_t, \beta_t)$ we can see that $h(x) + t \in \Omega$, so that $F_t(x) \in \mathbb{C}^+$. If $x = \alpha_t$ or $x = \beta_t$ then $F_t(x) = 1$. This implies by the Stieltjes inversion that μ_t has a continuous density function $p_t(x)$ on $[\alpha_t, \beta_t]$, positive on (α_t, β_t) and vanishing at the edges.

Finally, let us study F_t at 0. From $F_t(z) = h^{-1}(h(z) + t)$ we see that $\lim_{x \to 0^-} F_t(x) = 0$. The exponential form of $h(F_t(z)) = h(z) + t$ reads $F_t(z)e^{-F_t(z)} = ze^{-z+t}$. This has a unique analytic solution $F_t(z)$ at 0 having a convergent power series $F_t(z) = e^t z + O(z^2), z \to 0$ with real coefficients. Therefore, F_t is analytic at 0 and $G_{\mu_t}(z) = e^{-t}/z + O(1/z^2), z \to 0$, showing that $\mu_t(\{0\}) = e^{-t}$. Altogether, we have

$$\mu_t = \rho_t = e^{-t}\delta_0 + p_t(x)\chi_{[\alpha_t, \beta_t]}(x)\,dx,$$

where p_t is continuous on $[\alpha_t, \beta_t]$, positive in the interior and vanishing at the edges. In fact, $p(x)$ can be expressed with the Lambert W function as

$$p_t(x) = \frac{1}{\pi}\Im\frac{1}{W_{-1}(-xe^{-x+t})}\,dx;$$

see the original article [116] and [25] for further information.

Theorem 5.23. *Let $(\mu_t)_{t\ge0}$ be a monotone convolution semigroup, $A = A^{(\gamma,\sigma)}$ be its infinitesimal generator, and $n \in \mathbb{N}$. The following statements are equivalent:*

(1) $\displaystyle\int_{\mathbb{R}} x^{2n}\,\mu_t(dx) < +\infty$ *for some $t > 0$;*

(2) $\displaystyle\int_{\mathbb{R}} x^{2n}\,\mu_t(dx) < +\infty$ *for all $t > 0$;*

(3) $\displaystyle\int_{\mathbb{R}} x^{2n}\,\sigma(dx) < +\infty.$

Moreover, if the above conditions are satisfied then for all $t \geq 0$

$$\kappa_1(\mu_t) = m_1(\mu_t) = t\left(\gamma + \int_{\mathbb{R}} x\,\sigma(dx)\right),$$
$$\kappa_p(\mu_t) = t\int_{\mathbb{R}} x^{p-2}(1+x^2)\,\sigma(dx), \qquad 2 \leq p \leq 2n.$$

Proof. Throughout the proof we use the simplified symbols $F_t := F_{\mu_t}, G_t := G_{\mu_t}$, and also $m_p(t) := \int_{\mathbb{R}} x^p\,\mu_t(dx)$ whenever $\int_{\mathbb{R}} |x|^p\,\mu_t(dx) < +\infty$.

(1) $\Longrightarrow$ (2) is a direct consequence of Proposition 5.14 and of the semigroup relation $\mu_t = \mu_{t-s} \rhd \mu_s, 0 \leq s \leq t$; note that we do not need the part $\int_{\mathbb{R}} x^{2n}\,(\mu \rhd \nu)(dx) < +\infty \Longrightarrow \int_{\mathbb{R}} x^{2n}\,\mu(dx) < +\infty$, whose proof has been postponed to Appendix.

(2) $\Longrightarrow$ (3). First observe that $t \mapsto m_p(t)$ is measurable for any $1 \leq p \leq 2n$.

Step 1 to (2) $\Longrightarrow$ (3). We first show that $t \mapsto m_p(t)$ is locally integrable with respect to the Lebesgue measure. For this purpose, we will show more strongly that $m_p(t)$ is a polynomial in t. The starting point is the formula in Proposition 5.14:

$$m_p(t+s) = m_p(t) + m_p(s) + \sum_{\ell=1}^{p-1} \sum_{\substack{k_0,k_1,\dots,k_\ell \geq 0\\ k_0+k_1+\cdots+k_\ell = p-\ell}} m_\ell(t) m_{k_0}(s)\cdots m_{k_\ell}(s) \tag{5.22}$$

for $1 \leq p \leq 2n$. As shown in (3.6), the polynomials $\mathbf{m}_1(t), \mathbf{m}_2(t), \ldots$ satisfy the same relation:

$$\mathbf{m}_p(t+s) = \mathbf{m}_p(t) + \mathbf{m}_p(s) + \sum_{\ell=1}^{p-1} \sum_{\substack{k_0,k_1,\dots,k_\ell \geq 0\\ k_0+k_1+\cdots+k_\ell = p-\ell}} \mathbf{m}_\ell(t) \mathbf{m}_{k_0}(s)\cdots \mathbf{m}_{k_\ell}(s) \tag{5.23}$$

for $p \in \mathbb{N}$. Here we select (A, φ) and $x_i \in A$ in Proposition 3.12 so that $\varphi(x_i^p) = m_p(1), 0 \leq p \leq 2n$. We will show that $m_p(t) = \mathbf{m}_p(t)$ for all $t \geq 0$ and $1 \leq p \leq 2n$ by induction on p.

For $p = 1$, formula (5.22) is just $m_1(t+s) = m_1(t) + m_1(s)$, i.e., Cauchy's functional equation. Since $t \mapsto m_1(t)$ is measurable, $m_1(t)$ should be linear, i.e., $m_1(t) = m_1(1)t = \varphi(x_1)t = \mathbf{m}_1(t)$. Assume that $m_i(t) = \mathbf{m}_i(t)$ for all $t \geq 0$ and $1 \leq i \leq p-1$. Subtracting (5.23) from (5.22) yields

$$m_p(t+s) - \mathbf{m}_p(t+s) = [m_p(t) - \mathbf{m}_p(t)] + [m_p(s) - \mathbf{m}_p(s)],$$

which is again Cauchy's functional equation. Therefore $m_p(t) - \mathbf{m}_p(t) = (m_p(1) - \mathbf{m}_p(1))t$. Since $m_p(1) = m_p(\mu_1) = \varphi(x_i^p) = \mathbf{m}_p(1)$, we conclude $m_p(t) = \mathbf{m}_p(t)$ as desired.

Step 2 to (2) $\Longrightarrow$ (3). We fix an arbitrary $T > 0$. From equality (5.20) and the asymptotic expansion (4.34), we get as $z = iy, y \to +\infty$,

$$\begin{aligned} A(z) &= \frac{G_T(z) - 1/z}{\int_0^T \frac{\partial}{\partial z} G_t(z)\,dt} \\ &= \frac{\frac{m_1(T)}{z^2} + \cdots + \frac{m_{2n}(T)}{z^{2n+1}} + o(|z|^{-(2n+1)})}{\left(-\frac{1}{z^2}\right)\left(1 + \frac{2\int_0^T m_1(t)\,dt}{z} + \cdots + \frac{(2n+1)\int_0^T m_{2n}(t)\,dt}{z^{2n}} - z^2 \int_0^T R_t'(z)\,dt\right)}, \end{aligned} \tag{5.24}$$

where $R_t'(z)$ is the z-derivative of the remainder term in (4.34), i.e.,

$$R_t'(z) := -\frac{2n+1}{z^{2n+2}} \int_{\mathbb{R}} \frac{x^{2n+1}}{z-x}\mu_t(dx) - \frac{1}{z^{2n+1}} \int_{\mathbb{R}} \frac{x^{2n+1}}{(z-x)^2}\mu_t(dx).$$

Since $m_{2n}(t)$ is a polynomial in t, we have $\int_0^T (\int_{\mathbb{R}} x^{2n}\,\mu_t(dx))dt < +\infty$, which easily implies that $\int_0^T R_t'(iy)\,dt = o(y^{-(2n+2)})$ by the dominated convergence theorem. Therefore, using the geometric series expansion $1/(1+\zeta) = 1 - \zeta + \zeta^2 + \cdots$ to the denominator of (5.24) and recollecting terms, we find reals $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ such that

$$A(z) = -\alpha_1 - \frac{\alpha_2}{z} - \cdots - \frac{\alpha_{2n}}{z^{2n-1}} + Q(z) \tag{5.25}$$

where $Q(iy) = o(y^{-(2n-1)}), y \to +\infty$. By Proposition 4.43, we have $m_{2n}(\sigma) < +\infty$.

(3) $\Longrightarrow$ (1). Since $m_{2n}(\sigma) < +\infty$, again Proposition 4.43 yields the expansion (5.25) in each sector domain: $Q(z) = o(|z|^{-(2n-1)})$ as $z \in \nabla_\gamma, z \to \infty$ for each fixed $\gamma > 0$. Then the integral equation (5.13) reads

$$F_t(z) = z - \alpha_1 t - \sum_{\ell=1}^{2n-1} \alpha_{\ell+1} \int_0^t F_s(z)^{-\ell}\,ds + \int_0^t Q(F_s(z))\,ds, \qquad z = iy. \tag{5.26}$$

As discussed in the proof of Theorem 5.23, thanks to the tightness of $\{\mu_t : t \in [0,T]\}$ for each fixed $T > 0$ and $\gamma > 0$, there is $y_0 > 0$ such that $\{F_t(iy) : 0 \le t \le T, y > y_0\} \subseteq \nabla_\gamma$, and also the asymptotic behavior $F_t(iy) = iy(1 + o(1))$ holds uniformly on $t \in [0,T]$. From this observation, we can deduce the following uniform estimates over $0 \le t \le T$:

$$\int_0^t Q(F_s(iy))\, ds = o(y^{-(2n-1)}), \tag{5.27}$$

$$\int_0^t F_s(iy)^{-\ell}\, ds = t(iy)^{-\ell}(1 + o(1)). \tag{5.28}$$

Plugging (5.27) and (5.28) for $\ell = 1$ into (5.26) yields

$$F_t(iy) = z - \alpha_1 t - \frac{\alpha_2 t}{iy} + o(y^{-1}),$$

which is again uniform, i.e., the modulus of the remainder term $o(y^{-1})$ is bounded by a function $f(y)$ independent of $t \in [0,T]$ such that $yf(y) \to 0$ as $y \to +\infty$. Plugging this improved estimate into (5.26) then gives

$$F_t(iy) = iy - \alpha_1 t - \frac{\alpha_2 t}{iy} - \frac{\alpha_3 t + (\alpha_1\alpha_2 t^2)/2}{(iy)^2} + o(y^{-2}).$$

Repeating these arguments amounts to

$$F_t(iy) = iy - b_1(t) - \frac{b_2(t)}{iy} - \cdots - \frac{b_{2n}(t)}{(iy)^{2n-1}} + o(y^{-(2n-1)}), \qquad y \to +\infty.$$

for some polynomials $b_1(t), b_2(t), ..., b_{2n}(t)$ with real coefficients. From Proposition 4.43 we conclude $\int_{\mathbb{R}} x^{2n}\, \mu_t(dx) < +\infty$ for all $0 \le t \le T$.

On σ and monotone cumulants. In the proof of Step 2 above, we used

$$G_t(z) - \frac{1}{z} = A(z) \int_0^t \frac{\partial}{\partial z} G_s(z)\, ds.$$

Substituting the truncated Laurent series for $G_t(z)$ and $A(z)$ and comparing the coefficients yields exactly the relations (5.21) up to the order $2n$. Then the remaining proof is identical to the proof of Proposition 5.21. ☐

5.3. **Infinitely divisible distributions.** The concept of infinitely divisible distribution is closely related to convolution semigroups.

Definition 5.24. A probability measure μ on $\mathbb{R}$ is said to be **monotonically infinitely divisible** if for every $n \in \mathbb{N}$ there exists a probability measure $\mu_{1/n}$ such that μ is the n-fold monotone convolution of $\mu_{1/n}$.

It is obvious that each member of a monotone convolution semigroup $(\mu_t)_{t\ge 0}$ is monotonically infinitely divisible, as μ_t is the n-fold monotone convolution of $\mu_{t/n}$. This observation can be enhanced to the following.

Theorem 5.25. *Let μ be a probability measure on $\mathbb{R}$ having a determinate moment sequence. The following are equivalent.*

(1) *μ is infinitely divisible with respect to $\rhd$.*
(2) *there is a monotone convolution semigroup $(\mu_t)_{t\ge 0}$ such that $\mu_1 = \mu$.*
(3) *the sequence $(\kappa_n(\mu))_{n\ge 2}$ of monotone cumulants from order two is positive semi-definite: for every $p \in \mathbb{N}$ and $c_1, c_2, ..., c_p \in \mathbb{R}$ one has*

$$\sum_{m,n=1}^{p} c_m c_n \kappa_{m+n}(\mu) \ge 0.$$

(4) *there is a sequence of probability measures $(\nu_N)_{N\ge 1}$ with finite moments of all orders and strictly increasing positive integers $(\ell_N)_{N\ge 1}$ such that*

$$\lim_{N\to\infty} \int_{\mathbb{R}} x^n\, (\nu_N)^{\rhd \ell_N}(dx) = \int_{\mathbb{R}} x^n\, \mu(dx), \qquad n \in \mathbb{N}.$$

Proof. (2) $\Longrightarrow$ (1) is obvious from $\mu = \mu_1 = (\mu_{\frac{1}{N}})^{\rhd N}$ for all $N \in \mathbb{N}$.

(1) $\Longrightarrow$ (4) is also obvious as one can select $\nu_N := \mu_{1/N}$ and $\ell_N := N$.

(4) $\Longrightarrow$ (3). We show that

$$\kappa_n(\mu) = \lim_{N\to\infty} \ell_N \int_{\mathbb{R}} x^n\, \nu_N(dx), \qquad n \in \mathbb{N}. \tag{5.29}$$

To see this first we observe that

$$\kappa_n((\nu_N)^{\rhd \ell_N}) = \ell_N \kappa_n(\nu_N),$$

the left-hand side of which converges to $\kappa_n(\mu)$ since κ_n is a polynomial of moments up to order n. Also, recall that

$$m_n(\nu_N) = \kappa_n(\nu_N) + Q_n(\kappa_1(\nu_N), ..., \kappa_{n-1}(\nu_N)),$$

where the universal polynomial Q_n has no constant or linear term. Therefore, we obtain

$$\begin{aligned}\lim_{N\to\infty} \ell_N m_n(\nu_N) &= \lim_{N\to\infty} \ell_N \kappa_n(\nu_N) + \lim_{N\to\infty} \ell_N Q_n(\kappa_1(\nu_N), ..., \kappa_{n-1}(\nu_N)) \\ &= \lim_{N\to\infty} \ell_N \kappa_n(\nu_N) = \kappa_n(\mu)\end{aligned}$$

as desired. From (5.29) we obtain

$$\begin{aligned}\sum_{m,n=1}^{p} c_m c_n \kappa_{m+n}(\mu) &= \lim_{N\to\infty} \ell_N \sum_{m,n=1}^{p} c_m c_n \int_{\mathbb{R}} x^{m+n}\, \nu_N(dx) \\ &= \lim_{N\to\infty} \ell_N \int_{\mathbb{R}} \left| \sum_{n=1}^{p} c_n x^n \right|^2 \nu_N(dx) \geq 0.\end{aligned}$$

(3) $\Longrightarrow$ (2). By the assumption, the sequence $(\kappa_{n+1}(\mu))_{n\geq 0}$ is positive semi-definite, so that by [3, Theorem 2.1.1] or [134, Theorem 3.8], there exists a finite Borel measure ρ on $\mathbb{R}$ with finite moments of all orders such that

$$\kappa_n(\mu) = \int_{\mathbb{R}} x^{n-2}\, \rho(dx), \qquad n \geq 2.$$

We set

$$A(z) := -\kappa_1(\mu) + \int_{\mathbb{R}} \frac{1}{x-z} \rho(dx), \qquad z \in \mathbb{C}^+.$$

Since this is a Nevanlinna function with $\sphericalangle \lim_{z\to\infty} A(z)/z = 0$, by Theorem 5.16, there corresponds a monotone convolution semigroup $(\mu_t)_{t\geq 0}$. Theorem 5.23 ensures that every μ_t has finite moments of all orders. By the last statement of Theorem 5.23, the monotone cumulant $\kappa_n(\mu_t)$ coincides with $t\kappa_n(\mu)$ for all $t \geq 0$ and $n \in \mathbb{N}$; in particular, $\kappa_n(\mu_1) = \kappa_n(\mu)$. This means that μ_1 and μ have the same moment sequence. Since μ has a determinate moment sequence, we conclude that $\mu_1 = \mu$. □

5.4. **Notes.** The definition of additive monotone convolution of probability measures in Theorem 5.1 is due to Muraki [115, Definition 3.2]. Example 5.4 is a special case of examples in [112, Section 8]. Młotkowski introduced "Fuss-Catalan distributions", which behave nicely with respect to monotone convolution [111, Proposition 4.5]. The definition of multiplicative monotone convolution in Theorem 5.5 appeared in Arizmendi and Hasebe [9, Proposition 3.2] but its weak continuity in Proposition 5.6 is a new result. Both convolutions have unbounded operator models constructed by Franz [66]. Note that for multiplicative monotone convolution, Franz's model was restricted to the case where both measures are supported on $[0, +\infty)$, but the same technique is applicable to the general case of Theorem 5.5.

Theorem 5.16 was due to Muraki [115]. Our proof heavily depends on Berkson–Porta's work while the original proof was more straightforward. The exposition of Theorem 5.23 followed [76, Theorem 4.8]. Theorem 5.25 builds upon Hasebe [78, Theorem 8.5]. Theorem 5.25 for compactly supported μ was due to Muraki [115, Section 5] except condition (3). Belinschi proved the equivalence of (1) and (2) in Theorem 5.25 for arbitrary probability measures, as well as the uniqueness of the monotone convolution semigroup into which μ embeds [22]. Theorem 5.25 is an analogue of the free probability result [120, Theorem 13.16] in the case when μ is compactly supported; however, the proof of Theorem 5.25 is more complicated even if μ has compact support. The main difficulty is the absence of a priori bounds for monotone cumulants of the form $|\kappa_n(\mu)| \leq C^n$ for compactly supported measures μ; compare with the bounds for free cumulants [120, Lemma 13.13]. In the case of monotonically infinitely divisible distributions with compact support, this bound comes a posteriori as a result of Propositions 5.19, 5.21 and Theorem 5.25. For a general probability measure μ with compact support, it is still unknown whether the bound $|\kappa_n(\mu)| \leq C^n$ holds for some $C > 0$ or not.

Monotone convolution semigroups for multiplicative convolutions are also studied in the literature; see e.g. [30, 65]. There is a certain parallelism between additive and multiplicative cases, which was systematically studied in [5].

A remarkable feature of additive and multiplicative monotone convolutions is a connection to additive free convolution $\boxplus$ and multiplicative free convolution $\boxtimes$: for probability measures λ on $[0, +\infty)$ and μ, ν on $\mathbb{R}$ there exist probability measures $\rho = \rho_{\mu,\nu}$ and $\sigma = \sigma_{\lambda,\mu}$ on $\mathbb{R}$ such that

$$\mu \boxplus \nu = \mu \rhd \rho, \tag{5.30}$$

$$\lambda \boxtimes \mu = \lambda \circlearrowright \sigma; \tag{5.31}$$

see [20, 34] for additive convolution and [10] for multiplicative convolution. These relations have been used for the study of regularity properties of free convolutions, see e.g. [10, 23, 90]. Formulas (5.30) and (5.31) have an elegant interpretation in terms of graph products; see Accardi, Lenczewski and Sałapata [2] (additive case) and Lenczewski [99] (multiplicative case). Jekel and Liu's tree independence also allows an interpretation. In the context of Loewner theory, the monotone convolution hemigroups associated with free convolution hemigroups are studied e.g. in [67, 80, 88, 134]. Other notable connections between free probability and monotone probability can be found in Franz [65], Skoufranis [138], Cébron, Dahlqvist, Gabriel and Gilliers [40, 42], and Mingo and Tseng [109].

6. Monotone convolution hemigroups and Loewner theory

As already mentioned, convolution semigroups correspond to Lévy processes that are continuous-time analogues of random walk of identically distributed increments. We turn our attention to random walk whose increments are still independent but not necessarily identically distributed. Let $(X_i)_{i=1}^\infty$ be independent, $\mathbb{R}$-valued random variables. We denote by ν_i the distribution of X_i. Let us consider the random walk

$$S_n := X_1 + X_2 + \cdots + X_n; \qquad S_0 := 0.$$

The distribution of S_n is given by $\mu_n := \nu_1 * \nu_2 * \cdots * \nu_n$. For $n > m$ we have the relation $\mu_n = \mu_m * \nu_{m+1} * \cdots * \nu_n$, which involves ν_i's. To obtain a closed relation of distributions, it is more convenient to consider the increments

$$S_{m,n} := S_n - S_m, \quad 0 \le m \le n.$$

Let $\mu_{m,n}$ be the law of $S_{m,n}$. The obvious identity $S_{\ell,m} + S_{m,n} = S_{\ell,n}$ gives rise to the distributional relation

$$\mu_{\ell,m} * \mu_{m,n} = \mu_{\ell,n}, \qquad 0 \le \ell \le m \le n, \tag{6.1}$$

$$\mu_{n,n} = \delta_0, \qquad n \in \mathbb{N}_0. \tag{6.2}$$

The law of X_i can be recovered from the two parameter family $(\mu_{m,n})_{0\le m\le n}$ as $\mu_{i-1,i}$. Conversely, given a family $(\mu_{m,n})_{0\le m\le n}$ of probability measures on $\mathbb{R}$ with relations (6.1) and (6.2), it comes from a random walk.

The above discrete-time setup can be well extended to the continuous-time case. A family $(\mu_{s,t})$ of probability measures, index by real numbers $0 \le s \le t < \infty$, is called a convolution hemigroup if $\mu_{t,t} = \delta_0$ and $\mu_{s,t} * \mu_{t,u} = \mu_{s,u}$ for all $0 \le s \le t \le u$ and $(s,t) \mapsto \mu_{s,t}$ is weakly continuous. A convolution hemigroup corresponds to a stochastic process called an additive process or a process with independent increments. The reader is referred to [130] for further information. Replacing the convolution $*$ with monotone convolution, we are lead to the following.

Definition 6.1. Let $\triangle := \{(s,t) \in \mathbb{R}^2 : 0 \le s \le t < +\infty\}$. A family $(\mu_{s,t})_{(s,t)\in\triangle}$ of probability measures on $\mathbb{R}$ is called a **monotone convolution hemigroup** if

(i) $\triangle \ni (s,t) \mapsto \mu_{s,t}$ is weakly continuous, i.e., for every bounded continuous function f on $\mathbb{R}$, the function $(s,t) \mapsto \int_{\mathbb{R}} f(x)\mu_{s,t}(dx)$ is continuous.

(ii) $\mu_{s,u} = \mu_{s,t} \rhd \mu_{t,u}$ for all $0 \le s \le t \le u < +\infty$,

(iii) $\mu_{s,s} = \delta_0$ for all $s \ge 0$.

If a monotone convolution hemigroup satisfies $\mu_{s,t} = \mu_{0,t-s}$ for all $0 \le s \le t$, then it is reduced to the convolution semigroup: $\mu_{0,s} \rhd \mu_{0,t} = \mu_{0,s+t}$ holds. Conversely, given a monotone convolution semigroup (μ_t), the measures $\mu_{s,t} := \mu_{t-s}$ form a convolution hemigroup. Thus monotone convolution hemigroups generalize semigroups.

A monotone convolution hemigroup $(\mu_{s,t})$ can be described by its reciprocal Cauchy transforms.

Definition 6.2. A family of holomorphic self-maps $(F_{s,t})_{(s,t)\in\triangle}$ on $\mathbb{C}^+$ is called a **$\mathcal{P}$-reverse evolution family** ($\mathcal{P}$-REF for short)* if

(R1) $\sphericalangle \lim_{z\to\infty} F_{s,t}(z)/z = 1$ for all $(s,t) \in \triangle$,

(R2) $\triangle \ni (s,t) \mapsto F_{s,t}$ is continuous with respect to the locally uniform convergence on $\mathbb{C}^+$,

(R3) $F_{s,t} \circ F_{t,u} = F_{s,u}$ for all $0 \le s \le t \le u < +\infty$,

(R4) $F_{s,s}(z) = z$ for all $s \ge 0$ and $z \in \mathbb{C}^+$.

We often impose the additional condition that

(R5) for each $(s,t) \in \triangle$ there exist $m_{s,t} \in \mathbb{R}$ and $v_{s,t} \in [0,+\infty)$ such that

$$F_{s,t}(z) = z - m_{s,t} + \frac{v_{s,t}}{z} + o(z^{-1}), \qquad z = iy, y \to +\infty, \tag{6.3}$$

and the maps $(s,t) \mapsto m_{s,t}$ and $(s,t) \mapsto v_{s,t}$ are continuous.

We call $(F_{s,t})$ satisfying (R1)–(R5) a $\mathcal{P}_2$**-REF**. Moreover, if $m_{s,t} = 0$ for all (s,t), we call $(F_{s,t})$ a $\mathcal{P}_2^0$**-REF**.

*$\mathcal{P}$ stands for the fact that each function $F_{s,t}$ corresponds to a probability measure.

Remark 6.3. (a) The continuity $(s,t) \mapsto F_{s,t}$ is equivalent to the weaker condition that the map $\triangle \ni (s,t) \mapsto F_{s,t}(z) \in \mathbb{C}^+$ is continuous for all $z \in \mathbb{C}^+$, thanks to Proposition 4.33.

(b) In fact, the continuities of $(s,t) \mapsto m_{s,t}$ and $(s,t) \mapsto v_{s,t}$ in condition (R5) follow from the other conditions. This fact, however, requires rather long arguments and we refer the interested reader to [80].

(c) There is a one-to-one correspondence between the set of the monotone convolution hemigroups and the set of the $\mathcal{P}$-REFs. From Proposition 4.43, the $\mathcal{P}_2$-REFs exactly correspond to the monotone convolution hemigroups with finite first and second moments continuous with respect to (s,t). Let $(F_{s,t})_{(s,t)\in\triangle}$ be a $\mathcal{P}_2$-REF and $(\mu_{s,t})_{(s,t)\in\triangle}$ be the associated monotone convolution hemigroup. Formulas (5.9) and (5.10) together with (4.31) and (4.32) show that, with notation in (6.3),

$$\begin{aligned} m_{s,u} &= m_1(\mu_{s,u}) = m_1(\mu_{s,t}) + m_1(\mu_{t,u}) = m_{s,t} + m_{t,u}, \\ v_{s,u} &= \mathrm{Var}(\mu_{s,u}) = \mathrm{Var}(\mu_{s,t}) + \mathrm{Var}(\mu_{t,u}) = v_{s,t} + v_{t,u}, \qquad 0 \le s \le t \le u. \end{aligned}$$

(d) If $(F_{s,t})$ is a $\mathcal{P}_2$-REF then each $F_{s,t}$ has an integral formula

$$F_{s,t}(z) = z - m_{s,t} + \int_{\mathbb{R}} \frac{1}{x-z}\,\rho_{s,t}(dx),$$

where $\rho_{s,t}$ is a finite Borel measure such that $\rho_{s,t}(\mathbb{R}) = v_{s,t}$. We can see that the map $(s,t) \mapsto \rho_{s,t}$ is weakly continuous. Let $(s_n,t_n),(s,t) \in \triangle$ and $(s_n,t_n) \to (s,t)$. Let $\rho_n := \rho_{s_n,t_n}$ and $\rho := \rho_{s,t}$. If $\rho = 0$ then $\rho_n(\mathbb{R}) = v_{s_n,t_n} \to v_{s,t} = 0$, so that $\rho_n \to 0$ weakly. If $\rho \neq 0$ then $v_{s,t} > 0$ and we set $\overline{\rho} := \rho/v_{s,t}$ and $\overline{\rho}_n := \rho/v_{s_n,t_n}$. As $G_{\rho_n}(z) = z - m_{s_n,t_n} - F_{s_n,t_n}(z)$ converges to $G_\rho(z)$, the Cauchy transform of the normalized measure $G_{\overline{\rho}_n}(z)$ also converges to $G_{\overline{\rho}}(z)$ for each $z \in \mathbb{C}^+$. Therefore, by Proposition 4.33, $\overline{\rho}_n$ converges weakly to $\overline{\rho}$, which in turn implies that ρ_n converges weakly to ρ.

(e) A family $(F_{s,t})_{(s,t)\in\triangle}$ of holomorphic self-maps of $\mathbb{C}^+$ satisfying (R2)–(R4) is called a reverse evolution family. Such a family of holomorphic self-maps is well developed in Loewner theory, e.g. in [38]. The reason of the term "reverse" is that from the viewpoint of dynamics on $\mathbb{C}^+$, the alternative condition $F_{t,u} \circ F_{s,t} = F_{s,u}$ has a more natural interpretation that a point z at time s arrives at the point $F_{s,t}(z)$ at time t and then $F_{s,t}(z)$ arrives at the point $F_{t,u}(F_{s,t}(z))$ at time u, which coincides with $F_{s,u}(z)$. Such a family is called an evolution family.

Moreover, we consider a one-parameter family of holomorphic functions, which turns out to have a one-to-one correspondence with the $\mathcal{P}$-REFs and monotone convolution hemigroups.

Definition 6.4. A family $(F_t)_{t\ge0}$ of holomorphic self-maps of $\mathbb{C}^+$ is called a **$\mathcal{P}$-decreasing Loewner chain** ($\mathcal{P}$-DLC for short) if the following conditions (L1)–(L5) are satisfied:

(L1) $\sphericalangle \lim_{z\to\infty} F_t(z)/z = 1$ for every $t \ge 0$,

(L2) $t \mapsto F_t$ is continuous with respect to the locally uniform convergence,

(L3) F_t is injective on $\mathbb{C}^+$ for each $t \ge 0$,

(L4) the range $F_t(\mathbb{C}^+)$ is non-increasing with respect to $t \ge 0$,

(L5) $F_0(z) = z$ for all $z \in \mathbb{C}^+$.

We also consider the condition that

(L6) for every $t \ge 0$ there exist $m_t \in \mathbb{R}, v_t \in [0,+\infty)$ such that

$$F_t(z) = z - m_t + \frac{v_t}{z} + o(z^{-1}), \qquad z = iy,\ y \to +\infty, \tag{6.4}$$

and also the functions $t \mapsto m_t$ and $t \mapsto v_t$ are continuous.

We call (F_t) a **$\mathcal{P}_2$-DLC** if (L1)–(L6) are satisfied. Moreover, if $m_t = 0$ for all $t \ge 0$ then we call (F_t) a **$\mathcal{P}_2^0$-DLC**.

Example 6.5. (a) Let $(\mu_t)_{t\ge0}$ be a monotone convolution semigroup and $f\colon [0,+\infty) \to \mathbb{R}$ be a continuous nondecreasing function. Then $\mu_{s,t} := \mu_{f(t)-f(s)}$ form a monotone convolution hemigroup.

(b) The semicircle distribution of mean 0 and variance t has the reciprocal Cauchy transform

$$F_t(z) = \frac{z + \sqrt{z^2 - 4t}}{2}.$$

One can check that $(F_t)_{t\ge0}$ is a $\mathcal{P}_2^0$-DLC. First of all, F_t is of the form (6.4) with $m_t = 0$ and $v_t = t$. It remains to check conditions (L3) and (L4) since the others are easy. Condition (L3) can be directly confirmed: assuming $z, w \in \mathbb{C}^+$ and $F_t(z) = F_t(w)$, we obtain $z = w$ after algebraic calculations. In fact, we can show more strongly that F_t extends to a continuous injective function $\tilde{F}_t\colon \mathbb{C}^+ \cup \mathbb{R} \to \mathbb{C}^+ \cup \mathbb{R}$. Regarding condition (L4), from the previous consideration and Carathéodory's theorem for Jordan domains, the boundary of the domain $F_t(\mathbb{C}^+)$ is $\tilde{F}_t(\mathbb{R})$. For $x > 2\sqrt{t}$, the point $\tilde{F}_t(x) = (x+\sqrt{x^2-4t})/2$ moves over the half-line $[2\sqrt{t}, +\infty)$. For $x < -2\sqrt{t}$, note that the point $\sqrt{(x+0i)^2-4t}$ has argument π by the definition of square root, so that $\tilde{F}_t(x) = (x - \sqrt{x^2-4t})/2$ and its trajectory is the half-line $(-\infty, -2\sqrt{t})$. For $|x| \le 2\sqrt{t}$ by the Stieltjes inversion we have $\tilde{F}_t(x) = (x + i\sqrt{4t-x^2})/2$, which moves over the semi-circle $\{u+iv : u^2+v^2 = 4t, v > 0\}$. In conclusion,

$$F_t(\mathbb{C}^+) = \{u + iv \in \mathbb{C}^+ : u^2 + v^2 > 4t\},$$

which is decreasing with respect to $t \ge 0$.

This section establishes a correspondence between $\mathcal{P}_2$-REFs and $\mathcal{P}_2$-DLCs, and an integro-differential/integral equation for them, which generalizes the differential equation known in case the time-dependence is absolutely continuous.

6.1. **Reverse evolution families and Loewner chains.** We establish a bijection between the $\mathcal{P}_2$-REFs and $\mathcal{P}_2$-DLCs.

Lemma 6.6. *Let $\mathbb{C}^+_\beta := \{z \in \mathbb{C}^+ : \Im(z) > \beta\}$ for $\beta > 0$. Let μ be a probability measure with finite second moment. Then for any $\beta > 0$ one has*

$$F_\mu(z) = z + O(1), \qquad z \in \mathbb{C}^+_\beta,\ z \to \infty. \tag{6.5}$$

Moreover, F_μ is injective on $\mathbb{C}^+_\sigma$, where $\sigma := \sqrt{\mathrm{Var}(\mu)}$, and the range $F_\mu(\mathbb{C}^+_\sigma)$ contains $\mathbb{C}^+_{2\sigma}$.

Proof. Because the measure τ in Proposition 4.43 has finite second moment, the Nevanlinna formula for F_μ can be written in terms of $\rho(dx) := (1+x^2)\tau(dx)$ as

$$F_\mu(z) = z - m + \int_{\mathbb{R}} \frac{\rho(dx)}{x - z}, \tag{6.6}$$

where $m = m_1(\mu)$ and $\rho(\mathbb{R}) = \mathrm{Var}(\mu) = \sigma^2$. For $z \in \mathbb{C}^+_\beta$, the function $1/|x-z|$ is bounded by $1/\beta$, so that the dominated convergence theorem yields that the integral term in (6.6) goes to zero, and hence (6.5) holds true.

For every $z, w \in \mathbb{C}^+_\sigma$ with $z \ne w$ we have

$$\begin{aligned} |F_\mu(z) - F_\mu(w)| &= |z-w| \left| 1 + \int_{\mathbb{R}} \frac{\rho(dx)}{(x-z)(x-w)} \right| \\ &\ge |z-w| \left[1 - \int_{\mathbb{R}} \frac{\rho(dx)}{|x-z||x-w|} \right] \\ &\ge |z-w| \left[1 - \frac{\rho(\mathbb{R})}{\Im(z)\Im(w)} \right] > 0, \end{aligned}$$

thereby verifying the injectivity.

For all z with $\Im(z) = \sigma$ we have

$$\Im[F_\mu(z)] = \Im(z) + \int_{\mathbb{R}} \frac{\Im(z)}{(x - \Re(z))^2 + (\Im(z))^2} \rho(dx) \le \Im(z) + \frac{\sigma^2}{\Im(z)} = 2\sigma.$$

Combining this estimate and (6.5) yields that for every $w \in \mathbb{C}^+_{2\sigma}$ the curve $\{F_\mu(z) : z \in C\}$, where C is the boundary of a large square D in $\mathbb{C}^+_\sigma$ that has an edge $[-R, R] + i\sigma$, surrounds the point w. By the argument principle, the equation $F_\mu(z) = w$ has a solution $z \in D$. $\square$

Proposition 6.7. *Let $(F_{s,t})$ be a $\mathcal{P}_2$-REF. Then each map $F_{s,t}$ is injective on $\mathbb{C}^+$.*

Proof. Let $z, w \in \mathbb{C}^+$ with $z \ne w$ and $0 \le s \le t$. We select $\varepsilon > 0$ so that $z, w \in \mathbb{C}^+_\varepsilon$ and take $s = s_0 < s_1 < s_2 < \cdots < s_n = t$ such that $\min_{0 \le i \le n-1} |v_{s_i, s_{i+1}}| < \varepsilon^2$, which is possible by the continuity of $(r, u) \mapsto v_{r,u}$ and the fact $v_{r,r} = 0$. Let

$$\begin{aligned} z_k &:= F_{s_k, s_{k+1}} \circ \cdots \circ F_{s_{n-1}, s_n}(z), \\ w_k &:= F_{s_k, s_{k+1}} \circ \cdots \circ F_{s_{n-1}, s_n}(w), \qquad 0 \le k \le n-1. \end{aligned}$$

Note that $F_{s,t}(z) = z_0$ and $F_{s,t}(w) = w_0$. By the inequality $\Im[F_{s_{n-1}, s_n}(z)] \ge \Im(z)$ (see (4.17)) and by Lemma 6.6, the points z_{n-1} and w_{n-1} are distinct and lie in $\mathbb{C}^+_\varepsilon$; recall that $v_{s,t}$ is the variance of the probability measure associated with $F_{s,t}$. By induction on k in the decreasing direction, we can prove z_k, w_k are distinct and lie in $\mathbb{C}^+_\varepsilon$ for all $k = n-1, n-2, ..., 0$, so that $F_{s,t}(z) = z_0 \ne w_0 = F_{s,t}(w)$. $\square$

Theorem 6.8. *There is a one-to-one correspondence between $\mathcal{P}_2$-REFs and $\mathcal{P}_2$-DLCs given by the maps*

$$(F_{s,t})_{(s,t) \in \triangle} \mapsto (F_{0,t})_{t \ge 0}, \tag{6.7}$$

$$(F_t)_{t \ge 0} \mapsto (F_s^{-1} \circ F_t)_{(s,t) \in \triangle}. \tag{6.8}$$

Proof. Given $(F_{s,t})$, let $F_t := F_{0,t}$. Conditions (L1), (L2) and (L6) are obvious. By Proposition 6.7, each F_t is injective on $\mathbb{C}^+$. Moreover, $F_t \circ F_{t,u} = F_u$ holds for $0 \le t \le u$, which implies that $F_u(\mathbb{C}^+) = F_t(F_{t,u}(\mathbb{C}^+)) \subseteq F_t(\mathbb{C}^+)$, i.e., $t \mapsto F_t(\mathbb{C}^+)$ is non-increasing. Thus $(F_t)_{t\ge 0}$ is a $\mathcal{P}_2$-DLC.

Conversely, given a $\mathcal{P}_2$-DLC $(F_t)_{t\ge0}$, the assumption $F_s(\mathbb{C}^+) \supseteq F_t(\mathbb{C}^+), s \le t$ allows us to define the composed map $F_{s,t} := F_s^{-1} \circ F_t$ as a self-map of $\mathbb{C}^+$. It is well known that the inverse map of a holomorphic function is also holomorphic, so that $F_{s,t}$ is holomorphic. Conditions (R3) and (R4) are obvious.

Condition (R1). For each $s \ge 0$, recall from Lemma 6.6 that F_s^{-1} is defined on $\mathbb{C}^+_{2\sigma}$, where $\sigma^2 := v_s$ is the variance of the underlying probability measure. From (6.5), $w \to \infty$ implies $z = F_s^{-1}(w) \to \infty$, so that we have

$$\lim_{\substack{w\to\infty \\ w\in\mathbb{C}^+_{2\sigma}}} \frac{F_s^{-1}(w)}{w} = \lim_{\substack{z\to\infty \\ z\in\mathbb{C}^+_{\sigma}}} \frac{z}{F_s(z)} = 1.$$

Then

$$\sphericalangle \lim_{z\to\infty} \frac{F_{s,t}(z)}{z} = \sphericalangle \lim_{z\to\infty} \frac{F_s^{-1}(F_t(z))}{F_t(z)} \cdot \frac{F_t(z)}{z} = 1,$$

which verifies condition (R1).

Condition (R2). It suffices to show the continuity of $(s,t) \mapsto F_{s,t}(z)$ at each $z_0 \in \mathbb{C}^+$, see Remark 6.3. We use the Lagrange inversion formula. Let $(s_0, t_0) \in \triangle$. Since $F_{t_0}(z_0) \in F_{s_0}(\mathbb{C}^+)$, there exists an open disk D such that $\overline{D} \subseteq \mathbb{C}^+$ and $F_{t_0}(z_0) \in F_{s_0}(D)$. By the continuity of $t \mapsto F_t$ and since $F_{s_0}(D)$ is open, we can find open intervals $I \ni s_0$ and $J \ni t_0$ such that $F_t(z_0) \in F_s(D)$ for all $s \in I, t \in J$. Therefore, by the Lagrange inversion formula,

$$F_{s,t}(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{wF_s'(w)}{F_s(w) - F_t(z_0)}dw, \qquad s \in I,\ t \in J,\ s \le t.$$

The continuity $(s,t) \mapsto F_{s,t}(z_0)$ at (s_0,t_0) is now a consequence of the dominated convergence theorem.

Condition (R5). If we denote by μ_t and $\mu_{s,t}$ the underlying probability measures for F_t and $F_{s,t}$, respectively, then $\mu_s \rhd \mu_{s,t} = \mu_t$. Since μ_t has finite second moment due to (6.4), $\mu_{s,t}$ also has finite second moment from Proposition 5.14. Therefore, formula (6.3) holds and from Remark 6.3 (c) the numbers $m_{s,t}$ and $v_{s,t}$ satisfy

$$m_{s,t} = m_1(\mu_{s,t}) = m_1(\mu_t) - m_1(\mu_s) = m_t - m_s,$$
$$v_{s,t} = \mathrm{Var}(\mu_{s,t}) = \mathrm{Var}(\mu_t) - \mathrm{Var}(\mu_s) = v_t - v_s.$$

This implies the continuity of $(s,t) \mapsto m_{s,t}, v_{s,t}$, so that (R5) holds true. $\square$

Example 6.9. Let F_t be the reciprocal Cauchy transform of the semicircle distribution of mean 0 and variance $t \ge 0$. The family $(F_t)_{t\ge0}$ is a $\mathcal{P}_2^0$-DLC, see Example 6.5. This can be more easily shown from Theorem 6.8 by finding the corresponding REF. First, a formal algebraic calculation yields a formal inverse function $F_s^{-1}(z) = z + s/z$. Therefore, the corresponding REF should be $F_s^{-1} \circ F_t$, which is

$$F_{s,t}(z) := \frac{1}{2}\left(1 + \frac{s}{t}\right)z + \frac{1}{2}\left(1 - \frac{s}{t}\right)\sqrt{z^2 - 4t}, \qquad 0 \le s \le t.$$

We can check this is a Nevanlinna function with $F_{s,t}(iy)/(iy) \to 1$, so that it is the reciprocal Cauchy transform of a probability measure $\mu_{s,t}$. We can also check that $F_{s,t}(z) = z - (t-s)/z + O(z^{-2}), z \to \infty$ and $F_{s,t} \circ F_{t,u} = F_{s,u}, 0 \le s \le t \le u$. Therefore $(F_{s,t})$ is a $\mathcal{P}_2^0$-REF, and so $F_t := F_{0,t}$ form a $\mathcal{P}_2^0$-DLC.

6.2. **Integral/Integro-differential equations.** Infinitesimal descriptions are helpful to better understand reverse evolution families; later we will see in Section 7.1 that an infinitesimal description is useful for constructing an operator model for monotone additive processes. For example, suppose that the limit

$$A(s,z) = \lim_{h\to0^+} \frac{F_{s-h,s}(z) - z}{h}$$

exists. Then taking the derivative of $F_{s,u} = F_{s,t} \circ F_{t,u}$ with respect to s at $s = t$ yields the non-autonomous ODE

$$\frac{\partial F_{t,u}}{\partial t} = A(t, F_{t,u}(z)).$$

In the following, we establish a refined description of $\mathcal{P}_2^0$-REFs. In general, $F_{s,t}$ need not be differentiable in time, but still an integral/integro-differential equation holds. As a key lemma we use the following version of Radon-Nikodym's theorem that generalizes Lebesgue's differentiation theorem. The reader is referred to [59, Theorem 2, Section 1.6] for a proof.

Lemma 6.10. *Let μ and ν be Borel measures on $\mathbb{R}$ that are finite on any compact subset of $\mathbb{R}$. Suppose that μ is absolutely continuous with respect to ν. Then the limit*

$$\frac{d\mu}{d\nu}(x) := \lim_{h\to0^+} \frac{\mu(x-h,x+h)}{\nu((x-h,x+h))} \in [0,+\infty)$$

exists at ν-a.e. x, and it serves as a Radon-Nikodym derivative, i.e.,

$$\mu(B) = \int_B \frac{d\mu}{d\nu}(x)\,\nu(dx), \qquad B \in \mathcal{B}(\mathbb{R}).$$

Theorem 6.11. *Let $(F_{s,t})_{(s,t)\in\triangle}$ be a $\mathcal{P}_2^0$-REF having the asymptotic behavior* (6.3)*. Let τ be the Lebesgue-Stieltjes measure on $[0,+\infty)$ associated with the non-decreasing continuous function $t \mapsto v_{0,t}$. There exists a probability kernel $\dot\rho$ from $[0,+\infty)$ to $\mathbb{R}$ such that for all $0 \le s \le t$*

$$F_{s,t}(z) = z + \int_s^t \left[\int_{\mathbb{R}} \frac{1}{x - F_{a,t}(z)}\,\dot\rho(a,dx)\right]\tau(da), \tag{6.9}$$

$$F_{s,t}(z) = z + \int_s^t \frac{\partial F_{s,b}}{\partial z}(z)\left[\int_{\mathbb{R}} \frac{1}{x-z}\,\dot\rho(b,dx)\right]\tau(db). \tag{6.10}$$

Either of (6.9) *and* (6.10) *implies the uniqueness of $\dot\rho$ in the sense that if another probability kernel $\dot\sigma$ exists, then we must have $\dot\rho(t,\cdot) = \dot\sigma(t,\cdot)$ for τ-a.e. $t \ge 0$.*

Proof. From Proposition 4.43, for each (s,t), $F_{s,t}$ is of the form

$$F_{s,t}(z) = z + \int_{\mathbb{R}} \frac{\rho_{s,t}(dx)}{x-z}$$

and $v_{s,t} = \rho_{s,t}(\mathbb{R})$. We set $v_t := v_{0,t}$. The fact $F_{0,0} = \mathrm{id}$ implies $v_0 = 0$. From Remark 6.3 (c), we have $v_s + v_{s,t} = v_t$; in particular, $t \mapsto v_t$ is non-decreasing.

Let us consider the finite Borel measure $\overline{\rho}_{r,s}$ on $\widehat{\mathbb{R}}$ defined by

$$\overline{\rho}_{r,s}(\{\infty\}) = 0, \qquad \overline{\rho}_{r,s}|_{\mathbb{R}} := \begin{cases} \frac{\rho_{r,s}}{v_{r,s}}, & \text{if } v_{r,s} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\overline{\rho}_{r,s}(\widehat{\mathbb{R}}) \le 1$, for each $s \ge 0$ we can find a sequence $h_n \to 0^+$ such that $\overline{\rho}_{s-h_n,s+h_n}$ weakly converges to a Borel measure $\overline{\rho}_s$ on $\widehat{\mathbb{R}}$ with $\overline{\rho}_s(\widehat{\mathbb{R}}) \le 1$; be aware that h_n depends on s. Note that the following convergence holds whenever $v_{s+h} > v_{s-h}$ for all $h > 0$:

$$\frac{F_{s-h_n,s+h_n}(z) - z}{v_{s+h_n} - v_{s-h_n}} = \int_{\mathbb{R}} \frac{\overline{\rho}_{s-h_n,s+h_n}(dx)}{x-z} \to \int_{\mathbb{R}} \frac{\overline{\rho}_s(dx)}{x-z} = -G_{\overline{\rho}_s|_{\mathbb{R}}}(z), \qquad h \to 0^+. \tag{6.11}$$

Proof of (6.10). Observe that

$$\begin{aligned} F_{r,t}(z) - F_{r,s}(z) &= F_{r,s}(F_{s,t}(z)) - F_{r,s}(z) \\ &= [F_{s,t}(z) - z]\left[1 + \int_{\mathbb{R}} \frac{\rho_{r,s}(dx)}{(x-z)(x-F_{s,t}(z))}\right] \end{aligned} \tag{6.12}$$

and

$$|F_{s,t}(z) - z| = \left|\int_{\mathbb{R}} \frac{\rho_{s,t}(dx)}{x-z}\right| \le \frac{v_t - v_s}{\Im(z)}.$$

Therefore, for any fixed $r \ge 0$ and $z \in \mathbb{C}^+$, the function $f(s) := F_{r,s}(z)$ satisfies

$$|f(t) - f(s)| \le C_{t,z}(v_t - v_s), \qquad r \le s \le t,$$

where $C_{t,z} := 1/\Im(z) + v_t/\Im(z)^3$. In particular, for any fixed $T > 0$, f is of bounded variations on $[r,T]$, and so there exists a complex Borel measure ν on $(r,T]$ such that $\nu((s,t]) = f(t) - f(s)$ for $r \le s \le t \le T$. The inequality $|\nu((s,t])| \le C_{T,z}\tau((s,t])$ extends to

$$|\nu(B)| \le C_{T,z}\tau(B), \qquad B \in \mathcal{B}((r,T]). \tag{6.13}$$

To show (6.13), we consider the set $\mathcal{M} := \{B \in \mathcal{B}((r,T]) : |\nu(B)| \le C_{T,z}\tau(B)\}$, which is a monotone class. We also consider the algebra $\mathcal{A}$ consisting of the empty set and finite unions of disjoint intervals of the form $(s,t]$ $(r \le s \le t \le T)$. It is easy to check that $\mathcal{M}$ contains $\mathcal{A}$. By the monotone class theorem (Theorem 4.10), $\mathcal{M}$ contains $\sigma(\mathcal{A}) = \mathcal{B}((r,T])$.

Because of the arbitrariness of T, we can extend ν to a complex measure on $[r,+\infty)$ with domain the set of bounded Borel subsets. Inequality (6.13) implies that ν is absolutely continuous with respect to τ. By Lemma 6.10, the limit

$$D^{v(s)}F_{r,s}(z) := \lim_{h\to 0^+} \frac{\nu((s-h,s+h))}{\tau((s-h,s+h))} = \lim_{h\to 0^+} \frac{F_{r,s+h}(z) - F_{r,s-h}(z)}{v_{s+h} - v_{s-h}} \in \mathbb{C} \tag{6.14}$$

exists at τ-a.e. $s \in (r,+\infty)$. Let $J_{r,z}$ be the set of all $s \in (r,+\infty)$ such that this limit exists. Dividing (6.12) by $v_t - v_s$, replacing (s,t) with $(s-h,s+h)$ and passing to the limit $h \to 0^+$ yields

$$\lim_{h\to 0^+} \frac{F_{s-h,s+h}(z) - z}{v_{s+h} - v_{s-h}} = \frac{D^{v(s)}F_{r,s}(z)}{1 + \int_{\mathbb{R}} (x-z)^{-2}\,\rho_{r,s}(dx)} = \frac{D^{v(s)}F_{r,s}(z)}{\frac{\partial}{\partial z}F_{r,s}(z)}, \qquad s \in J_{r,z},\ z \in \mathbb{C}^+. \tag{6.15}$$

Note here that the weak convergence $\rho_{r,s-h} \to \rho_{r,s}$ shown in Remark 6.3 (d) was used in the first equality above.

On the other hand, at every $s \in J_{r,z}$, the inequality $v_{s+h} > v_{s-h}$ holds for all $h > 0$, and so we can deduce from (6.11) that

$$\lim_{h\to 0^+} \frac{F_{s-h,s+h}(z) - z}{v_{s+h} - v_{s-h}} = \lim_{h\to 0^+} \int_{\mathbb{R}} \frac{\overline{\rho}_{s-h,s+h}(dx)}{x - z} = \int_{\mathbb{R}} \frac{\overline{\rho}_s(dx)}{x-z}, \qquad s \in J_{r,z},\ z \in \mathbb{C}^+. \tag{6.16}$$

Combining (6.15) and (6.16) yields

$$D^{v(s)} F_{r,s}(z) = \frac{\partial F_{r,s}}{\partial z}(z) \int_{\mathbb{R}} \frac{1}{x-z}\, \overline{\rho}_s(dx), \qquad s \in J_{r,z},\ z \in \mathbb{C}^+. \tag{6.17}$$

Applying Lemma 6.10 to the measure ν and $B = [r,t]$ and using (6.17), we get

$$\begin{aligned} F_{r,t}(z) - z &= \int_r^t D^{v(s)} F_{r,s}(z)\, \tau(ds) \\ &= \int_r^t \frac{\partial F_{r,s}}{\partial z}(z) \left[\int_{\mathbb{R}} \frac{1}{x-z}\, \overline{\rho}_s(dx) \right] \tau(ds), \qquad z \in \mathbb{C}^+,\ 0 \le r \le t. \end{aligned} \tag{6.18}$$

Here we take any countable subset $A \subseteq \mathbb{C}^+$ having an accumulation point in $\mathbb{C}^+$, e.g. $A = \{i + 1/n : n \in \mathbb{N}\}$, and set $J := \bigcap_{z\in A} J_{0,z}$. For any $s \in J$ and $z \in A$ the convergence in (6.16) holds. Therefore, by Proposition 4.26, the convergence in (6.16) holds for all $z \in \mathbb{C}^+$ and $s \in J$. This implies that $J \ni s \mapsto G_{\overline{\rho}_s|_{\mathbb{R}}}(z)$ is measurable for each $z \in \mathbb{C}^+$, so that Lemma 5.11 implies that $J \ni s \mapsto \overline{\rho}_s(B)$ is measurable for each $B \in \mathcal{B}(\mathbb{R})$.

By the dominated convergence theorem applied to (6.18) (the estimates in (5.15) are helpful), we can see that

$$v_t = \lim_{y\to+\infty} [F_{0,t}(iy) - iy] iy = \int_0^t \overline{\rho}_s(\mathbb{R}) \tau(ds).$$

Since $\overline{\rho}_s(\mathbb{R}) \le 1$ and $\tau([0,t]) = v_t$, we must have $\overline{\rho}_s(\mathbb{R}) = 1$ for τ-a.e. $s \ge 0$. We can then define $\dot\rho(s,\cdot) := \overline{\rho}_s|_{\mathbb{R}}$ whenever $\overline{\rho}_s(\mathbb{R}) = 1$ and $s \in J$, and otherwise define $\dot\rho(s,\cdot) := \delta_0$. This is a probability kernel. Now formula (6.18) yields the desired formula (6.10).

Proof of (6.9). Observe that for $0 \le r \le s \le t$

$$F_{s,t}(z) - F_{r,t}(z) = F_{s,t}(z) - F_{r,s}(F_{s,t}(z)) = \int_{\mathbb{R}} \frac{\rho_{r,s}(dx)}{F_{s,t}(z) - x}, \tag{6.19}$$

and so $|F_{s,t}(z) - F_{r,t}(z)| \le (v_s - v_r)/\Im(z)$. As before, for any (t,z) there exists a complex Borel measure μ on $[0,t]$ such that $\mu((r,s]) = F_{s,t}(z) - F_{r,t}(z)$ for $0 \le r \le s \le t$ and

$$|\mu(B)| \le \frac{\tau(B)}{\Im(z)}, \qquad B \in \mathcal{B}((0,t]). \tag{6.20}$$

Inequality (6.20) implies that μ is absolutely continuous with respect to τ. By Lemma 6.10 the limit

$$D_{v(s)} F_{s,t}(z) := \lim_{h\to 0^+} \frac{F_{s+h,t}(z) - F_{s-h,t}(z)}{v_{s+h} - v_{s-h}} \in \mathbb{C} \tag{6.21}$$

exists at τ-a.e. $s \in (0,t)$. At any $s \in J$ where this limit exists, (6.19) and (6.16) yield

$$D_{v(s)} F_{s,t}(z) = \lim_{h\to 0^+} \int_{\widehat{\mathbb{R}}} \frac{\overline{\rho}_{s-h,s+h}(dx)}{F_{s+h,t}(z) - x} = \int_{\widehat{\mathbb{R}}} \frac{\overline{\rho}_s(dx)}{F_{s,t}(z) - x} = \int_{\mathbb{R}} \frac{\overline{\rho}_s(dx)}{F_{s,t}(z) - x}. \tag{6.22}$$

The last equality holds because the integrand vanishes at $x = \infty$. The second equality holds because of the continuity of $r \mapsto F_{r,t}$, i.e., in the triangular inequality

$$\begin{aligned} &\left| \int_{\widehat{\mathbb{R}}} \frac{\overline{\rho}_{s-h,s+h}(dx)}{F_{s+h,t}(z) - x} - \int_{\widehat{\mathbb{R}}} \frac{\overline{\rho}_s(dx)}{F_{s,t}(z) - x} \right| \\ &\le \underbrace{\left| \int_{\widehat{\mathbb{R}}} \frac{\overline{\rho}_{s-h,s+h}(dx)}{F_{s+h,t}(z) - x} - \int_{\widehat{\mathbb{R}}} \frac{\overline{\rho}_{s-h,s+h}(dx)}{F_{s,t}(z) - x} \right|}_{=:I_h^1} + \underbrace{\left| \int_{\widehat{\mathbb{R}}} \frac{\overline{\rho}_{s-h,s+h}(dx)}{F_{s,t}(z) - x} - \int_{\widehat{\mathbb{R}}} \frac{\overline{\rho}_s(dx)}{F_{s,t}(z) - x} \right|}_{=:I_h^2} \end{aligned}$$

the second term I_h^2 tends to zero due to (6.16) that holds for $s \in J$ and $z \in \mathbb{C}^+$, and the first term I_h^1 also converges to zero because

$$I_h^1 \le \int_{\widehat{\mathbb{R}}} \frac{|F_{s+h,t}(z) - F_{s,t}(z)|}{|F_{s+h,t}(z) - x||F_{s,t}(z) - x|}\, \overline{\rho}_{s-h,s+h}(dx) \le \frac{|F_{s+h,t}(z) - F_{s,t}(z)|}{\Im(z)^2}.$$

By Lemma 6.10 and (6.22), we have

$$F_{s,t}(z) - z = -\int_s^t D_{v(a)} F_{a,t}(z)\, \tau(da) = \int_s^t \left[\int_{\mathbb{R}} \frac{\overline{\rho}_a(dx)}{x - F_{a,t}(z)} \right] \tau(da),$$

which is nothing but (6.9).

Uniqueness of $\dot\rho$. For example, we assume that (6.9) holds for $\dot\rho$ and $\dot\sigma$. We fix $T>0$ and $z\in\mathbb{C}^+$ for some time. The complex measure

$$\lambda(B) := \int_B \left[\int_{\mathbb{R}} \frac{1}{x-F_{s,T}(z)}\,\dot\rho(s,dx)\right]\tau(ds), \qquad B\in\mathcal{B}([0,T]),$$

is absolutely continuous with respect to τ, so by Lemma 6.10 and uniqueness of Radon-Nikodym derivative, we obtain

$$\int_{\mathbb{R}} \frac{1}{x-F_{s,T}(z)}\,\dot\rho(s,dx) = \lim_{h\to 0^+}\frac{\lambda((s-h,s+h))}{\tau((s-h,s+h))} = \lim_{h\to0^+}\frac{F_{s-h,T}(z)-F_{s+h,T}(z)}{\tau((s-h,s+h))}$$

at τ-a.e. s. The same formula holds for $\dot\sigma$, so we obtain $G_{\dot\rho(s,\cdot)} = G_{\dot\sigma(s,\cdot)}$ on $F_{s,T}(\mathbb{C}^+)$ for τ-a.e. $s\in[0,T]$. By the identity theorem, the equality $G_{\dot\rho(s,\cdot)} = G_{\dot\sigma(s,\cdot)}$ holds on $\mathbb{C}^+$ and then we have $\dot\rho(s,\cdot)=\dot\sigma(s,\cdot)$ for τ-a.e. $s\in[0,T]$. A similar idea works for the case when we assume (6.10) instead. □

We also verify the converse direction: given τ and $\dot\rho$, solving these equations gives a unique $\mathcal{P}_2^0$-REF. The uniqueness is formulated in a stronger form.

Theorem 6.12. *Let τ be a Borel measure on $[0,+\infty)$ that is finite on any compact subset and that has no atom. Let $\dot\rho$ be a probability kernel and*

$$A(t,z) := \int_{\mathbb{R}} \frac{1}{x-z}\,\dot\rho(t,dx).$$

(i) *For each fixed $t\ge 0$ and $z\in\mathbb{C}^+$, the integral equation*

$$f(s) = z + \int_s^t A(r,f(r))\,\tau(dr), \qquad s\in[0,t] \tag{6.23}$$

has a unique solution $f(s)=f(s;t,z)\in\mathbb{C}^+, s\in[0,t]$, such that $[0,t]\ni s\mapsto f(s)\in\mathbb{C}^+$ is continuous.

(ii) *For each fixed $s\ge 0$, the integro-differential equation*

$$h(t,z) = z + \int_s^t \frac{\partial h}{\partial z}(r,z)A(r,z)\,\tau(dr), \qquad t\in[s,+\infty),\ z\in\mathbb{C}^+ \tag{6.24}$$

has a unique solution $h(t,z)=h(t,z;s)\in\mathbb{C}^+, t\in[s,+\infty), z\in\mathbb{C}^+$, such that $t\mapsto h(t,z)$ is continuous for each fixed z and $z\mapsto h(t,z)$ is holomorphic for each fixed t.

Moreover, $f(s;t,z)=h(t,z;s)$ holds for all $(s,t)\in\triangle, z\in\mathbb{C}^+$ and $F_{s,t}(z):=f(s;t,z)$ forms a $\mathcal{P}_2^0$-REF.

Proof. Uniqueness of a solution to (6.23). Let $t\ge 0$ and $z\in\mathbb{C}^+$ be fixed. Suppose that continuous functions $f_1,f_2\colon[0,t]\to\mathbb{C}^+$ satisfy (6.23). We easily obtain for $F(s):=f_1(s)-f_2(s)$

$$\begin{aligned}|F(s)| &\le \int_s^t \left[\int_{\mathbb{R}} \frac{|f_1(r)-f_2(r)|}{|x-f_1(r)||x-f_2(r)|}\dot\rho(r,dx)\right]\tau(dr)\\ &\le \frac{1}{\Im(z)^2}\int_s^t |F(r)|\,\tau(dr), \qquad s\in[0,t].\end{aligned}$$

Iterating this inequality yields

$$\begin{aligned}|F(s)| &\le \frac{1}{\Im(z)^{2n}}\int_{s\le s_1\le s_2\le\cdots\le s_n\le t} |F(s_n)|\,\tau^{\otimes n}(ds_1 ds_2\cdots ds_n)\\ &\le \frac{\tau([s,t])^n}{n!\Im(z)^{2n}}\sup_{r\in[0,t]}|F(r)|, \qquad s\in[0,t],\ n\in\mathbb{N},\end{aligned} \tag{6.25}$$

where we used the fact

$$\tau^{\otimes n}(\{(s_1,s_2,\dots,s_n): s\le s_1\le s_2\le\cdots\le s_n\le t\}) = \frac{\tau([s,t])^n}{n!}.$$

Since the right-hand side of (6.25) tends to zero as $n\to\infty$, we conclude $F\equiv 0$.

Existence of a solution to (6.23). The proof is based on Picard's iteration. We recursively define $F^0,F^1,F^2,\dots,$ by $F^0_{s,t}(z)\equiv z$ and

$$F^n_{s,t}(z) = z + \int_s^t A(r,F^{n-1}_{r,t}(z))\,\tau(dr), \qquad (s,t)\in\triangle,\ z\in\mathbb{C}^+,\ n\in\mathbb{N}. \tag{6.26}$$

Inductive arguments easily show that $F^n_{s,t}$ is holomorphic in $\mathbb{C}^+$ and satisfies $\Im[F^n_{s,t}(z)] \geq \Im(z)$. We then obtain from (6.26)

$$|F^n_{s,t}(z)| \leq |z| + \frac{\tau([s,t])}{\Im(z)},$$

$$|F^n_{s,t}(z) - F^n_{s',t'}(z)| \leq \frac{\tau([s,t]\Delta[s',t'])}{\Im(z)},$$

where $A\Delta B$ is the symmetric difference $(A \setminus B) \cup (B \setminus A)$. In particular, for each fixed $z \in \mathbb{C}^+$, the sequence of functions $f^z_n : \triangle \to \mathbb{C}^+$, $f^z_n(s,t) := F^n_{s,t}(z)$ is uniformly bounded and equicontinuous on each compact subset of $\triangle$; note that the latter follows from the (uniform) continuity of the function $t \mapsto \tau([0,t])$. We may therefore use Arzéla-Ascoli's theorem to find a subsequence $f^z_{n(k)}, k \in \mathbb{N}$, that converges to a function $f^z : \triangle \to \mathbb{C}^+$ locally uniformly. Passing to the limit in (6.26), the limit function $F_{s,t}(z) := f^z(s,t)$ satisfies equation (6.9) and hence $f(s) := F_{s,t}(z)$ satisfies (6.23).

The solution to (6.23) *forms a* $\mathcal{P}^0_2$*-REF.* As already proved, $F^n_{s,t}$ is a Nevanlinna function for each (s,t), so that its pointwise limit $F_{s,t}$ is also a Nevanlinna function; see Proposition 4.26. Moreover, the integral equation (6.9) and the dominated convergence theorem, together with the bound $1/|x - F_{a,t}(z)| \leq 1/\Im(z)$, yield $\lim_{y\to+\infty} F_{s,t}(iy)/(iy) = 1$ and

$$\lim_{y\to+\infty} iy[F_{s,t}(iy) - iy] = \tau([s,t]).$$

It remains to show $F_{s,t} \circ F_{t,u} = F_{s,u}$. For each fixed $z \in \mathbb{C}^+$ and $0 \leq t \leq u$, let $F_1, F_2 : [0,u] \to \mathbb{C}^+$ be defined by $F_1(s) := F_{s,u}(z)$ and

$$F_2(s) := \begin{cases} F_{s,t}(F_{t,u}(z)), & s \in [0,t], \\ F_{s,u}(z), & s \in (t,u], \end{cases}$$

which is continuous. Recalling the equation

$$F_{s,t}(z) = z + \int_s^t A(r, F_{r,t}(z))\,\tau(dr), \qquad z \in \mathbb{C}^+,\ 0 \leq s \leq t,$$

we have, for $s \in [0,t]$,

$$\begin{aligned} F_2(s) &= F_{t,u}(z) + \int_s^t A(r, F_2(r))\,\tau(dr) \\ &= z + \int_t^u A(r, F_{r,u}(z))\,\tau(dr) + \int_s^t A(r, F_2(r))\,\tau(dr) \\ &= z + \int_s^u A(r, F_2(r))\,\tau(dr), \end{aligned}$$

and for $s \in (t,u]$,

$$F_2(s) = F_{s,u}(z) = z + \int_s^u A(r, F_{r,u}(z))\,\tau(dr) = z + \int_s^u A(r, F_2(r))\,\tau(dr).$$

Therefore, F_2 satisfies exactly the same equation satisfied by F_1. By the trajectory-wise uniqueness, we conclude that $F_1 = F_2$ on $[0,u]$.

Existence of a solution to (6.24). We already constructed a $\mathcal{P}^0_2$-REF $(F_{s,t})$ that solves (6.9). On the other hand, by Theorem 6.11, there exists a probability kernel $\dot{\sigma}$ for which (6.9) and (6.10) hold, where $\dot{\rho}$ is replaced by $\dot{\sigma}$. In the same theorem the uniqueness of $\dot{\rho}$ is verified, so that $\dot{\rho}(t,\cdot) = \dot{\sigma}(t,\cdot)$ for τ-a.e. $t \geq 0$. Thus, $(F_{s,t})$ is also a solution to (6.10), so that $h(t,z) := F_{s,t}(z)$ satisfies (6.24).

Uniqueness of a solution to (6.24). Let $s \geq 0$ be fixed. Let $h_1, h_2 : [s,+\infty) \times \mathbb{C}^+ \to \mathbb{C}^+$ be solutions to (6.24) with prescribed assumptions. Note then that $h_i, \partial_z h_i$ are continuous on $[s,+\infty) \times \mathbb{C}^+$ thanks to Lemma 4.28; in particular, the integral in (6.24) is well defined. Since $|A(t,z)| \leq 1/\Im(z)$, the function $H(t,z) := h_1(t,z) - h_2(t,z)$ satisfies

$$|H(t,z)| \leq \frac{1}{\Im(z)} \int_s^t \left|\frac{\partial H}{\partial z}(r,z)\right| \tau(dr). \tag{6.27}$$

By Cauchy's integral formula we obtain

$$\left|\frac{\partial H}{\partial z}(t,z)\right| = \frac{1}{2\pi}\left|\int_{C(z,\varepsilon)} \frac{H(t,w)}{(w-z)^2}\,dw\right| \leq \frac{1}{2\pi\varepsilon^2}\int_{C(z,\varepsilon)} |H(t,w)|\,|dw|, \tag{6.28}$$

where $C(z,\varepsilon)$ is the circle centered at z with radius $\varepsilon \in (0, \Im z)$. Combining (6.27) and (6.28) gives

$$|H(t,z)| \leq \frac{1}{2\pi\varepsilon^2\Im(z)} \int_{[s,t]\times C(z,\varepsilon)} |H(r_1,w_1)|\,\tau(dr_1)|dw_1|. \tag{6.29}$$

Choosing $\varepsilon = \Im z/(2n), n \in \mathbb{N}$, and iterating this inequality n times yields

$$|H(t,z)| \le \left(\frac{1}{2\pi\varepsilon^2}\right)^n \int_{[s,t]^n_{\ge} \times B_n} \frac{|H(r_n, w_n)|}{\Im(z)\Im(w_1)\cdots\Im(w_{n-1})} \tau(dr_1 \cdots dr_n)|dw_1|\cdots|dw_n|,$$

where $[s,t]^n_{\ge} := \{(r_1, r_2, ..., r_n) : s \le r_n \le r_{n-1} \le \cdots \le r_1 \le t\}$ and $B_n := \{(w_1, w_2, ..., w_n) \in (\mathbb{C}^+)^n : w_1 \in C(z,\varepsilon), w_2 \in C(w_1,\varepsilon), ..., w_n \in C(w_{n-1},\varepsilon)\}$. Since $w_1, w_2, ..., w_n$ belong to the compact subset $K_z := \{w \in \mathbb{C}^+ : |w - z| \le \Im(z)/2\}$, by setting $M_{t,z} := \sup_{r\in[s,t], w\in K_z} |H(r,w)|$ we obtain

$$|H(t,z)| \le \frac{M_{t,z}}{[2\pi\varepsilon^2(\Im(z)/2)]^n} \cdot \frac{\tau([s,t])^n}{n!} \cdot (2\pi\varepsilon)^n = \frac{M_{t,z} n^n}{n!} \cdot \left(\frac{4\tau([s,t])}{\Im(z)^2}\right)^n.$$

By Stirling's formula, for sufficiently large n we have $n! \ge \frac{\sqrt{2\pi n}}{2}(n/e)^n$, so that

$$|H(t,z)| \le \frac{2M_{t,z}}{\sqrt{2\pi n}} \cdot \underbrace{\left(\frac{4e\tau([s,t])}{\Im(z)^2}\right)^n}_{=:\alpha^n}.$$

If we take t close enough to s such that $\alpha < 1$, say for $s < t < s + \delta$, then letting $n \to \infty$ we obtain $H(t,z) = 0$. Then (6.29) reads

$$|H(t,z)| \le \frac{1}{2\pi\varepsilon^2\Im(z)} \int_{[s+\delta,t]\times C(z,\varepsilon)} |H(r_1, w_1)|\, \tau(dr_1)|dw_1|.$$

Repeating the above calculations, we can prove $H(t,z) = 0$ for all $s + \delta < t < s + \delta + \delta'$. Actually, we can take $\delta' = \delta$ so this procedure shows $H(t,z) = 0$ for all $t \ge s$. The reason we can choose $\delta' = \delta$ is that $t \mapsto \tau([0,t])$ is uniformly continuous on any fixed interval $[0,T]$, so that for any $\eta > 0$ there exists $\delta > 0$ such that $\tau([a,b]) < \eta$ whenever $a, b \in [0,T]$ and $|a - b| < \delta$. □

Corollary 6.13. *Let τ be a Borel measure on $[0,+\infty)$ that is finite on any compact subset and that has no atom. Let $\dot\rho$ be a probability kernel. Then there exists a unique $\mathcal{P}_2^0$-DLC $(F_t)_{t\ge0}$ such that*

$$F_t(z) = z + \int_0^t \frac{\partial F_s}{\partial z}(z) \left[\int_{\mathbb{R}} \frac{1}{x - z}\, \dot\rho(s, dx)\right] \tau(ds).$$

Overall, there is a one-to-one correspondence between the following four kinds of sets:

- monotone convolution hemigroups $(\mu_{s,t})_{(s,t)\in\triangle}$ such that each $\mu_{s,t}$ has vanishing mean and finite second moment that is continuous with respect to (s,t),
- $\mathcal{P}_2^0$-REFs,
- $\mathcal{P}_2^0$-DLCs,
- pairs $(\dot\rho, \tau)$, where $\dot\rho$ is a probability kernel from $[0,+\infty)$ to $\mathbb{R}$ and τ is an atomless locally finite Borel measure on $[0,+\infty)$.

We call $(\dot\rho, \tau)$ the **generator** of the other three objects. Actually, the generators can also be defined for $\mathcal{P}_2$-REFs, $\mathcal{P}_2$-DLCs and monotone convolution hemigroups with finite second moments; see [80].

In light of the generator, we offer a sufficient condition for a monotone convolution hemigroup to have locally uniform compact support.

Proposition 6.14. *Let $\dot\rho$ be a probability kernel from $[0,+\infty)$ to $\mathbb{R}$ and τ be an atomless, locally finite Borel measure on $[0,+\infty)$. Let $(\mu_{s,t})$ be the corresponding monotone convolution hemigroup. Suppose that for every $T > 0$ there exists $R_T > 0$ such that $\dot\rho(t,\cdot)$ is supported on $[-R_T, R_T]$ for all $t \in [0,T]$. Then for every $T > 0$ there exists $R'_T > 0$ such that $\mu_{s,t}$ is supported on $[-R'_T, R'_T]$ for all $0 \le s \le t \le T$.*

Proof. The proof is analogous to the proof of Proposition 5.19, part (3) $\Longrightarrow$ (1). For example, one can replace (5.18) with

$$|\tilde{A}(t,z)| \le C, \qquad |z| > R_T + 1,\ t \in [0,T].$$

The details are omitted. □

6.3. **Notes.** C. Loewner introduced Loewner chains in 1923 to attack the Bieberbach conjecture, which lead to the positive solution by de Branges in 1985; the reader interested in the history is referred to the monograph [15] and the survey article [94]. Loewner theory has also found applications to other fields; in particular, applications to SLE (Stochastic Loewner Evolution) made a significant success in physics and probability theory [98].

The results in Section 6 are adopted from Hasebe, Hotta and Murayama [80]. Proposition 6.7 and Theorem 6.8 hold for more general REFs and DLCs, but we restricted the results to the $\mathcal{P}_2$-ones to simplify the proof. Lemma 6.6 was proved by Maassen [105, Lemma 2.4]. Example 6.9 of the REF associated with the semicircle distributions was give by Biane [34]. The original proof of Theorem 6.11 given in [80] was a reduction of time-continuous Loewner chains to the absolutely continuous ones that are already well studied by Goryainov and Ba

[73] and Bauer [18]. Our proof is rather different and is more self-contained. Our proof of the uniqueness of a solution to equation (6.10) is different from Bauer's short proof in the absolutely continuous case. Actually, we could give a similar proof to Bauer's but that would require a "$D_{v(s)}$- ($D^{v(s)}$-)calculus", e.g., the Leibniz formula and the derivative of composite functions, which also require proofs. To avoid such an argument, we gave a tricky proof based on Picard's iteration.

In the absolutely continuous case, a more general Loewner theory has been established by Bracci, Contreras, Díaz-Madrigal and Gumenyuk [38, 49]. Schleißinger [134], Franz, Hasebe and Schleißinger [67], and Jekel [88] also proved results analogous to Theorems 6.11 and 6.12 in different but absolutely continuous setups.

At present, for a technical reason, we need the assumption of finite second moment to establish the integral/integro-differential equations. On the other hand, similar and more complete results were established for multiplicative monotone convolution hemigroups on the unit circle by Hasebe and Hotta [79].

7. Monotone additive processes

In probability theory, an additive process is a continuous-time stochastic process whose increments are independent but may have time-dependent distributions. We define and construct a monotone additive process.

Definition 7.1. Let (A, φ) be a unital C^*-probability space. A family of real random variables $(x_t)_{t\geq 0}$ in A is called a **monotone additive process**, or a process of monotonically independent increments, if the following conditions are satisfied.

(i) $x_0 = 0$.

(ii) $\triangle \ni (s,t) \mapsto \mu_{x_t - x_s}$ is weakly continuous.

(iii) for every $n \in \mathbb{N}$ and reals $0 = t_0 < t_1 < \cdots < t_n$, the elements (called increments)

$$x_{t_1} - x_{t_0},\ x_{t_2} - x_{t_1},\ \dots,\ x_{t_n} - x_{t_{n-1}}$$

are monotonically independent.

Proposition 7.2. *Let $(x_t)_{t\geq 0}$ be a monotone additive process in a unital C^*-probability space. Let $\mu_{s,t} := \mu_{x_t - x_s}$ for $0 \leq s \leq t$. Then $(\mu_{s,t})$ is a monotone convolution hemigroup.*

Proof. It is obvious that $\mu_{t,t} = \delta_0$. The weak continuity holds by definition. From the decomposition $x_u - x_s = (x_t - x_s) + (x_u - x_t)$ and since $x_t - x_s$ and $x_u - x_t$ are monotonically independent, we have

$$\mu_{s,u} = \mu_{x_u - x_s} = \mu_{x_t - x_s} \rhd \mu_{x_u - x_t} = \mu_{s,t} \rhd \mu_{t,u}, \qquad 0 \leq s \leq t \leq u.$$

□

A question is given a monotone convolution hemigroup, does there exist a monotone additive process that realizes the hemigroup? If the given hemigroup contains probability measures with unbounded support, the process cannot be realized in a unital C^*-probability space. We will therefore consider monotone convolution hemigroups consisting of probability measures with compact support. Then we can indeed construct a monotone additive process on a unital C^*-probability space. In fact, several constructions are known. Two of them are presented below.

7.1. A construction on monotone Fock spaces.

Here we define a continuous monotone Fock space on which a monotone additive process can be canonically constructed. Suppose that $(\mu_{s,t})_{(s,t)\in\triangle}$ is a monotone convolution hemigroup of mean zero and finite second moment with generator $(\dot\rho, \tau)$ such that $\dot\rho$ is supported on compact subsets locally uniformly, i.e., for every $T > 0$ there exists $R_T > 0$ such that $\dot\rho(t, \cdot)$ is supported on $[-R_T, R_T]$ for all $t \in [0, T]$. From Proposition 6.14, $(\mu_{s,t})$ is compactly supported locally uniformly. Let $(F_{s,t})$ be the associated $\mathcal{P}_2^0$-REF. Recall that equation (6.9) holds.

For notational convenience, let $\mathbb{R}_+ := [0, +\infty)$ and Θ be the Borel measure on $\mathbb{R}_+ \times \mathbb{R}$ defined by $\Theta(dtdx) := \dot\rho(t, dx)\tau(dt)$, i.e.,

$$\Theta(B) = \int_{[0,+\infty)} \left[\int_{\mathbb{R}} \chi_B(t,x)\, \dot\rho(t,dx)\right] \tau(dt), \qquad B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}).$$

Let $(\mathbb{R}_+)^n_> := \{(t_1, t_2, ..., t_n) \in (\mathbb{R}_+)^n : t_1 > t_2 > \cdots > t_n \geq 0\}$. We restrict the measure $\Theta^{\otimes n}$ on $(\mathbb{R}_+ \times \mathbb{R})^n \simeq (\mathbb{R}_+)^n \times \mathbb{R}^n$ to the subset $(\mathbb{R}_+)^n_> \times \mathbb{R}^n$ and define

$$\begin{aligned} H_n &:= L^2((\mathbb{R}_+)^n_> \times \mathbb{R}^n, \Theta^{\otimes n}) \\ &= \left\{ f : (\mathbb{R}_+)^n_> \times \mathbb{R}^n \to \mathbb{C} \;\middle|\; \int_{(\mathbb{R}_+)^n_> \times \mathbb{R}^n} |f(\mathbf{t};\mathbf{x})|^2\, \Theta^{\otimes n}(d\mathbf{t}d\mathbf{x}) < +\infty \right\} \end{aligned}$$

equipped with the inner product

$$\langle f, g\rangle := \int_{(\mathbb{R}_+)^n_> \times \mathbb{R}^n} \overline{f(\mathbf{t};\mathbf{x})} g(\mathbf{t};\mathbf{x}) \Theta^{\otimes n}(d\mathbf{t}d\mathbf{x}).$$

The algebraic monotone Fock space associated to Θ is the pre-Hilbert space

$$\mathcal{F}_>^0 := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H_n,$$

where Ω is a unit vector and the direct sum is the algebraic one, i.e., each element of $\mathcal{F}_>^0$ is a finite sum of elements of H_i's and $\mathbb{C}\Omega$. We also write $H_0 = \mathbb{C}\Omega$. The **monotone Fock space** is the norm closure

$$\mathcal{F}_> := \overline{\mathcal{F}_>^0}^{\|\cdot\|} = \left\{ (h_n)_{n\in\mathbb{N}_0} : h_n \in H_n \ (n \in \mathbb{N}_0), \sum_{n\geq 0} \|h_n\|_{H_n}^2 < +\infty \right\}.$$

We consider the C^*-probability space $(\mathbb{B}(\mathcal{F}_>), \varphi)$, where $\varphi = \langle \Omega, \cdot \Omega\rangle$. The identity operator on $\mathcal{F}_>$ is denoted as $\mathbf{1}$. Let us first introduce three kinds of operators on the algebraic monotone Fock space, and then extend them to the monotone Fock space.

Definition 7.3. The symbols $\mathbf{t} = (t_1, t_2, ..., t_n) \in (\mathbb{R}_+)_>^n$ and $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ are employed below.

(a) **Creation operators** $a^*(f), f \in H_1$: its restriction to H_n is a map to H_{n+1} defined by

$$\begin{aligned} [a^*(f)h](t, \mathbf{t}; x, \mathbf{x}) &:= f(t; x)h(\mathbf{t}; \mathbf{x}), \qquad t > t_1, \ x \in \mathbb{R}, \ h \in H_n, \ n \in \mathbb{N}, \\ a^*(f)\Omega &:= f. \end{aligned}$$

(b) **Annihilation operators** $a(f), f \in H_1$: its restriction to H_{n+1} is a map into H_n defined by

$$\begin{aligned} [a(f)h](\mathbf{t}; \mathbf{x}) &:= \int_{(t_1,+\infty)\times\mathbb{R}} \overline{f(t; x)} h(t, \mathbf{t}; x, \mathbf{x}) \Theta(dtdx), && h \in H_{n+1}, \ n \in \mathbb{N}, \\ a(f)h &:= \langle f, h\rangle_{H_1} \Omega, && h \in H_1, \\ a(f)\Omega &:= 0. \end{aligned}$$

(c) **Gauge operators** $\lambda(g), g \in L^\infty(\Theta)$: its restriction to H_n is a map into itself given by

$$\begin{aligned} [\lambda(g)h](\mathbf{t}; \mathbf{x}) &:= g(t_1; x_1)h(\mathbf{t}; \mathbf{x}), \qquad h \in H_n, \ n \in \mathbb{N}, \\ \lambda(g)\Omega &:= 0. \end{aligned}$$

Note that $f \mapsto a^*(f)$ is linear while $f \mapsto a(f)$ is antilinear.

Proposition 7.4. *For any $f \in H_1$ and $g \in L^\infty(\Theta)$, we have*

$$\|a^*(f)\| = \|f\|_{H_1}, \qquad \|a(f)\| = \|f\|_{H_1}, \qquad \|\lambda(g)\| = \|g\|_{L^\infty}.$$

Proof. On H_0, we have $\|a^*(f)\Omega\|_{\mathcal{F}_>^0} = \|f\|_{\mathcal{F}_>^0} = \|f\|_{H_1}\|\Omega\|_{\mathcal{F}_>^0}$. For $h \in H_n, n \geq 1$ we have by Fubini's theorem

$$\begin{aligned} \|a^*(f)h\|_{\mathcal{F}_>^0}^2 &= \int_{(\mathbb{R}_+)_>^n \times \mathbb{R}^n} |h(\mathbf{t}; \mathbf{x})|^2 \left(\int_{t_1}^{\infty} |f(t; x)|^2 \Theta(dtdx) \right) \Theta^{\otimes n}(d\mathbf{t}d\mathbf{x}) \\ &\leq \|f\|_{H_1}^2 \int_{(\mathbb{R}_+)_>^n \times \mathbb{R}^n} |h(\mathbf{t}; \mathbf{x})|^2 \Theta^{\otimes n}(d\mathbf{t}d\mathbf{x}) \\ &= \|f\|_{H_1}^2 \|h\|_{\mathcal{F}_>^0}^2. \end{aligned}$$

Therefore, by linearity we have $\|a^*(f)h\|_{\mathcal{F}_>^0} \leq \|f\|_{H_1}\|h\|_{\mathcal{F}_>^0}$ for all $h \in \mathcal{F}_>^0$. The equality holds for $h \in H_0$ and hence $\|a^*(f)\| = \|f\|_{H_1}$.

The formula $\|a(f)\| = \|f\|_{H_1}$ can also be shown by similar estimates. This is also a consequence of the following Proposition 7.5.

For the gauge operator, we can easily show

$$\|\lambda(g)h\|_{\mathcal{F}_>^0} \leq \|g\|_{L^\infty}\|h\|_{\mathcal{F}_>^0}, \qquad h \in H_n, \ n \in \mathbb{N}_0,$$

and so $\|\lambda(g)\| \leq \|g\|_{L^\infty}$. It is a standard result in functional analysis that the equality holds because $\lambda(g)$ is a multiplication operator on each $H_n, \ n \in \mathbb{N}$. □

The previous boundedness allows us to extend these operators to bounded operators on $\mathcal{F}_>$, which we still denote by the same symbols.

Proposition 7.5. *The creation operator $a^*(f)$ is the adjoint of the annihilation operator $a(f)$ for all $f \in H_1$.*

Proof. By linearity and continuity it suffices to show $\langle a^*(f)g,h\rangle = \langle g, a(f)h\rangle$ for all $g \in H_n, h \in H_m, m,n \in \mathbb{N}_0$. We may assume $m = n+1$ since otherwise these inner products are all zero. For $n \geq 1$ we have

$$\begin{aligned}\langle a^*(f)g,h\rangle &= \int_{(\mathbb{R}_+)^{n+1}_{>}\times\mathbb{R}^{n+1}} \overline{f(t;x)g(\mathbf{t};\mathbf{x})}h(t,\mathbf{t};x,\mathbf{x})\,\Theta^{\otimes(n+1)}(dtd\mathbf{t}dxd\mathbf{x}) \\ &= \int_{(\mathbb{R}_+)^{n}_{>}\times\mathbb{R}^{n}} \overline{g(\mathbf{t};\mathbf{x})}\left[\int_{(t_1,\infty)\times\mathbb{R}} \overline{f(t;x)}h(t,\mathbf{t};x,\mathbf{x})\,\Theta(dtdx)\right]\Theta^{\otimes n}(d\mathbf{t}d\mathbf{x}) \\ &= \langle g, a(f)h\rangle.\end{aligned}$$

For $n = 0$ we have

$$\langle a^*(f)\Omega,h\rangle = \langle f,h\rangle = \langle \Omega, a(f)h\rangle. \qquad \square$$

Proposition 7.6. *The following formulas hold for $f,k \in H_1$ and $g,h \in L^\infty(\Theta)$:*

$$a(f)\lambda(g) = a(f\overline{g}), \tag{7.1}$$

$$\lambda(g)a^*(k) = a^*(gk), \tag{7.2}$$

$$\lambda(g)\lambda(h) = \lambda(gh), \tag{7.3}$$

$$a(f)a^*(k) = \langle f,k\rangle p_\Omega + \lambda(\mathcal{I}(\overline{f}k)), \tag{7.4}$$

where p_Ω is the orthogonal projection onto $\mathbb{C}\Omega$ and $\mathcal{I}\colon L^1(\Theta)\to L^\infty(\Theta)$ is defined by

$$[\mathcal{I}(v)](t,x) := \int_{(t,\infty)\times\mathbb{R}} v(s,y)\,\Theta(dsdy),$$

which is constant on the second variable x.

Proof. These formulas can be checked by straightforward calculations. Here we only show the last formula. On $H_0 = \mathbb{C}\Omega$, we have

$$a(f)a^*(k)\Omega = a(f)k = \langle f,k\rangle\Omega = [\langle f,k\rangle p_\Omega + \lambda(\mathcal{I}(\overline{f}k))]\Omega$$

For $l \in H_n, n \in \mathbb{N}$ we have

$$\begin{aligned}[a(f)a^*(k)l](\mathbf{t};\mathbf{x}) &= \left[\int_{(t_1,+\infty)\times\mathbb{R}} \overline{f(t;x)}k(t;x)\,\Theta(dtdx)\right] l(\mathbf{t};\mathbf{x}) \\ &= [\mathcal{I}(\overline{f}k)](t_1)l(\mathbf{t};\mathbf{x}) \\ &= [\langle f,k\rangle p_\Omega + \lambda(\mathcal{I}(\overline{f}k))]l(\mathbf{t};\mathbf{x}). \qquad \square\end{aligned}$$

We will show that the family of operators $(x_t)_{t\geq 0}$ defined by

$$x_t := a(\chi_{[0,t]\times\mathbb{R}}) + a^*(\chi_{[0,t]\times\mathbb{R}}) + \lambda(\chi_{[0,t]}X), \tag{7.5}$$

where $X(x) := x$, is an additive monotone process that has the generator $(\dot\rho,\tau)$. For notational conciseness we set

$$a_{s,t} := a(\chi_{(s,t]\times\mathbb{R}}), \qquad a^*_{s,t} := a^*(\chi_{(s,t]\times\mathbb{R}}), \qquad \lambda_{s,t} := \lambda(\chi_{(s,t]}X).$$

As a first step, we calculate the resolvent of $x_t - x_s$. The following lemma is substantially based on the integral equation developed in Section 6.2.

Lemma 7.7. *Let $(F_{s,t})_{(s,t)\in\Delta}$ be the $\mathcal{P}_2^0$-REF associated with $(\dot\rho,\tau)$. We fix $0\leq s\leq t$ and $z \in \mathbb{C}\setminus\mathbb{R}$ and define a function $\tilde F \in L^\infty(\Theta)$ by*

$$\tilde F(r) := \begin{cases} F_{s,t}(z), & 0\leq r\leq s, \\ F_{r,t}(z), & s\leq r\leq t, \\ z, & r\geq t \end{cases}$$

and the operator

$$\tilde\lambda := \lambda(F_{s,t}(z) - \tilde F(\cdot) + \chi_{(s,t]}X).$$

Then we have for sufficiently large $|z|$

$$[z\mathbf{1} - (x_t - x_s)]^{-1} = [\mathbf{1} - (F_{s,t}(z)\mathbf{1} - \tilde\lambda)^{-1}a^*_{s,t}]^{-1}(F_{s,t}(z)\mathbf{1} - \tilde\lambda)^{-1}[\mathbf{1} - a_{s,t}(F_{s,t}(z)\mathbf{1} - \tilde\lambda)^{-1}]^{-1}.$$

Proof. We can directly compute the inverse operator

$$(F_{s,t}(z)\mathbf{1} - \tilde\lambda)^{-1} = \frac{1}{F_{s,t}(z)}p_\Omega + \lambda\left(\frac{1}{\tilde F - \chi_{(s,t]}X}\right). \tag{7.6}$$

Using the relations in Proposition 7.6 and $p_\Omega a^*_{s,t} = 0$ we obtain

$$\begin{aligned} a_{s,t}(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}a^*_{s,t} &= a_{s,t}a^*\left(\chi_{(s,t]}\frac{1}{\tilde{F} - \chi_{(s,t]}X}\right) \\ &= \left[\int_{(s,t]\times\mathbb{R}} \frac{1}{F_{r,t}(z) - x}\Theta(drdx)\right] p_\Omega + \lambda\left(\mathcal{I}\left(\chi_{(s,t]}\frac{1}{\tilde{F} - \chi_{(s,t]}X}\right)\right). \end{aligned}$$

This can be simplified more because, thanks to (6.9)

$$\int_{(s,t]\times\mathbb{R}} \frac{1}{F_{r,t}(z) - x}\Theta(drdx) = z - F_{s,t}(z)$$

and similarly,

$$\mathcal{I}\left(\chi_{(s,t]}\frac{1}{\tilde{F} - \chi_{(s,t]}X}\right)(r) = z - \tilde{F}(r).$$

Indeed, for $r \le s$ we have

$$\begin{aligned} \mathcal{I}\left(\chi_{(s,t]}\frac{1}{\tilde{F} - \chi_{(s,t]}X}\right)(r) &= \int_{[r,\infty)\times\mathbb{R}} \chi_{(s,t]}(u)\frac{1}{\tilde{F}(u) - \chi_{(s,t]}(u)x}\Theta(dudx) \\ &= \int_{(s,t]}\left[\int_{\mathbb{R}} \frac{1}{F_{u,t}(z) - x}\dot{\rho}(u, dx)\right]\tau(du) \\ &= -F_{s,t}(z) + z = -\tilde{F}(r) + z. \end{aligned}$$

The other cases $s \le r \le t$ and $r \ge t$ can be calculated analogously. Altogether, we obtain

$$\begin{aligned} a_{s,t}(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}a^*_{s,t} &= [z - F_{s,t}(z)]p_\Omega + \lambda(z - \tilde{F}(\cdot)) \\ &= [z - F_{s,t}(z)]\mathbf{1} + \tilde{\lambda} - \lambda_{s,t}. \end{aligned}$$

Then

$$\begin{aligned} z\mathbf{1} - (x_t - x_s) &= z\mathbf{1} - a^*_{s,t} - a_{s,t} - \lambda_{s,t} \\ &= (z\mathbf{1} - F_{s,t}(z)\mathbf{1} + \tilde{\lambda} - \lambda_{s,t}) + (F_{s,t}(z)\mathbf{1} - \tilde{\lambda}) - a^*_{s,t} - a_{s,t} \\ &= (F_{s,t}(z)\mathbf{1} - \tilde{\lambda}) - a^*_{s,t} - a_{s,t} + a_{s,t}(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}a^*_{s,t} \\ &= [\mathbf{1} - a_{s,t}(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}](F_{s,t}(z)\mathbf{1} - \tilde{\lambda})[\mathbf{1} - (F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}a^*_{s,t}]. \end{aligned}$$

Taking the inverse of the above formula yields the desired formula. Note here that $F_{s,t}(z) = z + o(z)$ and the norm of $\tilde{\lambda}$ is uniformly bounded as $z \to \infty$, so that all the inverses exist as bounded operators. □

Proposition 7.8. *The distribution of $x_t - x_s$ is $\mu_{s,t}$ for all $0 \le s \le t$. Moreover, for any $y, y' \in \{a^*(f), a(f), \lambda(g) : f \in L^2([0,s)\times\mathbb{R},\Theta), g \in L^\infty([0,s)\times\mathbb{R},\Theta)\}$ and $n \in \mathbb{N}$, we have*

$$y(x_t - x_s)^n y' = \varphi[(x_t - x_s)^n]yy', \tag{7.7}$$

$$y(x_t - x_s)^n\Omega = \varphi[(x_t - x_s)^n]y\Omega. \tag{7.8}$$

Proof. Lemma 7.7 implies for z with large $|z|$

$$[z\mathbf{1} - (x_t - x_s)]^{-1} = \sum_{j,k=0}^{\infty}[(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}a^*_{s,t}]^j(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}[a_{s,t}(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}]^k, \tag{7.9}$$

and so

$$\begin{aligned} &\varphi([z\mathbf{1} - (x_t - x_s)]^{-1}) \\ &= \sum_{j,k=0}^{\infty}\langle[a_{s,t}(\overline{F_{s,t}(z)}\mathbf{1} - \tilde{\lambda}^*)^{-1}]^j\Omega, (F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}[a_{s,t}(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}]^k\Omega\rangle. \end{aligned}$$

Here from (7.6) we deduce that

$$a_{s,t}(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}\Omega = 0. \tag{7.10}$$

Therefore, only $j = k = 0$ gives a nonzero contribution, i.e.,

$$\varphi([z\mathbf{1} - (x_t - x_s)]^{-1}) = \langle\Omega, (F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}\Omega\rangle = \frac{1}{F_{s,t}(z)} =: G_{s,t}(z), \tag{7.11}$$

showing that the analytic distribution of $x_t - x_s$ equals $\mu_{s,t}$.

We turn to the proof of (7.7). Using (7.6) we can check

$$(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}y' = G_{s,t}(z)y'$$

for any y' that is either creation, annihilation or gauge operators supported on $[0, s)$. The point is that when multiplying a function h in H_n by y', the resulting function $y'h$ vanishes when the first variable t_1 is larger than s. Then the next operator $(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1}$ acts as

$$\frac{1}{F_{s,t}(z)} p_\Omega + \lambda\left(\frac{1}{F_{s,t}(z)}\right) = G_{s,t}(z)\mathbf{1}$$

because $\chi_{(s,t]}X$ is always zero on the support of the function $y'h$. From a similar consideration of support, we can further obtain

$$a_{s,t}(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1} y' = G_{s,t}(z) a_{s,t} y' = 0. \tag{7.12}$$

Taking the adjoint and complex conjugate yields

$$y(F_{s,t}(z)\mathbf{1} - \tilde{\lambda})^{-1} a_{s,t}^* = 0. \tag{7.13}$$

Combining (7.9), (7.12) and (7.13) we obtain

$$y[z\mathbf{1} - (x_t - x_s)]^{-1} y' = G_{s,t}(z) y y'.$$

Expanding the formula into Laurent series and comparing the coefficients of z^{-n-1} we obtain the desired (7.7). The last formula (7.8) can be obtained by similar calculations, combining (7.9), (7.10) and (7.13). □

Theorem 7.9. *The family of operators $(x_t)_{t\geq 0}$ defined by (7.5) is a monotone additive process such that the analytic distributions of the increments $x_t - x_s$ $(0 \leq s \leq t)$ form a monotone convolution hemigroup associated with the generator $(\dot{\rho}, \tau)$.*

Proof. It remains to prove the independent increment property. A fully general description requires heavy notation, so let us consider $0 = t_0 < t_1 < t_2 < t_3$, $m, n, p, q \in \mathbb{N}$ and calculate the example

$$\varphi[(x_{t_2} - x_{t_1})^m (x_{t_3} - x_{t_2})^n (x_{t_1} - x_{t_0})^p (x_{t_2} - x_{t_1})^q]. \tag{7.14}$$

Observe that $(x_{t_2} - x_{t_1})^m$ is the linear combination of words consisting of operators in $\mathcal{G}_{[0,t_2)}$ in Proposition 7.8. The same holds for $(x_{t_1} - x_{t_0})^p$. Then (7.7) implies

$$(x_{t_2} - x_{t_1})^m (x_{t_3} - x_{t_2})^n (x_{t_1} - x_{t_0})^p = \varphi[(x_{t_3} - x_{t_2})^n](x_{t_2} - x_{t_1})^m (x_{t_1} - x_{t_0})^p,$$

which obviously implies (7.14) equals

$$\varphi[(x_{t_3} - x_{t_2})^n]\varphi[(x_{t_2} - x_{t_1})^m (x_{t_1} - x_{t_0})^p (x_{t_2} - x_{t_1})^q]. \tag{7.15}$$

Recall here that $\varphi = \langle \Omega, \cdot\, \Omega\rangle$. We in turn use (7.8) to obtain

$$(x_{t_1} - x_{t_0})^p (x_{t_2} - x_{t_1})^q \Omega = \varphi[(x_{t_2} - x_{t_1})^q](x_{t_1} - x_{t_0})^p \Omega,$$

which implies (7.15) equals

$$\varphi[(x_{t_3} - x_{t_2})^n]\varphi[(x_{t_2} - x_{t_1})^q]\varphi[(x_{t_2} - x_{t_1})^m (x_{t_1} - x_{t_0})^p].$$

To compute the last factor $\varphi[(x_{t_2} - x_{t_1})^m (x_{t_1} - x_{t_0})^p]$, we can move the operators to the left side of the inner product as adjoints, and then we apply (7.8).

The general case can be shown analogously. For the interested reader, we note what has to be shown: Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n$. For each interval $I(k) := (t_{k-1}, t_k]$ we denote $x_{I(k)} := x_{t_k} - x_{t_{k-1}}$ for notational conciseness. Let $j_1, j_2, ..., j_m \in [n]$ with $j_1 \neq j_2, j_2 \neq j_3, ..., j_{m-1} \neq j_m$ and $P_i \in \mathbb{C}[x], i \in [m]$ without a constant term. In the setting above, if $\ell \in [m]$ is such that $j_{\ell-1} < j_\ell > j_{\ell+1}$ then

$$\begin{aligned}&\varphi[P_1(x_{I(j_1)})P_2(x_{I(j_2)}) \cdots P_m(x_{I(j_m)})] \\ &\quad = \varphi[P_\ell(x_{I(j_\ell)})]\varphi[P_1(x_{I(j_1)}) \cdots P_{\ell-1}(x_{I(j_{\ell-1})})P_{\ell+1}(x_{I(j_{\ell+1})}) \cdots P_m(x_{I(j_m)})].\end{aligned}$$

□

Example 7.10. In the case $\dot{\rho}(t, dx) = \delta_0(dx)$ and $\tau(dt) = dt$, the Hilbert space H_n is isomorphic to $L^2((\mathbb{R}_+)^n_>, dt_1 dt_2 \cdots d$ The function X is zero and hence $\lambda_{s,t} = 0$ and $x_t = a^*(\chi_{[0,t]}) + a(\chi_{[0,t]})$. This is called a **monotone Brownian motion**. The distribution of $x_t - x_s$ is the arcsine law with mean 0 and variance $t - s$ that appeared in the monotone CLT.

The above proof heavily depended on the resolvent of the increment $x_t - x_s$. We can show monotone independence for bigger subalgebras by combinatorial methods. For an interval $I \subseteq [0, +\infty)$, let A_I be the ($*$-)subalgebra of $\mathbb{B}(\mathcal{F}_>)$ generated by the set of operators

$$\mathcal{G}_I := \{a^*(f), a(f), \lambda(g) : f \in L^2(I \times \mathbb{R}, \Theta), g \in L^\infty(I \times \mathbb{R}, \Theta)\}.$$

Lemma 7.11. *Let $I := [s, +\infty)$ be a half-axis for some $s \geq 0$. The $*$-subalgebra $A_I + \mathbb{C}\mathbf{1}$ coincides with the linear span of the elements of the form*

$$w = a^*(f_1)a^*(f_2) \cdots a^*(f_m)[\lambda(h) + \alpha\mathbf{1}]a(g_1)a(g_2) \cdots a(g_n), \tag{7.16}$$

where $f_i, g_i \in L^2(I \times \mathbb{R}, \Theta)$, $h \in L^\infty(I \times \mathbb{R}, \Theta)$, $m, n, \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$.

Proof. Let B_I denote the linear span of the elements (7.16). Obviously, $B_I \subseteq A_I + \mathbb{C}\mathbf{1}$. To show the converse inclusion, since B_I contains the generator set $\mathcal{G}_I$ for A_I, it suffices to show that B_I is a $*$-subalgebra. Moreover, for this it suffices to show that $xB_I, B_I x \subseteq B_I$ for any $x \in \mathcal{G}_I$. In addition, since B_I and $\mathcal{G}_I$ are closed under $*$, only showing $xB_I \subseteq B_I$ suffices. We check this case-by-case.

The inclusion $xB_I \subseteq B_I$ is obvious for $x = a^*(f)$ and for $x = \lambda(g)$; the latter is because of (7.2) and (7.3). Therefore, it remains to check that $a(f)w \in B_I$, where w is the operator (7.16).

Case $m \geq 2$. Using Proposition 7.6 and $p_\Omega a^*(f_2) = 0$ we get

$$a(f)a^*(f_1)a^*(f_2) = [\langle f, f_1\rangle p_\Omega + \lambda(k)]a^*(f_2) = \lambda(k)a^*(f_2) = a^*(kf_2),$$

where $k := \mathcal{I}(\overline{f} f_1)$. Since kf_2 is supported on $I \times \mathbb{R}$, it follows that $a(f)w \in B_I$.

Case $m = 1$. Combining the decomposition $\mathbf{1} = p_\Omega + \lambda(\chi_{\mathbb{R}_+\times\mathbb{R}})$ and the previous calculations yields

$$a(f)a^*(f_1) = [\langle f, f_1\rangle p_\Omega + \lambda(k)] = \langle f, f_1\rangle \mathbf{1} + \lambda(k - \langle f, f_1\rangle \chi_{\mathbb{R}_+\times\mathbb{R}}).$$

Observe that $\tilde{k} := k - \langle f, f_1\rangle \chi_{\mathbb{R}_+\times\mathbb{R}}$ is supported on $I \times \mathbb{R}$. Setting $\beta := \langle f, f_1\rangle$, the first three letters of the word $a(f)w$ equals

$$a(f)a^*(f_1)[\lambda(h) + \alpha\mathbf{1}] = [\lambda(\tilde{k}) + \beta\mathbf{1}][\lambda(h) + \alpha\mathbf{1}] = \lambda(\tilde{k}h + \alpha\tilde{k} + \beta h) + \alpha\beta\mathbf{1},$$

so that $a(f)w \in B_I$.

Case $m = 0$. From Proposition 7.6, the first two letters of the word $a(f)w$ can be calculated as

$$a(f)[\lambda(h) + \alpha\mathbf{1}] = a(f\overline{h} + \overline{\alpha} f),$$

so that $a(f)w \in B_I$. □

Proposition 7.12. *Let $s \geq 0$. For any $x, x' \in A_{[0,s)}$ and $y \in A_{[s,+\infty)} + \mathbb{C}\mathbf{1}$, we have*

$$xyx' = \varphi(y)xx', \tag{7.17}$$

$$xy\Omega = \varphi(y)x\Omega. \tag{7.18}$$

Proof. The second formula follows from the first one because (7.4) shows $p_\Omega \in A_{[0,s)}$ unless $L^2([0,s)\times\mathbb{R}, \Theta) = \{0\}$, in which case the second formula is obvious. It suffices to show (7.17) for $y = w$ in the form (7.16) and $x, x' \in \mathcal{G}_{[0,s)}$.

Case $m \geq 1$. Then $y\Omega = 0$ and hence $\varphi(y) = 0$. Since x is one of $a^*(f), a(f), \lambda(g)$ where f, g are supported on $[0, s) \times \mathbb{R}$ and f_1 is supported on $[s, +\infty) \times \mathbb{R}$, we can conclude $xa^*(f_1) = 0$, so that $xy = 0$.

Case $m = 0$ and $n \geq 1$. We have $y\Omega = 0$ and hence $\varphi(y) = 0$. On the other hand, the argument symmetric with the previous case $m \geq 1$ shows $xyx' = 0$.

Case $m = 0$ and $n = 0$. Then $y = \lambda(h) + \alpha\mathbf{1}$ and $\varphi(y) = \alpha$. One can show that $x\lambda(h)x' = 0$ from Proposition 7.6 and the fact that the support of h is contained in $[s, +\infty) \times \mathbb{R}$. Thus we obtain $xyx' = \alpha xx' = \varphi(y)xx'$. □

Theorem 7.13. *Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n$. Then the $*$-subalgebras*

$$A_{[t_0,t_1)}, \quad A_{[t_1,t_2)}, \quad \dots \quad , \quad A_{[t_{n-1},t_n)}$$

are monotonically independent.

Proof. Almost the same as the proof of Theorem 7.9. □

Note that this theorem implies the monotone independence of the increments of (x_t), which was already proved in Theorem 7.9.

7.2. **A construction from Markov processes.** The second construction of monotone additive processes is based on classical Markov processes. Basic facts on Markov processes are reviewed briefly here. For further information on Markov processes, the reader is referred to e.g., [92, 128].

Recall from Lemma 5.9 that the composition of two probability kernels k, l on $\mathbb{R}$

$$(kl)(x, B) := \int_{\mathbb{R}} k(x, dy)l(y, B), \qquad x \in \mathbb{R}, \ B \in \mathcal{B}(\mathbb{R}),$$

is also a probability kernel.

Definition 7.14. A family $(k_{s,t})_{(s,t)\in\triangle}$ of probability kernels on $\mathbb{R}$ is called **transition kernels** if $k_{s,s}(x, \cdot) = \delta_x$ for all $x \in \mathbb{R}, s \geq 0$ and the following **Chapman–Kolmogorov equation** holds:

$$k_{s,t}k_{t,u} = k_{s,u}, \qquad 0 \leq s \leq t \leq u. \tag{7.19}$$

Definition 7.15. Let $(k_{s,t})_{(s,t)\in\triangle}$ be transition kernels on $\mathbb{R}$. A family of $\mathbb{R}$-valued measurable functions $(X_t)_{t\ge 0}$ on a measurable space $(\Omega,\mathcal{F})$, together with a family of σ-subfields $\mathcal{F}_I\subseteq\mathcal{F}$ indexed by the closed intervals I of $[0,\infty)$ and a family of probability measures $\mathbb{P}^{(s,x)}$ on $(\Omega,\mathcal{F}_{[s,\infty)})$ $(s\ge 0, x\in\mathbb{R})$, is called a **Markov process having the transition kernels** $(k_{s,t})_{(s,t)\in\triangle}$ if

- $\mathcal{F}_I\subseteq\mathcal{F}_J$ whenever $I\subseteq J$;
- for every bounded Borel measurable function $f\colon\mathbb{R}\to\mathbb{R}$, $0\le s\le t\le u$ and $x\in\mathbb{R}$,
$$\mathbb{E}^{(s,x)}[f(X_u)|\mathcal{F}_{[s,t]}]=\int_{\mathbb{R}} f(y)k_{t,u}(X_t,dy),\qquad \mathbb{P}^{(s,x)}\text{-a.s.}; \tag{7.20}$$
- $\mathbb{P}^{(s,x)}[X_s=x]=1$ for all $s\ge 0$ and $x\in\mathbb{R}$.

The following is a rather standard result.

Proposition 7.16. *Let $(k_{s,t})_{(s,t)\in\triangle}$ be transition kernels on $\mathbb{R}$. Then there exists a Markov process that has the transition kernels $(k_{s,t})_{(s,t)\in\triangle}$.*

Proof. A standard construction is called the coordinate process that we present below. Let $\Omega=\mathbb{R}^{[0,\infty)}$ be the set of all functions $\omega\colon[0,\infty)\to\mathbb{R}$. Let $\mathcal{C}\subseteq 2^\Omega$ be the set of the cylinder sets $\{\omega\in\Omega:\omega(t_1)\in A_1,\omega(t_2)\in A_2,...,\omega(t_n)\in A_n\}$, $0\le t_1<t_2<\cdots<t_n, A_1,A_2,..,A_n\in\mathcal{B}(\mathbb{R}), n\in\mathbb{N}$. Let $\mathcal{F}\subseteq 2^\Omega$ be the σ-field generated by $\mathcal{C}$. The coordinate process $X_t\colon\Omega\to\mathbb{R}$ is defined by $X_t(\omega):=\omega(t)$. The σ-subfield $\mathcal{F}_I$ is the σ-field generated by $X_t^{-1}(A), t\in I, A\in\mathcal{B}(\mathbb{R})$.

We consider the family of probability measures $\mu^{(s,x)}_{t_1,t_2,...,t_n}$ on $\mathbb{R}^n$, indexed by $s\le t_1<t_2<\cdots<t_n$ $(n\in\mathbb{N})$, defined by the iterated integrals
$$\mu^{(s,x)}_{t_1,t_2,...,t_n}(A):=\int_{\mathbb{R}}k_{s,t_1}(x,dx_1)\int_{\mathbb{R}}k_{t_1,t_2}(x_1,dx_2)\cdots\int_{\mathbb{R}}\chi_A(x_1,x_2,...,x_n)k_{t_{n-1},t_n}(x_{n-1},dx_n).$$
By the Chapman-Kolmogorov equation, these probability measures satisfy the consistency
$$\begin{aligned}&\mu^{(s,x)}_{t_1,t_2,...,t_n}(A_1\times\cdots\times A_{i-1}\times\mathbb{R}\times A_{i+1}\times\cdots\times A_n)\\&=\mu^{(s,x)}_{t_1,...,t_{i-1},t_{i+1},...,t_n}(A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_n),\qquad 1\le i\le n,\end{aligned}$$
and so Kolmogorov's extension theorem (see, e.g. [56]) guarantees that there exists a probability measure $\mathbb{P}^{(s,x)}$ on $\mathcal{F}_{[s,\infty)}$ such that
$$\mathbb{P}^{(s,x)}[X_{t_1}\in A_1,X_{t_2}\in A_2,...,X_{t_n}\in A_n]=\mu^{(s,x)}_{t_1,t_2,...,t_n}(A_1\times A_2\times\cdots\times A_n)$$
for all $s\le t_1<t_2<\cdots<t_n$ and $A_1,A_2,...,A_n\in\mathcal{B}(\mathbb{R})$.

All the requirements for a Markov process are obvious except (7.20). To show this, by the well known characterization of the conditional expectation, it suffices to show
$$\mathbb{E}^{(s,x)}[f(X_u)\chi_F]=\mathbb{E}^{(s,x)}\left[\int_{\mathbb{R}}f(y)k_{t,u}(X_t,dy)\chi_F\right],\qquad F\in\mathcal{F}_{[s,t]}. \tag{7.21}$$
It further suffices to show that the above holds for all $F\in\mathcal{G}:=\{\bigcap_{i=1}^n X_{r_i}^{-1}(A_i): n\in\mathbb{N}, s\le r_i\le t, A_i\in\mathcal{B}(\mathbb{R})\}$. Indeed, as soon as the set $\mathcal{F}':=\{F\in\mathcal{F}_{[s,t]}:(7.21)\text{ holds}\}$ contains $\mathcal{G}$, since $\mathcal{G}$ is a π-system and $\mathcal{F}'$ is a λ-system, by the π-λ theorem (Theorem 4.9), $\mathcal{F}'$ contains $\sigma(\mathcal{G})=\mathcal{F}_{[s,t]}$.

To finish the proof, let $F=\bigcap_{i=1}^n X_{r_i}^{-1}(A_i)\in\mathcal{G}$ with $s\le r_1<r_2<\cdots<r_n\le t$. To avoid heavy notation we only consider $n=2$; the general case is similar. We then have
$$\begin{aligned}&\mathbb{E}^{(s,x)}\left[\int_{\mathbb{R}}f(y)k_{t,u}(X_t,dy)\chi_F\right]\\&=\mathbb{E}^{(s,x)}\left[\int_{\mathbb{R}}f(y)\chi_{A_1}(X_{r_1})\chi_{A_2}(X_{r_2})k_{t,u}(X_t,dy)\right]\\&=\int_{\mathbb{R}^2}\mu^{(s,x)}_{r_1,r_2,t}(dx_1dx_2dz)\int_{\mathbb{R}}\chi_{A_1}(x_1)\chi_{A_2}(x_2)f(y)k_{t,u}(z,dy)\\&=\int_{\mathbb{R}}\chi_{A_1}(x_1)k_{s,r_1}(x,dx_1)\int_{\mathbb{R}}\chi_{A_2}(x_2)k_{r_1,r_2}(x_1,dx_2)\int_{\mathbb{R}}k_{r_2,t}(x_2,dz)\int_{\mathbb{R}}f(y)k_{t,u}(z,dy) &(7.22)\\&=\int_{\mathbb{R}}\chi_{A_1}(x_1)k_{s,r_1}(x,dx_1)\int_{\mathbb{R}}\chi_{A_2}(x_2)k_{r_1,r_2}(x_1,dx_2)\int_{\mathbb{R}}f(y)k_{r_2,u}(x_2,dy) &(7.23)\\&=\mathbb{E}^{(s,x)}[\chi_{A_1}(X_{r_1})\chi_{A_2}(X_{r_2})f(X_u)]=\mathbb{E}^{(s,x)}[\chi_F f(X_u)],\end{aligned}$$
where the Chapman-Kolmogorov equation was used from (7.22) to (7.23). ∎

Remark 7.17. The above construction of Markov process does not tell us about how the sample path $t \mapsto X_t(\omega)$ looks like for each $\omega \in \Omega$. To have good sample paths, usually some kind of continuity, one has to "modify" the above Markov process. For our purpose, sample paths do not matter and the above construction is enough.

We introduce a suitable Markov process for constructing a monotone additive process. Let $(\mu_{s,t})_{(s,t)\in\triangle}$ be a monotone convolution hemigroup. We take $0 \le s \le t \le u$. Let $k_{s,t}(x,\cdot) := (\delta_x \rhd \mu_{s,t})(\cdot)$. Recall from Proposition 5.12 that $k_{s,t}$ is a probability kernel and for every probability measure μ, we have

$$\mu \rhd \mu_{t,u} = \int_{\mathbb{R}} k_{t,u}(y,\cdot)\mu(dy).$$

Selecting $\mu = \delta_x \rhd \mu_{s,t} = k_{s,t}(x,\cdot)$ yields

$$\delta_x \rhd \mu_{s,u} = \int_{\mathbb{R}} k_{t,u}(y,\cdot)k_{s,t}(x,dy),$$

which reads $k_{s,u} = k_{s,t}k_{t,u}$, i.e., the Chapman-Kolmogorov equation. The weak continuity of $(s,t) \mapsto k_{s,t}(x,\cdot)$ follows from

$$F_{k_{s,t}(x,\cdot)}(z) = F_{\mu_{s,t}}(z) - x$$

and Proposition 4.33. By Proposition 7.16 there exists a Markov process $(X_t)_{t\ge 0}$ that has the constructed transition kernels $(k_{s,t})$. We then set $\mathcal{H} := L^2(\Omega, \mathcal{F}, \mathbb{P}^{(0,0)})$ and work on the C^*-probability space $(\mathbb{B}(\mathcal{H}), \varphi)$, where $\varphi(a) := \langle \chi_\Omega, a\chi_\Omega\rangle$. The identity operator on $\mathcal{H}$ is denoted as $\mathbf{1}$. For notational simplicity we denote $\mathbb{P} := \mathbb{P}^{(0,0)}$, $\mathbb{E} := \mathbb{E}^{(0,0)}$ and $\mathcal{F}_t := \mathcal{F}_{[0,t]}$. Also for analytic transforms, we set the shorthand symbols $F_{s,t}(z) := F_{\mu_{s,t}}(z)$ and $G_{s,t}(z) := G_{\mu_{s,t}}(z)$.

For the sake of simplicity, we assume that each $\mu_{s,t}$ has compact support. Let $p_t \in \mathbb{B}(\mathcal{H})$ be the conditional expectation $p_t Z := \mathbb{E}[Z|\mathcal{F}_t], Z \in \mathcal{H}$. It is known that conditional expectations onto σ-subfields are orthogonal projections on the L^2 space, so each p_t is an orthogonal projection. The multiplication operator on $\mathcal{H}$ by the random variable X_t is denoted by m_t, i.e., $m_t(Z) := X_t Z$. Since $\mu_{0,t}$ has compact support, $X_t \in L^\infty$ and so the operator m_t is bounded.

We set

$$y_t := p_t m_t, \qquad t \ge 0. \tag{7.24}$$

Note that $p_t m_t = m_t p_t$ because

$$p_t m_t(Z) = \mathbb{E}[X_t Z|\mathcal{F}_t] = X_t\mathbb{E}[Z|\mathcal{F}_t] = m_t p_t(Z).$$

We will show that $(y_t)_{t\ge 0}$ is a monotone additive process.

Lemma 7.18. *For $z \in \mathbb{C}^+$ and $0 \le s \le t$ it holds that*

$$p_s(z\mathbf{1} - m_t)^{-1}p_s = (F_{s,t}(z)\mathbf{1} - m_s)^{-1}p_s.$$

Proof. For $Z \in \mathcal{H}$, keeping in mind that $p_s(Z)$ is $\mathcal{F}_s$-measurable, we have

$$\begin{aligned}
p_s(z\mathbf{1} - m_t)^{-1}p_s(Z) &= \mathbb{E}\left[\frac{1}{z - X_t}p_s(Z)\middle|\mathcal{F}_s\right] = \mathbb{E}\left[\frac{1}{z - X_t}\middle|\mathcal{F}_s\right]p_s(Z) \\
&= \int_{\mathbb{R}} \frac{1}{z-y}k_{s,t}(X_s,dy)p_s(Z) = G_{k_{s,t}(X_s,\cdot)}(z)p_s(Z) \\
&= \frac{1}{F_{s,t}(z) - X_s}p_s(Z) = (F_{s,t}(z)\mathbf{1} - m_s)^{-1}p_s(Z).
\end{aligned}$$

□

Proposition 7.19. *For $z \in \mathbb{C}^+$ and $0 \le s \le t$ it holds that*

$$p_s[z\mathbf{1} - (y_t - y_s)]^{-1}p_s = G_{s,t}(z)p_s. \tag{7.25}$$

In particular, the distribution of $y_t - y_s$ with respect to $\varphi = \langle \chi_\Omega, \cdot\chi_\Omega\rangle_{\mathcal{H}}$ equals $\mu_{s,t}$, and

$$p_s(y_t - y_s)^n p_s = \varphi[(y_t - y_s)^n]p_s, \qquad n \in \mathbb{N}. \tag{7.26}$$

Proof. By analytic continuation, it suffices to show the formula for $z \in \mathbb{C}^+$ with large $|z|$. We first observe

$$p_s[z\mathbf{1} - (y_t - y_s)]^{-1}p_s = p_s[z\mathbf{1} - (m_t - m_s p_s)]^{-1}p_s. \tag{7.27}$$

Indeed, by series expansion

$$p_s[z\mathbf{1} - (y_t - y_s)]^{-1}p_s = \sum_{n\ge 0}\frac{p_s(m_t p_t - m_s p_s)^n p_s}{z^{n+1}}.$$

If we look at the expansion of $(m_t p_t - m_s p_s)^n p_s Z$ ($Z \in \mathcal{H}$), p_t always acts on a random variable that is $\mathcal{F}_s$-measurable, so that p_t acts as the identity operator. This verifies (7.27).

We continue the calculation (7.27) as follows:

$$\begin{aligned}
p_s[z\mathbf{1} - (m_t - m_s p_s)]^{-1} p_s &= p_s[\mathbf{1} + (z\mathbf{1} - m_t)^{-1} m_s p_s]^{-1} (z\mathbf{1} - m_t)^{-1} p_s \\
&= \sum_{n\geq 0} (-1)^n p_s[(z\mathbf{1} - m_t)^{-1} m_s p_s]^n (z\mathbf{1} - m_t)^{-1} p_s \\
&= \sum_{n\geq 0} (-1)^n p_s m_s^n [p_s (z\mathbf{1} - m_t)^{-1} p_s]^{n+1} \qquad (7.28)\\
&= \sum_{n\geq 0} p_s (-m_s)^n [(F_{s,t}(z)\mathbf{1} - m_s)^{-1} p_s]^{n+1} \qquad (7.29)\\
&= \sum_{n\geq 0} (-m_s)^n (F_{s,t}(z)\mathbf{1} - m_s)^{-n-1} p_s \\
&= [F_{s,t}(z)\mathbf{1} - m_s - (-m_s)]^{-1} p_s \\
&= G_{s,t}(z) p_s.
\end{aligned}$$

Here, (7.28) holds because m_s commutes with both p_s and $(z\mathbf{1} - m_t)^{-1}$, and (7.29) is obtained from Lemma 7.18.

For the last statement, observe that

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{z-u}\, \mu_{y_t - y_s}(du) &= \varphi([z\mathbf{1} - (y_t - y_s)]^{-1}) = \varphi(p_s[z\mathbf{1} - (y_t - y_s)]^{-1} p_s) \\
&= G_{s,t}(z) = \int_{\mathbb{R}} \frac{1}{z-u}\, \mu_{s,t}(du),
\end{aligned}$$

so that $\mu_{y_t - y_s} = \mu_{s,t}$. The relation (7.26) follows by comparing the coefficients of z^{-n-1} in the series expansion of (7.25). □

Theorem 7.20. *The process $(y_t)_{t\geq 0}$ defined in* (7.24) *is a monotone additive process such that each increment $y_t - y_s$ has the given distribution $\mu_{s,t}$.*

Proof. Let us consider $0 = t_0 < t_1 < t_2 < t_3$, $m, n, p, q, r \in \mathbb{N}$ and calculate the example

$$\varphi[(y_{t_2} - y_{t_1})^m (y_{t_3} - y_{t_2})^n (y_{t_1} - y_{t_0})^p (y_{t_2} - y_{t_1})^q (y_{t_1} - y_{t_0})^r]. \qquad (7.30)$$

Observe that $(y_{t_2} - y_{t_1})^m p_{t_2} = (y_{t_2} - y_{t_1})^m$ and $p_{t_2}(y_{t_1} - y_{t_0})^p = (y_{t_1} - y_{t_0})^p$ because of the tower property of conditional expectations $p_{t_2} p_{t_i} = p_{t_i}$ for $i = 0, 1, 2$. Hence, we are allowed to replace the factor $(y_{t_3} - y_{t_2})^n$ with $p_{t_2}(y_{t_3} - y_{t_2})^n p_{t_2}$, which is equal to $\varphi[(y_{t_3} - y_{t_2})^n] p_{t_2}$ by Proposition 7.19. This implies that (7.30) equals

$$\varphi[(y_{t_3} - y_{t_2})^n] \varphi[(y_{t_2} - y_{t_1})^m (y_{t_1} - y_{t_0})^p (y_{t_2} - y_{t_1})^q (y_{t_1} - y_{t_0})^r].$$

The general case can be shown analogously. □

Remark 7.21. The above construction of monotone additive processes is independent of the integral or integro-differential equation developed in Section 6.2. In the case of monotone convolution semigroup, i.e., $\mu_{s,t} = \mu_{0,t-s}$ for all $0 \leq s \leq t$, the Markov process $(X_t)_{t\geq 0}$ is a Feller process and its generator can be expressed in terms of the parameter (γ, σ) in (5.16); see [67] for further details.

7.3. **Notes.** The study of Fock spaces in noncommutative probability can be traced back to Boson and Fermion Fock spaces in quantum physics. Hudson and Parthasarathy developed a quantum version of Itô calculus on the Boson Fock space [87, 124]. In free probability, the corresponding space is the full (or free) Fock space [143], on which free stochastic calculus was initiated by Kümmerer and Speicher [96]. The Boson, Fermion and full Fock spaces are interpolated by the q-Fock space of Bożejko and Speicher [37]. Fock spaces have provided a canonical construction of independent random variables and continuous-time processes of independent increments. In particular, the q-Fock space and its relatives have offered remarkable von Neumann algebras that are still actively studied [95, 110]. Concerning monotone probability, there seems to be no notable von Neumann algebras building upon monotonically independent random variables so far. On the other hand, C^*-algebras related to the monotone Fock space have been investigated in the literature, see e.g. [52, 53, 54].

The construction of additive processes on the monotone Fock space followed Jekel [88], who developed Loewner theory and monotone convolution hemigroups in a more general operator-valued setting, where φ is an algebra-valued functional called a conditional expectation. Here we have presented the results in the simplified setting of $\mathbb{C}$-valued functional. Our results also contain advancements because we only assume the continuity (not absolute continuity) about the time parameter. The special case of monotone Brownian motion $x_t = a^*(\chi_{[0,t]}) + a(\chi_{[0,t]})$ first appeared in Lu [104] and Muraki [114]. Muraki considered a discrete monotone Fock space in [113, 116]. In free probability, a parallel construction of free additive processes on the full Fock space is known, see [120, Exercise 13.19] that assumes the stationarity of the distributions. The term "gauge operator" appears in [120] for an analogous operator on the full Fock space; note that the terms "gauge process" and "conservation process" are used in [87] and [124] respectively for the operator on the Boson Fock space corresponding to our $\lambda(\chi_{[0,t]} X)$.

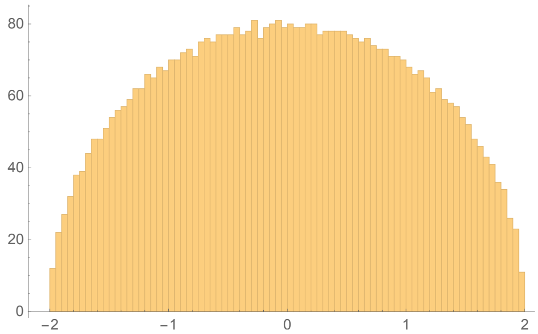

FIGURE 3. A histogram for the eigenvalues of G_N with $N = 5000$ and bin size 0.05. One can observe that the graph of the histogram looks like the probability density function of the semicircle distribution. In fact, one can show that when the bin size $\delta_N \to 0^+$ ($N \to \infty$) is appropriately selected, then the graph of the normalized histogram (so that the area equals one) converges to $(1/(2\pi))\sqrt{4-x^2}$. A remarkable sufficient condition $N^{-2/3}\log N \ll \delta_N \ll 1$ is given in [58, Corollary 4.2].

The operator $\lambda(g)$ is called the multiplication operator in [88]. Concerning notation, it is common to denote $\ell(f)$ for the creation operator and $\ell^*(f)$ for the annihilation operator on the full Fock space [120, 143]. On the other hand, the symbol $a^*(f)$ or $a^\dagger(f)$ is commonly used for the creation operator on the Boson Fock space and $a(f)$ for the annihilation, which we have followed.

The construction of monotone additive processes from the Markov processes $(X_t)_{t\geq 0}$ is due to [67]. In the original paper unbounded operator processes are treated. Our construction is limited to the bounded case, which greatly simplifies the proofs and formulations; already the definition of monotone additive processes is more involved in the unbounded operator setting. The Markov process $(X_t)_{t\geq 0}$ was first considered by Biane [34] in connection to free additive processes and subordination functions.

There are other constructions of monotone additive processes as solutions to quantum stochastic differential equations, see [24, 64] and [67]. Hamdi constructed a multiplicative monotone unitary Brownian motion as a solution to a quantum SDE [74].

Classical stochastic processes related to monotone independence are studied in the literature: a discrete-time analogue of the Markov process (X_t), i.e., Markov chains, can be similarly defined and is studied by Letac and Malouche [103], Wang and Wendler [148]; Biane mentions that the Markov process $(X_t)_{t\geq 0}$ associated with $F_t(z) = \sqrt{z^2 - 2t}$ (the reciprocal Cauchy transform of the arcsine law with mean 0 an variance t) is the Azéma martingale [34]; Belton studied a semimartingale having the monotone Poisson distribution [25, 26].

8. Monotone independence in random matrix and graph theory

So far we have studied monotone independence from the viewpoint of analogy to probability theory. Here we discuss different aspects of monotone independence: large random matrices and a graph product.

For a square matrix X_N of size N, recall from Example 1.6 (a) that the **empirical eigenvalue distribution** is the probability measure on $\mathbb{R}$

$$\mu_{X_N} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(X_N)},$$

where $\lambda_i(X_N), i = 1, 2, ..., N$ are the eigenvalues of X_N counting multiplicities. For many random matrix models X_N, the empirical eigenvalue distributions converge weakly to a nonrandom compactly supported probability measure μ on $\mathbb{C}$ as $N \to \infty$. In addition, the largest modulus of eigenvalues often converge to the largest modulus of the support of μ. For example, the **normalized Gaussian Unitary Ensemble (GUE)** is a random matrix $G_N = (g(i,j))_{i,j\in[N]}$ that satisfies the following conditions:

- $g(i,j) = \overline{g(j,i)}$ for all i, j, i.e., G_N is Hermitian,
- the random variables $\{\Re g(i,j), \Im g(i,j), g(i,i) : 1 \leq i,j \leq N, i > j\}$ are independent,
- $\Re g(i,j)$ and $\Im X(i,j)$ are distributed as $\mathrm{N}(0, 1/(2N))$ if $i \neq j$, and $g(i,i)$ is distributed as $\mathrm{N}(0, 1/N)$.

It is known that the empirical eigenvalue distribution of G_N converges weakly to Wigner's semicircle distribution $\frac{1}{2\pi}\sqrt{4-x^2}\,\chi_{[-2,2]}(x)\,dx$ a.s., see e.g. [142, Theorem 2.4.2]. Moreover, the largest eigenvalue converges to 2 a.s., see e.g. [142, Section 2.3]. A simulation is shown in Figure 3.

In general, the weak convergence of the empirical eigenvalue distributions does not guarantee that the largest modulus of eigenvalues converges to the largest modulus of the support of the limiting measure. This is because, in the convergence of empirical eigenvalue distributions, a relatively small number of eigenvalues, e.g. of order

$o(N)$, will disappear in the large N limit. For example, suppose that $X_N, N \in \mathbb{N}$ are Hermitian matrices and μ_{X_N} weakly converges to a probability measure μ. Then for any sequence of positive integers $(\ell(N))_{N=1}^\infty$ with $\ell(N) = o(N)$ and a bounded continuous function f on $\mathbb{R}$ we have

$$\int_{\mathbb{R}} f(x)\,\mu_{X_N}(dx) = \frac{1}{N}\sum_{i=1}^{\ell(N)} f(\lambda_i(X_N)) + \frac{1}{N}\sum_{i=\ell(N)+1}^{N} f(\lambda_i(X_N)).$$

The first sum goes to zero since $N \to \infty$ as

$$\left|\frac{1}{N}\sum_{i=1}^{\ell(N)} f(\lambda_i(X_N))\right| \le \|f\|_{L^\infty}\frac{\ell(N)}{N} \to 0,$$

and therefore, the first $\ell(N)$ eigenvalues do not contribute to the limit μ. To put it differently, the weak convergence $\mu_{X_N} \to \mu$ carries no information about the first $\ell(N)$ eigenvalues.

Eigenvalues located outside the support of the limiting measure in the large N limit are called **outliers**. As an application of monotone independence, we will analyze some random matrix models that have outliers in the large N limit.

8.1. **Weingarten calculus on the unitary group.** Let U be a **Haar unitary random matrix** of size N, i.e., it is a random variable taking values in the group U_N of unitary matrices of size N such that the distribution on U_N induced by U is the normalized Haar measure. We use several known results on expectations of moments of entries of U. Let $\mathfrak{S}_k$ denote the symmetric group on $[k]$. For each $\sigma \in \mathfrak{S}_k$ and matrices $A_1, A_2, ..., A_k \in \mathrm{M}_N(\mathbb{C})$, let $\mathrm{Tr}_\sigma[A_1, A_2, ..., A_k]$ denote the product of traces according to the cycle decomposition of σ: if $\sigma = c_1c_2...c_\ell$ where $c_i = (k_i(1), k_i(2), ..., k_i(p_i))$ are cyclic permutations, then

$$\mathrm{Tr}_\sigma(A_1, A_2, ..., A_k) := \prod_{i=1}^{\ell} \mathrm{Tr}(A_{k_i(1)}A_{k_i(2)}\cdots A_{k_i(p_i)}).$$

The number ℓ is determined uniquely by σ and is denoted $\ell(\sigma)$. For example for $\sigma = (1,4,6)(2)(3,5)$ the above definition reads

$$\mathrm{Tr}_\sigma[A_1, A_2, ..., A_6] := \mathrm{Tr}(A_1A_4A_6)\mathrm{Tr}(A_2)\mathrm{Tr}(A_3A_5).$$

There exists a function $\mathrm{Wg}\colon \left(\bigcup_{k\in\mathbb{N}}\mathfrak{S}_k\right)\times\mathbb{N}\to\mathbb{R}$, called the **Weingarten function** for the unitary groups, such that

$$\begin{aligned}&\mathbb{E}[\mathrm{Tr}_\sigma(A_1UB_1U^*, A_3UB_2U^*, \dots, A_kUB_kU^*)]\\ &= \sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\mathfrak{S}_k\\ \sigma_1\sigma_2\sigma_3=\sigma}} \mathrm{Tr}_{\sigma_1}(A_1, A_2, \dots, A_k)\mathrm{Tr}_{\sigma_2}(B_1, B_2, \dots, B_k)\mathrm{Wg}(\sigma_3, N)\end{aligned} \tag{8.1}$$

for any $k, N \in \mathbb{N}$ and nonrandom square matrices $A_i, B_i \in \mathrm{M}_N(\mathbb{C})$, see [48].

Let C_p be the Catalan number

$$\mathrm{C}_p := \frac{(2p)!}{p!(p+1)!}, \qquad p \in \mathbb{N}.$$

For $\sigma \in \mathfrak{S}_k$ let $|\sigma|$ be the minimal number of transpositions such that σ can be expressed as the product of them. The relation $\ell(\sigma) = k - |\sigma|$ holds true. Let $\sigma = c_1\cdots c_{\ell(\sigma)}$ be the cycle decomposition of $\sigma \in \mathfrak{S}_k$ and then let

$$\mathrm{m}(\sigma) := \prod_{i=1}^{\ell(\sigma)} (-1)^{|c_i|}\mathrm{C}_{|c_i|}. \tag{8.2}$$

The Weingarten function satisfies

$$\mathrm{Wg}(\sigma, N) = N^{-k-|\sigma|}(\mathrm{m}(\sigma) + O(N^{-2})), \qquad \sigma \in \mathfrak{S}_k, \tag{8.3}$$

see [48, Corollary 2.7].

8.2. **Asymptotic monotone independence of large random matrices.** Since Voiculescu's pioneering work on asymptotic free independence of large random matrices [145], the method of noncommutative probability has been applied to a wide range of theoretical and practical problems on random matrices. One of results in this direction is asymptotic monotone independence for large random matrices.

Proposition 8.1. *Let $U(N)$ be an $N \times N$ Haar unitary random matrix and $(A(i,N) : i \in I)$, $(B(j,N) : j \in J)$ be families of $N \times N$ nonrandom matrices for $N \in \mathbb{N}$. Suppose that the limits*

$$\lim_{N\to\infty} \mathrm{Tr}(A(i_1,N)^{\varepsilon_1}A(i_2,N)^{\varepsilon_2}\cdots A(i_k,N)^{\varepsilon_k}) \in \mathbb{C}, \tag{8.4}$$

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}(B(j_1,N)^{\varepsilon_1}B(j_2,N)^{\varepsilon_2}\cdots B(j_k,N)^{\varepsilon_k}) \in \mathbb{C} \tag{8.5}$$

exist for any $k \in \mathbb{N}$, $i_1, i_2, ..., i_k \in I$, $j_1, j_2, ..., j_k \in J$ and $\varepsilon_1, \varepsilon_2, ..., \varepsilon_k \in \{1, *\}$. Then for any noncommutative polynomials $P_r(x_i : i \in I) \in \mathbb{C}\langle x_i : i \in I\rangle$ ($r \in [k]$) and $Q_s(y_j : j \in J) \in \mathbb{C}\langle 1, y_j : j \in J\rangle$ ($s \in [k]$), the matrices $A_r = A_r(N) := P_r(A(i,N) : i \in I), B_s = B_s(N) := Q_s(U(N)B(j,N)U(N)^* : j \in J)$ satisfy the following estimates:

$$\mathbb{E}[\mathrm{Tr}(A_1B_1 \cdots A_kB_k)] = \mathrm{Tr}[A_1 \cdots A_k] \prod_{p=1}^{k} \left[\frac{1}{N}\mathrm{Tr}(B_p)\right] + O(N^{-1}), \tag{8.6}$$

$$\mathbb{E}\left[|\mathrm{Tr}(A_1B_1A_2B_2 \cdots A_kB_k) - \mathbb{E}[\mathrm{Tr}(A_1B_1 \cdots A_kB_k)]|^4\right] = O(N^{-2}). \tag{8.7}$$

Proof. Let γ be the circular permutation $\gamma = (1, 2, \ldots, k)$.

Proof of (8.6). The left hand side of the desired formula is exactly (8.1) for $\sigma = \gamma$. By the assumptions on the convergence of traces, the following estimates hold:

$$\mathrm{Tr}_{\sigma_1}(A_1, A_2, \ldots, A_k) = O(1), \tag{8.8}$$

$$\mathrm{Tr}_{\sigma_2}(B_1, B_2, \ldots, B_k) = O(N^{k-|\sigma_2|}). \tag{8.9}$$

Since $\mathrm{Wg}(\sigma_3, N) = O(N^{-k-|\sigma_3|})$, for a triple $\sigma_1, \sigma_2, \sigma_3$ such that $\sigma_1\sigma_2\sigma_3 = \gamma$, the contribution of the summand is $O(N^{-|\sigma_2|-|\sigma_3|})$. Therefore, the leading term of (8.1) is of order $O(1)$ and it appears when $|\sigma_2| = |\sigma_3| = 0$, i.e., only when $\sigma_2 = \sigma_3 = 1_k$ and $\sigma_1 = \gamma$. Since $\mathrm{m}(\gamma) = 1$ and so $\mathrm{Wg}(\gamma, N) = N^{-k}(1 + O(N^{-2}))$, we obtain the desired formula (8.6).

Proof of (8.7)*, Step 1: reduction of the problem.* We prove a slightly stronger result: taking additional polynomials of matrices $A_r = P_r(A(i,N) : i \in I), B_r = Q_r(U(N)B(j,N)U(N)^* : j \in J), k+1 \le r \le 4k$ and setting

$$X_i = \mathrm{Tr}\big(A_{(i-1)k+1}B_{(i-1)k+1} \cdots A_{(i-1)k+k}B_{(i-1)k+k}\big)\,, \qquad i = 1, 2, 3, 4, \tag{8.10}$$

$$\mathring{X}_i = X_i - \mathbb{E}[X_i], \tag{8.11}$$

we prove

$$\mathbb{E}[\mathring{X}_1\mathring{X}_2\mathring{X}_3\mathring{X}_4] = O(N^{-2}). \tag{8.12}$$

This implies the desired (8.7) in the special case $X_1 = X_3 = \overline{X_2} = \overline{X_4}$. Note that the last condition $X_2 = \overline{X_1}$ can be satisfied by selecting $A_{k+1} := A_k^*, B_{k+1} := B_{k-1}^*, A_{k+2} := A_{k-1}^*, B_{k+2} := B_{k-2}^*, \ldots, B_{2k-1} := B_1^*, A_{2k} = A_1^*, B_{2k} = B_k^*$, which yields

$$\overline{X_1} = \mathrm{Tr}((A_1B_1 \cdots A_kB_k)^*) = \mathrm{Tr}_n(A_k^*B_{k-1}^*A_{k-1}^* \cdots A_2^*B_1^*A_1^*B_k^*) = X_2.$$

Let I_i be the interval $\{(i-1)k+1, (i-1)k+2, \ldots, (i-1)k+k\}$, γ_i the cyclic permutation $((i-1)k+1, (i-1)k+2, \ldots, (i-1)k+k)$ of $\mathfrak{S}_{I_i}$, and $\gamma^{\cup 4} := \gamma_1\gamma_2\gamma_3\gamma_4 \in \mathfrak{S}_{4k}$. Let us expand

$$\mathbb{E}[\mathring{X}_1\mathring{X}_2\mathring{X}_3\mathring{X}_4] = \sum_{J \subseteq [4]} \mathbb{E}_J, \quad \text{where} \tag{8.13}$$

$$\mathbb{E}_J := (-1)^{\#J}\mathbb{E}\left[\prod_{i \in J} X_i\right] \prod_{i \in [4]\setminus J} \mathbb{E}[X_i]. \tag{8.14}$$

For example, our notation means $\mathbb{E}_{\{1,3,4\}} = -\mathbb{E}[X_1\mathbb{E}[X_2]X_3X_4]$.

For each $J \subseteq [4]$, from the Weingarten formulas for $\mathbb{E}\left[\prod_{i\in J} X_i\right]$ and for $\mathbb{E}[X_i], i \in [4] \setminus J$, there exists a number $f_J(\sigma_1, \sigma_2, N)$ such that

$$\mathbb{E}_J = \sum_{\substack{\sigma_1,\sigma_2,\sigma_3 \in \mathfrak{S}_{4k} \\ \sigma_1\sigma_2\sigma_3 = \gamma^{\cup 4}}} \mathrm{Tr}_{\sigma_1}(A_1, A_2, \ldots, A_{4k})\mathrm{Tr}_{\sigma_2}(B_1, B_2, \ldots, B_{4k})f_J(\sigma_1, \sigma_2, N). \tag{8.15}$$

The function f_J is either a product of Weingarten functions with signs or zero. Take again $J = \{1, 3, 4\}$ for example. The product of Weingarten formulas (8.1) for $\sigma = \gamma_2$ and for $\sigma = \gamma_1\gamma_3\gamma_4$ gives permutations of $\mathfrak{S}_{I_2} \times \mathfrak{S}_{I_1\cup I_3\cup I_4}$, and so, for $\sigma_1, \sigma_2, \sigma_3 \in \mathfrak{S}_{4k}$ with $\sigma_1\sigma_2\sigma_3 = \gamma^{\cup 4}$,

$$f_J(\sigma_1, \sigma_2, N) = \begin{cases} -\mathrm{Wg}(\sigma_3|_{I_2}, N)\mathrm{Wg}(\sigma_3|_{I_1\cup I_3\cup I_4}, N), & \text{if } \sigma_1, \sigma_2 \text{ preserve } I_2 \text{ and } I_1 \cup I_3 \cup I_4, \\ 0, & \text{otherwise.} \end{cases}$$

The function f_J has a similar expression for general J.

We have the expression

$$\mathbb{E}[\mathring{X}_1\mathring{X}_2\mathring{X}_3\mathring{X}_4] = \sum_{\substack{\sigma_1,\sigma_2,\sigma_3 \in \mathfrak{S}_{4k} \\ \sigma_1\sigma_2\sigma_3 = \gamma^{\cup 4}}} \mathrm{Tr}_{\sigma_1}(A_1, \ldots, A_{4k})\mathrm{Tr}_{\sigma_2}(B_1, \ldots, B_{4k})f(\sigma_1, \sigma_2, N), \tag{8.16}$$

where

$$f(\sigma_1,\sigma_2,N):=\sum_{J\subseteq[4]} f_J(\sigma_1,\sigma_2,N). \tag{8.17}$$

From the assumption of convergence of traces, we have the estimates $\mathrm{Tr}_{\sigma_1}(A_1,\dots,A_{4k})=O(1)$ and $\mathrm{Tr}_{\sigma_2}(B_1,\dots,B_{4k})=O(N^{\ell(\sigma_2)})=O(N^{4k-|\sigma_2|})$. Combined with (8.3) these estimates yield

$$\mathrm{Tr}_{\sigma_2}(B_1,\dots,B_{4k})f(\sigma_1,\sigma_2,N)=O(N^{-|\sigma_2|-|\sigma_3|}). \tag{8.18}$$

We show the estimates

$$\mathrm{Tr}_{\sigma_2}(B_1,\dots,B_{4k})f(\sigma_1,\sigma_2,N)=O(N^{-2}), \tag{8.19}$$

which suffice to finish the proof.

Proof of (8.7)*, Step 2: Proof of* (8.19). Given $\sigma_1,\sigma_2,\sigma_3\in\mathfrak{S}_{4k}$ with $\sigma_1\sigma_2\sigma_3=\gamma^{\cup 4}$, we introduce an equivalence relation $\sim$ on $[4k]$: $i\sim j$ if there exists $\tau\in\mathrm{Grp}\langle\sigma_1,\sigma_2,\sigma_3\rangle$ such that $\tau(i)=j$. Since this group contains $\gamma^{\cup 4}$, every interval I_i must be a subset of some equivalence class. Then the permutations $\sigma_1,\sigma_2,\sigma_3$ associate a set partition $\pi(\sigma_1,\sigma_2,\sigma_3)=\{P_1,\dots,P_m\}$ of [4] such that the subsets

$$\bigcup_{i\in P_1} I_i,\ \bigcup_{i\in P_2} I_i,\dots,\ \bigcup_{i\in P_m} I_i\subseteq[4k]$$

are exactly the equivalence classes.

On the other hand, a subset $J\subseteq[4]$ also associates the set partition $\pi(J)=\{J,\{p\}:p\in[4]\setminus J\}$ of [4]. Since $f_J(\sigma_1,\sigma_2,N)$ vanishes if σ_1 or σ_2 do not preserve one of the subsets $\bigcup_{i\in J}I_i, I_p, p\in[4]\setminus J$, the only f_J's satisfying $\pi(J)\geq\pi(\sigma_1,\sigma_2,\sigma_3)$ contribute to f in the sum (8.17) and the other f_J's are zero. We discuss several cases according to $\pi(\sigma_1,\sigma_2,\sigma_3)$.

Case 1: $\pi(\sigma_1,\sigma_2,\sigma_3)=\{[4]\}$, or equivalently, the group $\mathrm{Grp}\langle\sigma_1,\sigma_2,\sigma_3\rangle$ acts on $[4k]$ transitively. In this case, f_J vanishes unless $J=[4]$, and so $f=f_{[4]}=\mathrm{Wg}(\sigma_3,N)$. The case $|\sigma_2|+|\sigma_3|=0$ is irrelevant because the condition $\sigma_1\sigma_2\sigma_3=\gamma^{\cup 4}$ contradicts transitivity. The case $|\sigma_2|+|\sigma_3|=1$ is also irrelevant because then one would be the identity and the other would be a transposition, and again the condition $\sigma_1\sigma_2\sigma_3=\gamma^{\cup 4}$ would contradict the transitivity. Therefore, only the case $|\sigma_2|+|\sigma_3|\geq 2$ occurs, and so $\mathrm{Tr}_{\sigma_2}(B_1,\dots,B_{4k})f(\sigma_1,\sigma_2,N)=O(N^{-2})$.

Case 2: $\pi(\sigma_1,\sigma_2,\sigma_3)$ is a pair partition, i.e., its each block has cardinality two. In this case again we have $f=f_{[4]}=\mathrm{Wg}(\sigma_3,N)$, and from a similar reasoning we must have $|\sigma_2|+|\sigma_3|\geq 2$ and hence $\mathrm{Tr}_{\sigma_2}(B_1,\dots,B_{4k})f(\sigma_1,\sigma_2,N)=O(N^{-2})$.

In the other cases, a cancellation occurs between Wg functions.

Case 3: $\pi(\sigma_1,\sigma_2,\sigma_3)$ has two blocks with cardinality 1 and 3. For example, let us consider the case $\pi(\sigma_1,\sigma_2,\sigma_3)=\{\{1,3,4\},\{2\}\}$. The equivalence classes are $I_1\cup I_3\cup I_4$ and I_2. The only indices J's for which f_J is non-zero are $J=\{1,3,4\},\{1,2,3,4\}$. In these cases we have

$$\begin{aligned} f_{\{1,3,4\}}&=-\mathrm{Wg}(\sigma_3|_{I_2},N)\mathrm{Wg}(\sigma_3|_{I_1\cup I_3\cup I_4},N),\\ f_{\{1,2,3,4\}}&=\mathrm{Wg}(\sigma_3,N).\end{aligned}$$

By (8.3) and the multiplicativity (8.2) of Moebius functions, we obtain

$$\begin{aligned} f&=f_{\{1,3,4\}}+f_{\{1,2,3,4\}}\\ &=N^{-4k-|\sigma_3|}(\underbrace{-\mathrm{m}(\sigma_3|_{I_2})\mathrm{m}(\sigma_3|_{I_1\cup I_3\cup I_4})+\mathrm{m}(\sigma_3)}_{=0}+O(N^{-2}))\\ &=O(N^{-4k-|\sigma_3|-2}).\end{aligned}$$

Thus $\mathrm{Tr}_{\sigma_2}(B_1,\dots,B_{4k})f(\sigma_1,\sigma_2,N)=O(N^{-|\sigma_2|-|\sigma_3|-2})=O(N^{-2})$. The other cases of $\pi(\sigma_1,\sigma_2,\sigma_3)$ are similar.

Case 4: $\pi(\sigma_1,\sigma_2,\sigma_3)$ has three blocks. Let us consider the example $\pi(\sigma_1,\sigma_2,\sigma_3)=\{\{1\},\{2\},\{3,4\}\}$. The indices J's for which f_J is non-zero are $J=\{3,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}$. We see that

$$\begin{aligned} f=&f_{\{3,4\}}+f_{\{1,3,4\}}+f_{\{2,3,4\}}+f_{\{1,2,3,4\}}\\ =&\mathrm{Wg}(\sigma_3|_{I_1},N)\mathrm{Wg}(\sigma_3|_{I_2},N)\mathrm{Wg}(\sigma_3|_{I_3\cup I_4},N)-\mathrm{Wg}(\sigma_3|_{I_2},N)\mathrm{Wg}(\sigma_3|_{I_1\cup I_3\cup I_4},N)\\ &-\mathrm{Wg}(\sigma_3|_{I_1},N)\mathrm{Wg}(\sigma_3|_{I_2\cup I_3\cup I_4},N)+\mathrm{Wg}(\sigma_3,N)\\ =&N^{-4k-|\sigma_3|}(\mathrm{m}(\sigma_3|_{I_1})\mathrm{m}(\sigma_3|_{I_2})\mathrm{m}(\sigma_3|_{I_3\cup I_4})-\mathrm{m}(\sigma_3|_{I_2})\mathrm{m}(\sigma_3|_{I_1\cup I_3\cup I_4})\\ &-\mathrm{m}(\sigma_3|_{I_1})\mathrm{m}(\sigma_3|_{I_2\cup I_3\cup I_4})+\mathrm{m}(\sigma_3)+O(N^{-2}))\\ =&O(N^{-4k-|\sigma_3|-2}).\end{aligned}$$

The other $\pi(\sigma_1,\sigma_2,\sigma_3)$'s can be handled in the same way.

Case 5: $\pi(\sigma_1,\sigma_2,\sigma_3)=\{\{1\},\{2\},\{3\},\{4\}\}$, i.e., every interval I_i is preserved by $\sigma_1,\sigma_2,\sigma_3$. In this case f_J for all the 16 subsets $J\subseteq\{1,2,3,4\}$ contribute to f. By the multiplicativity (8.2), the dominant contribution to f is the sum of 16 terms

$$\pm N^{-4k-|\sigma_3|}\mathrm{m}(\sigma_3|_{I_1})\mathrm{m}(\sigma_3|_{I_2})\mathrm{m}(\sigma_3|_{I_3})\mathrm{m}(\sigma_3|_{I_4}).$$

Exactly half of them have the minus sign, so their sum cancel and we obtain $f=O(N^{-4k-|\sigma_3|-2})$.

The above arguments finish the proof of (8.19). □

Theorem 8.2. *Under the assumptions and notations of Proposition 8.1, we have*

$$\lim_{N\to\infty}\mathrm{Tr}(A_1B_1A_2B_2\cdots A_kB_k)=\left[\lim_{N\to\infty}\mathrm{Tr}(A_1A_2\cdots A_k)\right]\prod_{p=1}^k\left[\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}(B_p)\right]\qquad a.s.$$

Proof. The assumptions (8.4) and (8.5) imply that the limits

$$\lim_{N\to\infty}\mathrm{Tr}(A_1\cdots A_k),\qquad\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}(B_p)$$

exist, and then the proven estimate (8.6) further implies

$$\mathbb{E}[\mathrm{Tr}(A_1B_1\cdots A_kB_k)]=\left[\lim_{N\to\infty}\mathrm{Tr}(A_1\cdots A_k)\right]\prod_{p=1}^k\left[\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}(B_p)\right].\tag{8.20}$$

On the other hand, taking the sum $\sum_{N=1}^\infty$ of the estimate (8.7) yields

$$\sum_{N=1}^\infty|\mathrm{Tr}(A_1B_1\cdots A_kB_k)-\mathbb{E}[\mathrm{Tr}(A_1B_1\cdots A_kB_k)]|^4<+\infty\quad\text{a.s.},$$

so that the summand converges to zero a.s. Combining this fact and (8.20) finishes the proof. □

Remark 8.3. To show the almost sure convergence, we computed the L^4-norm in Proposition 8.1. Of course the L^2-norm is easier to calculate but it only gives the estimate $O(N^{-1})$, which is insufficient to deduce the almost sure convergence. One can confirm this through a simple example for $k=1$: for a rank-one projection $P=\mathrm{diag}(1,0,0,...,0)$ and deterministic matrices $A,B\in\mathrm{M}_N(\mathbb{C})$ we can see

$$\begin{aligned}&\mathbb{E}[|\mathrm{Tr}(PUBU^*)-\mathbb{E}[\mathrm{Tr}(PUBU^*)]|^2]\\&=\mathbb{E}[|\mathrm{Tr}(PUBU^*)|^2]-|\mathbb{E}[\mathrm{Tr}(PUBU^*)]|^2\\&=\mathbb{E}[\mathrm{Tr}(PUBU^*)\mathrm{Tr}(PUB^*U^*)]-|\mathbb{E}[\mathrm{Tr}(PUBU^*)]|^2\\&=\mathrm{Tr}(P)^2|\mathrm{Tr}(B)|^2\mathrm{Wg}(1_2,N)+\mathrm{Tr}(P)^2\mathrm{Tr}(BB^*)\mathrm{Wg}((1,2),N)\\&+\mathrm{Tr}(P^2)|\mathrm{Tr}(B)|^2\mathrm{Wg}((1,2),N)+\mathrm{Tr}(P^2)\mathrm{Tr}(BB^*)\mathrm{Wg}(1_2,N)\\&-\mathrm{Tr}(P)^2|\mathrm{Tr}(B)|^2\mathrm{Wg}(1_1,N)^2.\end{aligned}$$

As soon as one applies (8.3) to the above expression, the first and fifth terms contain a cancellation and yield $O(N^{-2})$; the second term is $O(N^{-2})$; however, the third and forth terms do not cancel and contribute $O(N^{-1})$. In a similar manner, the L^3-norm is also of order $O(N^{-1})$.

Corollary 8.4 (Asymptotic monotone independence)**.** *Let $\ell\in\mathbb{N}$ be fixed, $\tilde{A}(i,N)$ $(i\in I,N>\ell)$ be deterministic $\ell\times\ell$ matrices such that $\lim_{N\to\infty}\tilde{A}(i,N)$ exists in $\mathrm{M}_\ell(\mathbb{C})$. Let*

$$A(i,N):=\begin{pmatrix}\tilde{A}(i,N)&O\\O&O\end{pmatrix}\in\mathrm{M}_N(\mathbb{C}).$$

Let $(B(j,N))_{j\in J}$ be a family of $N\times N$ matrices that satisfy the assumption (8.5)*. Let $U(N)$ be a Haar unitary random matrix of size N. We consider the partial trace*

$$\varphi_\ell(X):=\frac{1}{\ell}\sum_{i=1}^\ell X_{i,i},\qquad X=(X_{i,j})_{i,j\in[N]}.\tag{8.21}$$

Then for any noncommutative polynomials $P_r(x_i:i\in I)\in\mathbb{C}\langle x_i:i\in I\rangle$ and $Q_s(y_j:j\in J)\in\mathbb{C}\langle 1,y_j:j\in J\rangle$ and any $k\geq 2$, the matrices $A_r:=P_r(A(i,N):i\in I)$ $(r\in[k-1]),B_s:=Q_s(U(N)B(j,N)U(N)^*:j\in J)$ $(s\in[k])$ satisfy

$$\lim_{N\to\infty}\varphi_\ell(B_p)=\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}(B_p)\quad a.s.,\qquad p\in[k],\tag{8.22}$$

$$\lim_{N\to\infty}\varphi_\ell(B_1A_1B_2A_2\cdots A_{k-1}B_k)=\lim_{N\to\infty}\varphi_\ell(A_1A_2\cdots A_{k-1})\prod_{p=1}^k\left[\lim_{N\to\infty}\varphi_\ell(B_p)\right]\quad a.s.\tag{8.23}$$

The second convergence shows that the subsets $(A(i,N) : i \in I)$ and $(U(N)B(j,N)U(N)^* : j \in J)$ are almost surely "monotonically independent in the limit $N \to \infty$" with respect to φ_ℓ.

Proof. To show (8.22), we apply Theorem 8.2 to the rank-ℓ projection $P_\ell(N) = \operatorname{diag}(1,1,...,1,0,0,...,0)$ regarded as a family of single matrix, and the family $(B(j,N) : j \in J)$. Then Theorem 8.2 for $k = 1$, $A_1 = P_\ell(N)$ and B_1 replaced with B_p, is exactly the desired (8.22).

To show (8.23), we again apply Theorem 8.2 now for the families $(P_\ell(N), A(i,N) : i \in I)$ and $(B(j,N) : j \in J)$, the former of which satisfies the assumption (8.4). Then Theorem 8.2 with $A_1 := P_\ell(N)$, A_i replaced with A_{i-1} for $i \geq 2$, together with the proven (8.22), is exactly the desired (8.23). ☐

Remark 8.5. For $\ell = 1$, the factorization (8.23) holds without taking the limit $N \to \infty$; see Example 1.18.

Remark 8.6. Results of this section can be extended to the case when $(B(j,N) : j \in J)$ is a family of random matrices independent of the Haar unitary $U(N)$ with an additional assumption: Let us assume that for any $k \in \mathbb{N}$, $j_1, j_2, ..., j_k \in J$ and $\varepsilon_1, ..., \varepsilon_k \in \{1, *\}$,

$$\lim_{N\to\infty} \frac{1}{N}\operatorname{Tr}(B(j_1)^{\varepsilon_1} \cdots B(j_k)^{\varepsilon_k}) = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{Tr}(B(j_1)^{\varepsilon_1} \cdots B(j_k)^{\varepsilon_k})\right] \in \mathbb{C} \quad \text{a.s.} \tag{8.24}$$

with shorthand symbol $B(j) := B(j,N)$. Then we modify Proposition 8.1 as follows.

- We only take the expectation in (8.6) for the Haar unitary part:

$$\mathbb{E}_U[\operatorname{Tr}(A_1B_1 \cdots A_kB_k)] = \operatorname{Tr}[A_1 \cdots A_k] \prod_{p=1}^{k} \left[\frac{1}{N}\operatorname{Tr}(B_p)\right] + O(N^{-1}) f(B_1, ..., B_k, N). \tag{8.25}$$

  Here the term $f(B_1, ..., B_k, N)$ is a polynomial in N^{-1} and the normalized traces of B_i's that are bounded a.s. by the assumption (8.24).
- Instead of (8.7) we can show

$$\mathbb{E}\Big[|\operatorname{Tr}(A_1B_1A_2B_2 \cdots A_kB_k) - \mathbb{E}_U[\operatorname{Tr}(A_1B_1 \cdots A_kB_k)]|^4\Big] = O(N^{-2}). \tag{8.26}$$

  For this first we show (8.7) only by taking the expectation concerning the Haar unitary U and fixing the $B(j,N)$'s. Then the term $O(N^{-2})$ in (8.7) would include random variables of the form $\frac{1}{N}\operatorname{Tr}(B(j_1)^{\varepsilon_1} \cdots B(j_k)^{\varepsilon_k})$. Further taking the expectation with respect to $B(j,N)$'s and using the condition (8.24) we can deduce (8.26). The modified formulas (8.25) and (8.26) yield the same conclusions Theorem 8.2 and Corollary 8.4.

If $(B(j,N) : j \in J)$ is an independent family of normalized GUEs, then the above requirements are satisfied because a GUE has the same law as UDU^* where D is diagonal, U is Haar unitary and D, U are independent. Condition (8.24) is satisfied because the convergence of the expected traces is a consequence of Voiculescu's asymptotic freeness [145], and the almost sure convergence is known in Hiai and Petz [85, Corollary 4.3.6].

8.3. **Outliers of additive and multiplicative perturbations.** We consider finite-rank perturbations of random matrices. In this section we always assume that the eigenvalues $\lambda_i(X), i \in [N]$ of a Hermitian matrix $X \in \mathrm{M}_N(\mathbb{C})$ are arranged in the way

$$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_N(X).$$

Let μ_X be the empirical eigenvalue distribution of X and ν_X the analytic distribution of X with respect to the partial trace φ_ℓ defined in (8.21). As being an analytic distribution, ν_X is supported on $\operatorname{Sp}(X)$ that is the set of the eigenvalues of X.

We will use the following Weyl's inequalities. These can be proved from the min-max theorem and the reader is referred to [33, Chapter III.2] for the proofs.

Lemma 8.7. *Let X, Y be $N \times N$ Hermitian matrices.*

(i) *$\lambda_{i+j-1}(X+Y) \leq \lambda_i(X) + \lambda_j(Y)$ holds for $1 \leq i,j \leq N$ with $i+j-1 \leq N$.*

(ii) *If $X \leq Y$ then $\lambda_i(X) \leq \lambda_i(Y)$ for all $i \in [N]$.*

(iii) *$|\lambda_i(X) - \lambda_i(Y)| \leq \|X - Y\|$ for all $i \in [N]$, where $\|\cdot\|$ is the operator norm.*

Theorem 8.8. *Let $\ell \in \mathbb{N}$ be fixed, $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_\ell > 0$ and*

$$P = P(\theta_1, \theta_2, ..., \theta_\ell, N) := \operatorname{diag}(\theta_1, \theta_2, ..., \theta_\ell, 0, 0, ..., 0) \in \mathrm{M}_N(\mathbb{C}), \qquad N \geq \ell.$$

Let $U = U(N)$ be a Haar unitary random matrix of size N defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $B = B(N)$ be an $N \times N$ Hermitian deterministic matrix. Suppose that there exists a probability measure μ on $\mathbb{R}$

such that $\beta := \max \operatorname{supp}(\mu) < +\infty$, μ *has finite moments of all orders that satisfy Carleman's condition* (A.2), *and*

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}(B^k) = \int_{\mathbb{R}} x^k \,\mu(dx), \qquad k \in \mathbb{N}, \tag{8.27}$$

$$\lim_{N\to\infty} \lambda_1(B) = \beta. \tag{8.28}$$

Let $\gamma := F_\mu(\beta + 0)$ *that lies in* $[0, +\infty)$. *For each* $i \in [\ell]$ *the following hold.*

(i) *The empirical eigenvalue distribution of* $UBU^* + P$ *converges weakly to* μ *a.s.*

(ii) *If* $\theta_i > \gamma$, *then the equation* $F_\mu(x) = \theta_i$ *has a unique solution* $x = x_i \in (\beta, +\infty)$ *and* $\lim_{N\to\infty} \lambda_i(UBU^* + P) = x_i$ *a.s.*

(iii) *If* $\theta_i \le \gamma$ *then* $\lim_{N\to\infty} \lambda_i(UBU^* + P) = \beta$ *a.s.*

Remark 8.9. The result also holds when B is a random matrix independent of U and satisfying conditions (8.24), (8.27) and (8.28) almost surely.

Proof. Before going into the details, it would be helpful for the reader to have the key idea: by Corollary 8.4, the matrices P and UBU^* are asymptotically monotonically independent with respect to the state φ_ℓ. Then we can identify the limit distribution of $UBU^* + P$ with respect to φ_ℓ as the monotone convolution of ν_P and μ. If this distribution has an atom at a point in $(\beta, +\infty)$, then the matrix $UBU^* + P$ must have an eigenvalue near the point, which becomes an outlier. This point is exactly the solution x to the equation $F_\mu(x) = \theta_i$. However, this argument only shows the existence of an outlier and does not tell us the number or multiplicities of them. Then Weyl's inequalities provide sufficiently sharp estimates on the number of outliers. The details are as follows.

Weak convergence of μ_B *and* μ_{UBU^*+P}. Assumption (8.27) and the determinacy of the moment sequence of μ imply that the empirical eigenvalue distribution μ_B converges weakly to μ. Concerning μ_{UBU^*+P}, we expand $(UBU^* + P)^k$ into monomials. Each monomial except $(UBU^*)^k$ has at least one factor P, so that by Theorem 8.2 its evaluation by Tr converges almost surely to a finite value. Therefore, the evaluation by the normalized trace converges to zero. The above arguments yield

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}((UBU^* + P)^k) = \lim_{N\to\infty} \mathrm{Tr}((UBU^*)^k) = \lim_{N\to\infty} \mathrm{Tr}(B^k) \quad \text{a.s.}, \qquad k \in \mathbb{N}. \tag{8.29}$$

This implies the weak convergence $\mu_{UBU^*+P} \to \mu$ a.s.

General estimates on eigenvalues. As a preparation for proving (ii) and (iii) we derive some facts from Weyl's inequalities. First, the weak convergence $\mu_B \to \mu$ and (8.28) imply

$$\lim_{N\to\infty} \lambda_i(B) = \beta, \qquad i \in \mathbb{N}, \tag{8.30}$$

because otherwise μ would be supported on a smaller interval $(-\infty, \beta - \varepsilon)$.

Next, by Lemma 8.7 (i) we have

$$\lambda_{i+j-1}(UBU^* + P) \le \lambda_i(B) + \lambda_j(P), \qquad i + j - i \le N.$$

Since P has ℓ positive eigenvalues, we conclude $\lambda_{i+\ell}(UBU^* + P) \le \lambda_i(B)$ for all $1 \le i \le N - \ell$. Combining this and (8.30) yields

$$\limsup_{N\to\infty} \lambda_{i+\ell}(UBU^* + P) \le \beta, \qquad i \in \mathbb{N}. \tag{8.31}$$

Since the weak convergence limit of μ_{UBU^*+P} is also μ, we must have

$$\liminf_{N\to\infty} \lambda_i(UBU^* + P) \ge \beta \quad \text{a.s.}, \qquad i \in \mathbb{N}; \tag{8.32}$$

otherwise the measure μ would be supported on $(-\infty, \beta - \varepsilon)$ for some $\varepsilon > 0$, which would be a contradiction. The previous two inequalities imply

$$\lim_{N\to\infty} \lambda_{i+\ell}(UBU^* + P) = \beta \quad \text{a.s.}, \qquad i \in \mathbb{N}. \tag{8.33}$$

Asymptotic monotone independence with measurability issues. Corollary 8.4 can be applied to the families of single matrices $\{P(\Theta, N)\}$ and $\{B(N)\}$, which yields with notation $\tilde{B} := UBU^*$

$$\lim_{N\to\infty} \varphi_\ell(\tilde{B}^p) = \lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}(\tilde{B}^p) \quad \text{a.s.}, \qquad p \in \mathbb{N}_0, \tag{8.34}$$

$$\lim_{N\to\infty} \varphi_\ell(\tilde{B}^{p_0} P^{q_1} \tilde{B}^{p_1} P^{q_2} \cdots P^{q_k} \tilde{B}^{p_k}) = \lim_{N\to\infty} \varphi_\ell(P^{q_1+q_2+\cdots+q_k}) \prod_{i=0}^{k} \left[\lim_{N\to\infty} \varphi_\ell(\tilde{B}^{p_i}) \right] \quad \text{a.s.} \tag{8.35}$$

for all $k \in \mathbb{N}$, $q_i \in \mathbb{N}$ and $p_i \in \mathbb{N}_0$. A technical issue to note here is that the almost sure convergence above holds on an event Ω_Θ of probability one depending on $\Theta = (\theta_1, ..., \theta_\ell)$. Because we will change the parameter Θ later, let us consider the countable set

$$S := \{(\theta_1, \theta_2, ..., \theta_\ell) \in \mathbb{Q}^\ell : \theta_1 > \theta_2 > \cdots > \theta_\ell > 0\}$$

and an event $\Omega' \in \mathcal{F}$ with probability one defined by

$$\Omega' := \{\omega \in \Omega : (8.29), (8.31), (8.32), (8.34) \text{ and } (8.35) \text{ hold} \\ \text{for all } i,\, k \in \mathbb{N},\, p, p_i \in \mathbb{N}_0,\, q_i \in \mathbb{N} \text{ and } \Theta \in S\}.$$

For any sample $\omega \in \Omega'$ it holds from (8.35) that

$$\begin{aligned} \lim_{N\to\infty} \int_{\mathbb{R}} x^k \, \nu_{UBU^*+P}(dx) &= \lim_{N\to\infty} \varphi_\ell((UBU^* + P)^k) \\ &= \int_{\mathbb{R}} x^k (\nu_P \rhd \mu)(dx), \qquad k \in \mathbb{N}. \end{aligned}$$

Note that $\nu_P = \frac{1}{\ell}(\delta_{\theta_1} + \delta_{\theta_2} + \cdots + \delta_{\theta_\ell})$. By Proposition A.4, $\nu_P \rhd \mu$ has a determinate moment sequence, and hence by Proposition A.6, ν_{UBU^*+P} weakly converges to $\nu_P \rhd \mu$. According to Theorem 5.1, the limit distribution has the Cauchy transform

$$G_{\nu_P}(F_\mu(z)) = \frac{1}{\ell}\left(\frac{1}{F_\mu(z) - \theta_1} + \frac{1}{F_\mu(z) - \theta_2} + \cdots + \frac{1}{F_\mu(z) - \theta_\ell}\right).$$

Since μ is supported on $(-\infty, \beta]$, the reciprocal Cauchy transform F_μ has an analytic continuation to $\mathbb{C}\setminus(-\infty, \beta]$, strictly increasing and taking positive values on $(\beta, +\infty)$, and $\lim_{x\to+\infty} F_\mu(x) = +\infty$; see Proposition 4.39. This implies $\gamma = F_\mu(\beta + 0) \in [0, \infty)$ and, if $\theta_i > \gamma$, the existence of a unique solution x_i to $F_\mu(x) = \theta_i$ as stated in (ii).

Completing the proof. Now we are ready to finish the proofs of (ii) and (iii).

Case 1: $\Theta \in S, \theta_1 > \cdots > \theta_\ell > \gamma$. Since for each $i \in [\ell]$ the equation $F_\mu(x) = \theta_i$ has a solution $x = x_i > \beta$ with $x_1 > x_2 > \cdots > x_\ell > \beta$, the function $G_{\nu_P}(F_\mu(z))$ has a pole at x_i, so that the monotone convolution $\nu_P \rhd \mu$ has an atom at each x_i. We take $\varepsilon > 0$ so that $x_\ell > \beta + \varepsilon$. This implies that for sufficiently large N, the matrix $UBU^* + P$ has at least one eigenvalue close to x_i for each i, so that altogether at least ℓ eigenvalues on the interval $(\beta + \varepsilon, +\infty)$. On the other hand, (8.33) implies that $UBU^* + P$ has at most ℓ eigenvalues greater than $\beta + \varepsilon$, so that has exactly ℓ eigenvalues on $(\beta + \varepsilon, +\infty)$. This shows $\lim_{N\to\infty} \lambda_i(UBU^* + P) = x_i$ for all $i \in [\ell]$ and $\omega \in \Omega'$.

Case 2: $\Theta \in S$, $\theta_1 > \cdots > \theta_{\ell-1} > \gamma \geq \theta_\ell$. We again fix a sample $\omega \in \Omega'$. We consider θ_ℓ as a variable. In case 1, the function $(\gamma, \theta_{\ell-1}) \cap \mathbb{Q} \ni \theta_\ell \mapsto x_\ell(\theta_\ell) \in (\beta, +\infty)$ is continuous and $\lim_{\theta_\ell \downarrow \gamma, \theta_\ell \in \mathbb{Q}} x_\ell(\theta_\ell) = \beta$. By Lemma 8.7 (ii), the function $\mathbb{Q} \ni \theta_\ell \mapsto \lambda_\ell(UBU^* + P)$ is non-decreasing, so that for each $\theta_\ell \in (0, \gamma]$ we obtain $\limsup_{N\to\infty} \lambda_\ell(UBU^* + P) \leq \beta$. Combined with (8.32) this yields

$$\lim_{N\to\infty} \lambda_\ell(UBU^* + P) = \beta.$$

Hence, for each $\varepsilon > 0$, $UBU^* + P$ has at most $\ell - 1$ eigenvalues on $(\beta + \varepsilon, +\infty)$ for all large N. On the other hand, $\nu_P \rhd \mu$ has an atom at $x_i > \beta + \varepsilon$ for $i \in [\ell - 1]$. Therefore, for large N, $UBU^* + P$ must have exactly $\ell - 1$ eigenvalues on $(\beta + \varepsilon, +\infty)$. The weak convergence $\nu_{UBU^*+P} \to \nu_P \rhd \mu$ implies the convergence $\lambda_i(UBU^* + P) \to x_i$ for each $i \in [\ell - 1]$, finishing the proof of Case 2. Repeating similar arguments yields the statement in the case $\Theta \in S, \theta_i > \gamma \geq \theta_{i+1}$.

Case 3: $\Theta \in \mathbb{R}^\ell$, $\theta_1 \geq \cdots \geq \theta_{\ell-1} > 0$ *(the most general case).* We take a sequence $\Theta^{(n)} \in S$ such that $\Theta^{(n)}$ converges to Θ. By Lemma 8.7 (iii), we have the uniform estimate $|\lambda_i(UBU^* + P(\Theta)) - \lambda_i(UBU^* + P(\Theta^{(n)}))| \leq \max_{i\in[\ell]} |\theta_i - \theta_i^{(n)}|$, which finishes the proof. ☐

Example 8.10. Let μ be the standard semicircle distribution $(1/(2\pi))\sqrt{4 - x^2}\chi_{[-2,2]}(x)\, dx$. From (4.38) we have

$$F_\mu(z) = \frac{z + \sqrt{z^2 - 4}}{2}.$$

We see that $F_\mu(2 + 0) = 1$ and for each $\theta > 1$ the solution to $F_\mu(x) = \theta$ is given by $x = \theta + 1/\theta > 2$. Therefore,

$$\lim_{N\to\infty} \lambda_i(UBU^* + P) = \begin{cases} \theta_i + \dfrac{1}{\theta_i}, & \theta_i > 1, \\ 2, & 0 < \theta_i \leq 1. \end{cases}$$

See Figure 4 for a simulation.

For the multiplicative perturbation, we can obtain a similar result.

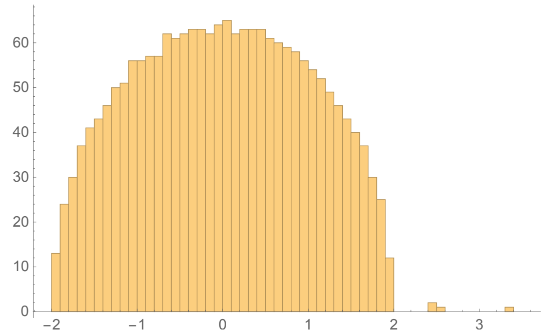

FIGURE 4. A histogram for the eigenvalues of $G+P$, where G is the normalized GUE of matrix size 2000, $P = \mathrm{diag}(3,2,2,2,1,0,0,\dots,0)$ and the bin size is selected to be $1/10$.

Theorem 8.11. *Let ℓ, P and U be as defined in Theorem 8.8. Let $B = B(N)$ be an $N\times N$ positive semi-definite deterministic matrix and*

$$Y_N := (I+P)^{\frac{1}{2}} UBU^* (I+P)^{\frac{1}{2}}.$$

We assume that the empirical eigenvalue distribution μ_B converges weakly to a probability measure μ supported on an interval $[0,\beta]$ with $0 \le \beta < +\infty$, and that $\lambda_1(B)$ converges to β as $N\to\infty$. The function η_μ has analytic continuation to $\mathbb{C}\setminus[1/\beta,+\infty)$, still denoted as η_μ. The limit $\delta := \eta_\mu(1/\beta - 0) \in (0,+\infty]$ exists and the following holds for each $i\in[\ell]$.

(i) *The empirical eigenvalue distribution of Y_N converges weakly to μ a.s.*

(ii) *If $1+\theta_i > 1/\delta$, then the equation $\eta_\mu(y) = 1/(1+\theta_i)$ has a unique solution $y = y_i \in (0,1/\beta)$ and $\lim_{N\to\infty}\lambda_i(Y_N) = 1/y_i$ a.s.*

(iii) *If $1+\theta_i \le 1/\delta$ then $\lim_{N\to\infty}\lambda_i(Y_N) = \beta$ a.s.*

Proof. The proof is quite analogous to that of Theorem 8.8. We omit the details and only mention the main differences. Let $B' := UBU^*$.

- Under our assumptions, we have the moment convergence
$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}(B^k) = \int_{[0,\beta]} x^k\,\mu(dx), \qquad k\in\mathbb{N}.$$
- The matrix $Y_N' = Y_N'(\Theta) := \sqrt{B'}(I+P)\sqrt{B'}$ has the same eigenvalues as Y_N. We then use the decomposition $Y_N' = B' + \sqrt{B'}P\sqrt{B'}$. Because the rank of the positive semi-definite matrix $\sqrt{B'}P\sqrt{B'}$ is at most ℓ, we can deduce $\lambda_{i+\ell}(Y_N) = \lambda_{i+\ell}(Y_N') \to \beta$ a.s. for each $i\in\mathbb{N}$.
- The almost sure limit of ν_{Y_N} is the multiplicative monotone convolution $\nu_{I+P} \circlearrowright \mu$, where
$$\nu_{I+P} = \frac{1}{\ell}\sum_{i=1}^{\ell}\delta_{1+\theta_i}.$$
- The function ψ_μ has an analytic continuation to $\mathbb{C}\setminus[1/\beta,+\infty)$, still denoted as ψ_μ, such that $\psi_\mu(\overline{z}) = \overline{\psi_\mu(z)}$. We can check that $\psi_\mu' > 0$ on $(0,1/\beta)$ and $\eta_\mu' = \psi_\mu'/(1+\psi_\mu)^2 > 0$ on $(0,1/\beta)$. Therefore, $\delta := \eta_\mu(1/\beta - 0)$ exists in $(0,+\infty]$. Note that $\eta_\mu(0) = 0$.
- The relation $\psi_{\nu_{I+P}\circlearrowright\mu} = \psi_{\nu_{I+P}}(\eta_\mu(z))$ reads
$$\psi_{\nu_{I+P}\circlearrowright\mu}(z) = \frac{1}{\ell}\sum_{i=1}^{\ell}\frac{(1+\theta_i)\eta_\mu(z)}{1-(1+\theta_i)\eta_\mu(z)}.$$
From this relation, if the equation $\eta_\mu(y) = 1/(1+\theta_i)$ has a solution $y = y_i\in(0,1/\beta)$ then $\psi_{\nu_{I+P}\circlearrowright\mu}$ has a pole at y_i and so the measure $\nu_{I+P}\circlearrowright\mu$ has an atom at $1/y_i$.
- Using the matrix Y_N' above, we can deduce the monotonicity of the map $\theta_i \mapsto \lambda_i(Y_N'(\Theta))$ and the estimate $|\lambda_i(Y_N'(\Theta)) - \lambda_i(Y_N'(\Theta^{(n)}))| \le \beta\max_{i\in[\ell]}|\theta_i - \theta_i^{(n)}|$. $\square$

Remark 8.12. We can also consider the more general case when B is Hermitian. For simplicity, assume that μ_B converges to μ supported on a compact interval $[\alpha,\beta]$ with $\alpha < 0 < \beta$. In this case the above method to estimate the number of outliers does not work because $Y_N' = \sqrt{B'}(I+P)\sqrt{B'}$ cannot be defined. However, we can still prove that if $\eta_\mu(y) = 1/(1+\theta_i)$ has a solution $y = y_i \in (0,1/\beta)$ then for any $\varepsilon > 0$ there exists $N_0\in\mathbb{N}$ such that $\min\{|\lambda_j(Y_N) - 1/y_i| : j\in[N]\} < \varepsilon$ a.s. for all $N \ge N_0$. This is because as soon as the measure $\nu_{I+P}\circlearrowright\mu$ has

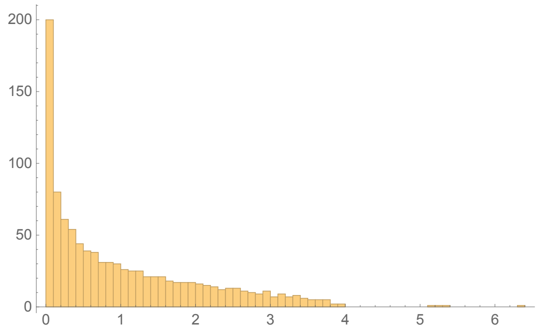

FIGURE 5. A histogram for the eigenvalues of $(I+P)^{\frac{1}{2}}G^*G(I+P)^{\frac{1}{2}}$, where G is the normalized GUE of size 1000 and $P = \mathrm{diag}(4,3,3,3,1,0,0,\dots,0)$ and the bin size is selected to be $1/10$. There are three eigenvalues near the theoretical limit $16/3 = 5.333...$ and one eigenvalue near 6.25.

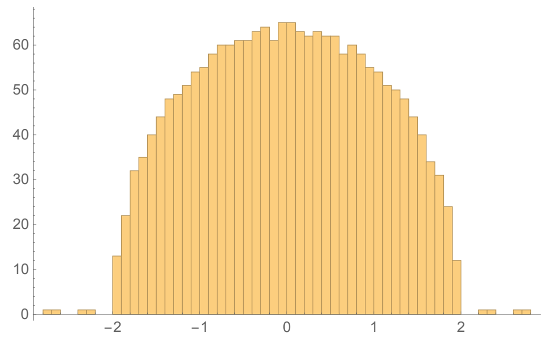

FIGURE 6. A histogram for the eigenvalues of $(I+P)^{\frac{1}{2}}G(I+P)^{\frac{1}{2}}$, where G is the normalized GUE of size 2000 and $P = \mathrm{diag}(5,5,3,3,1/2,0,0,\dots,0)$ and the bin size is selected to be $1/10$. This simulation suggests that the number of eigenvalues in $(2+\varepsilon,+\infty)$ in the large N limit is exactly the number of θ_i's larger than one.

an atom at $1/y_i$ there must exists at least one eigenvalue of Y_N close to $1/y_i$. Note that η_μ still has an analytic continuation to $(0,1/\beta)$ but it might not be increasing on $(0,1/\beta)$ anymore.

Example 8.13. Suppose that μ is the Marchenko–Pastur distribution

$$\mu(dx) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\chi_{(0,4)}(x)\,dx.$$

The Cauchy transform of μ is known in Example 4.48 as $G_\mu(z) = (z-\sqrt{z^2-4z})/(2z)$, which implies

$$F_\mu(z) = \frac{z+\sqrt{z^2-4z}}{2} \qquad \text{and} \qquad \eta_\mu(z) = \frac{1-\sqrt{1-4z}}{2}.$$

We see that $\eta_\mu(1/4-0) = 1/2$ and hence for each $\theta > 1$ the solution to $\eta_\mu(y) = 1/(1+\theta)$ is given by $y = \theta/(\theta+1)^2 < 1/4$. Therefore, for each $i \in \mathbb{N}$,

$$\lim_{N\to\infty} \lambda_i((I+P)^{\frac{1}{2}}UBU^*(I+P)^{\frac{1}{2}}) = \begin{cases} \theta_i + \dfrac{1}{\theta_i} + 2, & \theta_i > 1, \\ 4, & 0 < \theta_i \le 1. \end{cases}$$

See Figure 5 for a simulation where UBU^* is selected to be a Wishart matrix.

Example 8.14. Suppose that μ is the standard semicircle distribution. The η-transform is given by

$$\eta_\mu(z) = \frac{1-\sqrt{1-4z^2}}{2}.$$

We see that $\eta_\mu(1/2-0) = 1/2$ and hence for each $\theta > 1$ the solution to $\eta_\mu(y) = 1/(1+\theta)$ is given by $y = \sqrt{\theta}/(\theta+1) < 1/2$. If $\theta_i > 1$, from Remark 8.12, we know that there is an eigenvalue of $Y_N := (I+P)^{\frac{1}{2}}UBU^*(I+P)^{\frac{1}{2}}$ converging to $\sqrt{\theta_i} + 1/\sqrt{\theta_i}$. See Figure 6 for a simulation where UBU^* is selected to be the normalized GUE.

8.4. **Comb product of graphs.** As another application of monotone independence, we consider a certain graph product. A **directed graph** is a pair $G = (V,E)$, where V is a set and $E \subseteq V \times V$. For simplicity, let us assume that V is a finite set. Each element $v \in V$ is called a **vertex** and $(u,v) \in E$ is called an edge (from u to v). In particular, an edge $(v,v) \in E$ is called a **loop**. Let $\ell^2(V)$ be the finite-dimensional Hilbert space of the

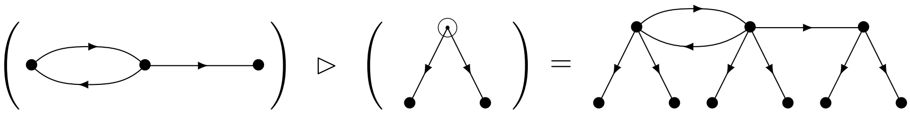

FIGURE 7. Comb product. The root of the second graph is denoted as $\odot$.

functions $f\colon V \to \mathbb{C}$ equipped with inner product $\langle f, g\rangle := \sum_{v\in V} \overline{f(v)}g(v)$. The **adjacency matrix** of G is a linear operator on $\ell^2(V)$ defined by

$$(A_G f)(v) := \sum_{u\in V, (u,v)\in E} f(u).$$

If we assume $V = \{1, 2, ..., N\}$ and identify f with the row vector $(f(1), f(2), ..., f(N))$, then A_G can be identified with the matrix whose (i, j) entry is 1 if $(i, j) \in E$ and zero otherwise, acting on the row vector f from the right.† The adjacency matrix contains all information about G.

Let o be a vertex of G and we set the vector state $\varphi_o\colon \mathbb{B}(\ell^2(V)) \to \mathbb{C}$ by $\varphi_o(a) := \langle\delta_o, a\delta_o\rangle$, where δ_o is the function defined by

$$\delta_o(v) = \begin{cases} 1, & v = o, \\ 0, & v \neq o. \end{cases}$$

Then the nth moment $\varphi(A_G^n)$ is exactly the number of directed paths of length n on G started from o and terminated at o.

Let $G_i = (V_i, E_i)$ be two finite directed graphs. We specify a vertex $o_2 \in V_2$ for G_2, called the **root**. The **comb product** (also called a rooted product) of G_1 and G_2 is a directed graph G with vertex set $V := V_1 \times V_2$ and edge set $E \subseteq V \times V$ defined as follows: for $(u_1, u_2), (v_1, v_2) \in V$,

$$((u_1, u_2), (v_1, v_2)) \in E \iff \begin{cases} (u_1, v_1) \in E_1 \text{ and } u_2 = v_2 = o_2, \text{ or} \\ u_1 = v_1 \text{ and } (u_2, v_2) \in E_2. \end{cases}$$

We write $G = G_1 \rhd G_2$. The comb product is the graph made by gluing copies of G_2 to each vertex of G_1 at o_2, see Figure 7.

Proposition 8.15. *Under the natural identification $\ell^2(V_1 \times V_2) = \ell^2(V_1) \otimes \ell^2(V_2)$, we have*

$$A_{G_1\rhd G_2} = A_{G_1} \otimes P_2 + I_1 \otimes A_{G_2},$$

where I_1 is the identity operator on $\ell^2(V_1)$ and P_2 is the orthogonal projection onto $\mathbb{C}\delta_{o_2}$.

Proof. It suffices to check the formula on simple tensors, i.e., functions of separated variables $f(v_1, v_2) = g(v_1)h(v_2)$:

$$\begin{aligned}
(A_{G_1\rhd G_2} f)(v_1, v_2) &= \sum_{\substack{(u_1,u_2)\in V_1\times V_2 \\ ((u_1,u_2),(v_1,v_2))\in E}} g(u_1)h(u_2) \\
&= \sum_{\substack{(u_1,u_2)\in V_1\times V_2 \\ (u_1,v_1)\in E_1, u_2=v_2=o_2}} g(u_1)h(u_2) + \sum_{\substack{(u_1,u_2)\in V_1\times V_2 \\ u_1=v_1, (u_2,v_2)\in E_2}} g(u_1)h(u_2) \\
&= \left(\sum_{\substack{u_1\in V_1 \\ (u_1,v_1)\in E_1}} g(u_1)\right) h(o_2)\delta_{o_2}(v_2) + g(v_1)\left(\sum_{\substack{u_2\in V_2 \\ (u_2,v_2)\in E_2}} h(u_2)\right) \\
&= (A_{G_1}g)(v_1)(P_2h)(v_2) + g(v_1)(A_{G_2}h)(v_2) \\
&= [(A_{G_1} \otimes P_2 + I_1 \otimes A_{G_2})f](v_1, v_2).
\end{aligned}$$

□

Observe that from Example 1.17 that $A_{G_1\rhd G_2}$ is the sum of monotonically independent random variables with respect to the state $\varphi_{(o_1,o_2)} = \langle\delta_{(o_1,o_2)}, \cdot\delta_{(o_1,o_2)}\rangle$.

We can consider the comb product of more than two graphs. In fact, the comb product satisfies the following associativity.

Proposition 8.16. *Let G_1, G_2, G_3 be finite directed graphs, and $o_2 \in V_2$ and $o_3 \in V_3$ be roots. We select (o_2, o_3) as the root of $G_2 \rhd G_3$. Then the bijection $V_1 \times (V_2 \times V_3) \simeq (V_1 \times V_2) \times V_3$ induces the isomorphism*

$$G_1 \rhd (G_2 \rhd G_3) \simeq (G_1 \rhd G_2) \rhd G_3.$$

†Multiplying a row vector by a matrix from the right is a common convention in some fields, e.g. in the theory of Markov chains.

Proof. It suffices to show that the adjacency matrices coincide on $\ell^2(V_1)\otimes\ell^2(V_2)\otimes\ell^2(V_3)$. On one hand we have

$$\begin{aligned} A_{G_1\rhd(G_2\rhd G_3)} &= A_{G_1}\otimes P_{2,3} + I_1\otimes A_{G_2\rhd G_3} \\ &= A_{G_1}\otimes (P_2\otimes P_3) + I_1\otimes (A_{G_2}\otimes P_3 + I_2\otimes A_{G_3}) \\ &\simeq A_{G_1}\otimes P_2\otimes P_3 + I_1\otimes A_{G_2}\otimes P_3 + I_1\otimes I_2\otimes A_{G_3}, \end{aligned} \tag{8.36}$$

where $P_{2,3}$ is the orthogonal projection from $\ell^2(V_2\times V_3)$ onto $\mathbb{C}\delta_{(o_2,o_3)}$, which is $P_2\otimes P_3$ under the identification $\ell^2(V_2\times V_3)=\ell^2(V_2)\otimes\ell^2(V_3)$. A similar calculation shows that $A_{(G_1\rhd G_2)\rhd G_3}$ has the same expression (8.36). □

If both G_1, G_2 have roots o_1, o_2 respectively, then the comb product $G_1\rhd G_2$ has the natural root (o_1,o_2), and thus the comb product provides an associative binary operation for rooted finite directed graphs $(G_1,o_1)\rhd(G_2,o_2):=(G_1\rhd G_2,(o_1,o_2))$.

Let (G,o) be a finite rooted undirected graph, where "undirected" means that $(u,v)\in E$ implies $(v,u)\in E$, or equivalently, A_G is a self-adjoint operator. Assume further that $(o,o)\notin E$ and

$$\deg(o) := \#\{v\in V : (v,o)\in E\}\geq 1.$$

Let $(G_N,o_N):=(G,o)^{\rhd N}$ be the N-fold comb product of (G,o). Having no loop at o implies that A_G has mean 0 with respect to φ_o. Since the adjacency matrix of G_N is the sum of monotonically independent, identically distributed random variables with mean 0, the monotone CLT (Theorem 3.15) implies

$$\lim_{N\to\infty}\varphi_{o_N}\left[\left(\frac{A_N}{\sqrt{\deg(o)N}}\right)^k\right] = \int_{-\sqrt{2}}^{\sqrt{2}} x^k \frac{dx}{\pi\sqrt{2-x^2}}. \tag{8.37}$$

Note that $\deg(o)$ is exactly the second moment of A_G with respect to φ_o. The convergence (8.37) means that the number of paths of length k on G_N started from o_N and terminated at o_N is

$$\deg(o)^{\frac{k}{2}}\left[\int_{-\sqrt{2}}^{\sqrt{2}} x^k\frac{dx}{\pi\sqrt{2-x^2}} + o(1)\right] N^{\frac{k}{2}}$$

as $N\to\infty$.

8.5. **Notes.** The change of the location of the largest eigenvalue of random matrices depending on perturbation is often called the BBP phase transition, named after the work of Baik, Ben Arous and Péché [16]. Forrester gives an excellent survey on rank one perturbations including earlier works [62].

The application of monotone independence to outliers is due to Cébron, Dahlqvist and Gabriel [40]. Our Corollary 8.4 is pointed out in [40] and Theorem 8.8 is an extension of a result stated in [40, Section 1.4]. Theorems 8.8 and 8.11 are close to the results of Benaych-Georges and Nadakuditi [29, Theorem 2.1, Theorem 2.6]. The difference is that our result for additive perturbation model allows the limit distribution to have unbounded support, while if the matrices $B(j,N)$ are random then we need a stronger assumption than [29] even if the limit distribution is compactly supported; see Remark 8.6. Note that we have not included (a fixed number of) negative diagonal entries in P but such an extension is also possible for the additive perturbation and the proof requires only small modifications. For the multiplicative perturbation, however, in case P contains negative eigenvalues, it is not clear how many negative eigenvalues the matrix $\sqrt{B'}P\sqrt{B'}$ would have. The reader is referred to e.g. [125, 21] for further results on outliers.

Formula (8.6) is a slight extension of a result of Shlyakhtenko [135]; note that a stronger statement is given in [45, Theorem 4.1] but it contains an error: the result would be correct if the matrices $A_j(n), B_j(n)$ are deterministic just as in our result (8.6). The almost sure version (Theorem 8.2) is proved by Collins, Hasebe and Sakuma [45]. The factorization formula (8.6) or Theorem 8.2 is abstracted to the notion of "cyclic monotone independence" in [45] and is further developed e.g. in [7, 11, 40, 46, 68]. Another random matrix model that satisfies asymptotic monotone independence is constructed and studied by Lenczewski [100, 101] and by Banna, Mingo and Tseng [17, 109].

The relation between the comb product of graphs and monotone independence appeared in Accardi, Ben Ghorbal and Obata's work [1]. Arizmendi, Hasebe and Lehner studied the empirical eigenvalue distribution of the adjacency matrix of $(G,o)^{\rhd N}$ in the large N limit [11]. The limit distribution is not universal and depends heavily on the original graph (G,o). Schleißinger studied comb products of non-identical rooted graphs and obtained an approximation of (continuous-time) monotone convolution hemigroups by discrete-time ones [131].

There are other graph products that correspond to other independences in noncommutative probability. The adjacency matrix of the free product of graphs is the sum of free independent operators. Accardi, Lenczewski and Sałapata decomposed the free product graph into the comb product of two subgraphs related to subordination of free convolution [2]. Lenczewski constructed a graph product that corresponds to conditionally monotone independence [102]. Garza-Vargas and Kulkarni constructed an amalgamated free product of graphs and applied it to the spectral analysis of Jacobi matrices of graphs [69]. Obata's monograph [122] contains other notable examples of graph products and references to earlier works.

Appendix A. Moments and weak convergence

In noncommutative probability, moments of random variables are fundamental concepts. We collect here supplementary results on moments.

Definition A.1. Let μ be a probability measure on $\mathbb{R}$ having finite moments of all orders. We say that μ has a **determinate moment sequence** if no other probability measures have the same moment sequence.

Example A.2. Let $-1 \le \varepsilon \le 1$. The probability measure

$$\mu_\varepsilon(dx) = \frac{1}{\sqrt{\pi}} x^{-1} e^{-(\log x)^2} [1 - \varepsilon \sin(2\pi \log x)] \chi_{(0,\infty)}(x)\, dx$$

has an indeterminate moment sequence because the moments are independent of ε. The measure μ_0 is the distribution of the random variable e^X where X has the distribution $\mathrm{N}(0, 1/2)$ and is called a lognormal distribution. See [3, p. 88] and [134, pp. 88–90] for more examples of indeterminate moment sequences.

We show a simple criterion for the determinacy of the moment problem.

Proposition A.3. *Let μ be a probability measure on $\mathbb{R}$ with compact support. Then the moment sequence of μ is determinate.*

Proof. Suppose that μ is supported on the compact interval $[-R, R]$. Suppose that ν is a probability measure on $\mathbb{R}$ having the same moment sequence.

Step 1: ν is also supported on $[-R, R]$. For this, let us observe the obvious bound

$$m_{2k}(\mu) = \int_{[-R,R]} x^{2k}\, \mu(dx) \le R^{2k}.$$

Suppose to the contrary that ν is not supported on $[-R, R]$. Then there would exist $R_2 > R$ such that $\nu(|x| > R_2) > 0$. Then

$$R^{2k} \ge m_{2k}(\nu) \ge \int_{|x|>R_2} x^{2k} \nu(dx) \ge R_2^{2k} \nu(|x| > R_2),$$

which would yield the contradiction

$$0 < \nu(|x| > R_2) \le \left(\frac{R}{R_2}\right)^{2k} \to 0, \qquad k \to \infty.$$

Step 2: $\mu = \nu$. We have now

$$\int_{[-R,R]} f(x)\, \mu(dx) = \int_{[-R,R]} f(x)\, \nu(dx) \tag{A.1}$$

for any polynomial f. By Weierstrass's theorem, the above holds for any continuous function f on $[-R, R]$. Then some standard technique in probability theory shows $\mu = \nu$; for example, we first approximate every indicator function χ_I over a closed interval $I \subseteq [-R, R]$ by an increasing sequence of nonnegative continuous functions, and then by the monotone convergence theorem, (A.1) imply $\mu(I) = \nu(I)$. By Proposition 4.11, we conclude $\mu = \nu$. □

A more general criterion is **Carleman's condition**

$$\sum_{n \ge 1} m_{2n}(\mu)^{-\frac{1}{2n}} = +\infty. \tag{A.2}$$

If a probability measure μ satisfies Carleman's condition then the moment sequence of μ is determinate; see [3, p. 85] or [134, Theorem 4.3]. It is easy to see that if μ has compact support then μ satisfies Carleman's condition. The normal distribution $\mathrm{N}(m, \sigma^2)$ and the exponential distribution $\lambda e^{-x/\lambda}\, dx, x > 0$ satisfy Carleman's condition.

Proposition A.4. *Let μ, ν be probability measures on $\mathbb{R}$. If μ has compact support and ν satisfies Carleman's condition, then $\mu \rhd \nu$ also satisfies Carleman's condition.*

Proof. From the moment formula (5.8) for monotone convolution and inequality (4.19), we have

$$\begin{aligned} m_{2n}(\mu \rhd \nu) &\leq \sum_{\ell=0}^{2n} \sum_{\substack{k_0,k_1,\dots,k_\ell \geq 0, \\ k_0+k_1+\cdots+k_\ell = 2n-\ell}} |m_\ell(\mu)||m_{k_0}(\nu)||m_{k_1}(\nu)|\cdots|m_{k_\ell}(\nu)| \\ &\leq \sum_{\ell=0}^{2n} \sum_{\substack{k_0,k_1,\dots,k_\ell \geq 0, \\ k_0+k_1+\cdots+k_\ell = 2n-\ell}} m_{2n}(\mu)^{\frac{\ell}{2n}} m_{2n}(\nu)^{\frac{k_0+k_1+\cdots+k_\ell}{2n}} \\ &= \sum_{\ell=0}^{2n} \binom{2n}{\ell} m_{2n}(\mu)^{\frac{\ell}{2n}} m_{2n}(\nu)^{\frac{2n-\ell}{2n}} \\ &= \left[m_{2n}(\mu)^{\frac{1}{2n}} + m_{2n}(\nu)^{\frac{1}{2n}}\right]^{2n}. \end{aligned}$$

Let $a_n := m_{2n}(\mu)^{\frac{1}{2n}}$ and $b_n := m_{2n}(\nu)^{\frac{1}{2n}}$. Suppose that μ is supported on $[-R,R]$. Then $a_n \leq R$. Note that due to (4.19) the sequence $(b_n)_{n\geq 1}$ is nondecreasing. If $b_1 = 0$ then $\nu = \delta_0$ and the statement is obvious. If $b_1 > 0$ is then we can find $c > 0$ such that

$$m_{2n}(\mu \rhd \nu)^{-\frac{1}{2n}} \geq \frac{1}{a_n + b_n} \geq \frac{1}{R + b_n} \geq \frac{c}{b_n}, \qquad n \in \mathbb{N}.$$

Since $\sum_{n\geq 1} 1/b_n = +\infty$ we have $\sum_{n\geq 1} m_{2n}(\mu \rhd \nu)^{-\frac{1}{2n}} = +\infty$. ☐

Next we discuss the relation between convergence of moments and weak convergence.

Lemma A.5. *Let $\mathcal{P}$ be a family of probability measures on $\mathbb{R}$. Let $f\colon \mathbb{R} \to [0,+\infty)$ be a measurable function such that $\lim_{|x|\to\infty} f(x) = +\infty$ and*

$$\sup_{\mu\in\mathcal{P}} \int_{\mathbb{R}} f(x)\,\mu(dx) < \infty. \tag{A.3}$$

Then $\mathcal{P}$ is tight.

Proof. Let $C \geq 0$ be the finite value (A.3). For any $\varepsilon > 0$ there exists $R > 0$ such that $\inf_{|x|>R} f(x) \geq \frac{C+1}{\varepsilon}$. For all $\mu \in \mathcal{P}$ we have

$$\begin{aligned} \mu(\mathbb{R} \setminus [-R,R]) &= \int_{|x|>R} \mu(dx) \leq \int_{|x|>R} \frac{\varepsilon}{C+1} f(x)\,\mu(dx) \\ &\leq \frac{\varepsilon}{C+1} \int_{\mathbb{R}} f(x)\,\mu(dx) \leq \frac{\varepsilon C}{C+1} \leq \varepsilon. \end{aligned}$$ ☐

Proposition A.6. *Let μ, μ_n $(n \in \mathbb{N})$ be probability measures on $\mathbb{R}$. Suppose that μ, μ_n all have finite moments of all orders, and*

$$\lim_{n\to\infty} m_k(\mu_n) = m_k(\mu), \qquad k \in \mathbb{N}.$$

If the moment sequence of μ is determinate, then μ_n converges weakly to μ.

Proof. Since $m_2(\mu_n)$ converges to $m_2(\mu)$, it is a bounded sequence and hence the assumption of Lemma A.5 is satisfied for $f(x) = x^2$, and so the sequence $(\mu_n)_{n\geq 1}$ is tight. Let $(\mu_{n(j)})_{j\geq 1}$ be a subsequence that weakly converges to a probability measure μ'.

Step 1: μ' has finite moments of all orders. For this we take a sequence of continuous functions $f_N\colon \mathbb{R} \to [0,1]$, $N = 1,2,3,\ldots$ such that

- f_N is supported on $[-2N, 2N]$,
- $f_N = 1$ on $[-N, N]$,
- $f_N(x) \uparrow 1$ as $N \to \infty$ at every $x \in \mathbb{R}$.

In the obvious inequality

$$\int_{\mathbb{R}} x^{2k} f_N(x) \mu_{n(j)}(dx) \leq \int_{\mathbb{R}} x^{2k} \mu_{n(j)}(dx) = m_{2k}(\mu_{n(j)})$$

passing to the limit $j \to \infty$ yields

$$\int_{\mathbb{R}} x^{2k} f_N(x) \mu'(dx) \leq m_{2k}(\mu).$$

Further passing to the limit $N \to \infty$, together with the monotone convergence theorem, shows

$$\int_{\mathbb{R}} x^{2k} \mu'(dx) \leq m_{2k}(\mu) < +\infty.$$

Thus μ' has finite moments of all orders.

Step 2: $m_k(\mu') = m_k(\mu)$ for all $k \in \mathbb{N}$. To show this we first observe that

$$\int_{|x|\geq N} |x|^k \mu_n(dx) \leq \int_{|x|\geq N} \left(\frac{|x|}{N}\right)^k |x|^k \mu_n(dx) \leq \frac{1}{N^k} m_{2k}(\mu_n) \leq C_k N^{-k},$$

where $C_k := \sup_{n\in\mathbb{N}} m_{2k}(\mu_n) < +\infty$. With this inequality we obtain

$$\begin{aligned}
|m_k(\mu') - m_k(\mu_{n(j)})| &\leq \underbrace{\left|\int_{\mathbb{R}} x^k f_N(x)\mu'(dx) - \int_{\mathbb{R}} x^k f_N(x)\mu_{n(j)}(dx)\right|}_{=:\varepsilon(N,j)} \\
&\quad + \left|\int_{\mathbb{R}} x^k(1 - f_N(x))\mu'(dx)\right| + \left|\int_{\mathbb{R}} x^k(1 - f_N(x))\mu_{n(j)}(dx)\right| \\
&\leq \varepsilon(N,j) + \int_{|x|\geq N} |x|^k \mu'(dx) + C_k N^{-k}.
\end{aligned}$$

As $\lim_{j\to\infty} \varepsilon(N,j) = 0$, taking N large enough and then j large enough, we obtain

$$m_k(\mu) = \lim_{j\to\infty} m_k(\mu_{n(j)}) = m_k(\mu').$$

Step 3. Since the moment sequence of μ is determinate, we have $\mu = \mu'$. The whole above argument shows that any subsequence of $(\mu_n)_{n\geq 1}$ has a further subsequence that weakly converges to μ. By Lemma 4.2, we conclude that the original sequence $(\mu_n)_{n\geq 1}$ converges weakly to μ. □

Appendix B. Inverse mapping of Cauchy transform

We study the inverse mapping of (reciprocal) Cauchy transform to complete the proof of Proposition 5.14.

Lemma B.1. *Let $E \subset \mathbb{C}$ be a convex subset and $f\colon E \to \mathbb{C}$ be the restriction of a holomorphic function defined on an open set containing E such that $\Im[e^{i\alpha} f'(z)] > 0$ holds on E for some constant $\alpha \in [0, 2\pi)$. Then f is injective on E.*

Proof. By considering $g(z) := e^{i\alpha} f(z)$, we may assume from the beginning that $\alpha = 0$. For any $z_0, z_1 \in D$, the following identity holds:

$$\begin{aligned}
f(z_1) - f(z_0) &= \int_0^1 \frac{d}{dt} f((1-t)z_0 + tz_1)\, dt \\
&= (z_1 - z_0) \int_0^1 f'((1-t)z_0 + tz_1)\, dt.
\end{aligned}$$

Since the number $C(z_0, z_1) = \inf\{\Im[f'((1-t)z_0 + tz_1)] : t \in [0,1]\}$ is positive, the inequality

$$|f(z_1) - f(z_0)| \geq C(z_0, z_1)|z_1 - z_0|$$

implies the injectivity of f. □

For a concise statement, we consider the domain

$$\nabla_{\gamma,\delta} := \nabla_\gamma \cap \{z : \Im(z) > \delta\} = \{z \in \mathbb{C}^+ : \Im(z) > \max\{\gamma|\Re(z)|, \delta\}\}, \qquad \gamma, \delta > 0.$$

Proposition B.2. *Let μ be a probability measures on $\mathbb{R}$. For every $0 < \varepsilon < \gamma < 1$ there exists $\delta_0 = \delta_0(\gamma, \varepsilon) > 0$ such that, for all $\delta \geq \delta_0$, the function F_μ is injective in $\nabla_{\gamma,\delta}$ and $\nabla_{\gamma+\varepsilon,(1+\varepsilon)\delta} \subseteq F_\mu(\nabla_{\gamma,\delta}) \subseteq \nabla_{\gamma-\varepsilon,(1-\varepsilon)\delta}$.*

Proof. Injectivity. We first establish

$$|F'_\mu(z) - 1| = o(1), \qquad z \to \infty,\ z \in \nabla_\gamma. \tag{B.1}$$

Using Proposition 4.34(3) we get

$$F'_\mu(z) = -F_\mu(z)^2 G'_\mu(z) = (1 + o(1)) \int_{\mathbb{R}} \frac{z^2}{(z-x)^2}\, \mu(dx)$$

as $z \to \infty, z \in \nabla_\gamma$. Therefore, it suffices to show that

$$\limsup_{\substack{z\to\infty \\ z\in\nabla_\gamma}} \left|\int_{\mathbb{R}} \frac{z^2}{(z-x)^2}\, \mu(dx) - 1\right| = 0. \tag{B.2}$$

To see this, we use the inequalities $|z^2/(z-x)^2| \le 1+\gamma^{-2}$ and $|z| \le \sqrt{1+\gamma^{-2}}\Im z$ for $z \in \nabla_\gamma$ to proceed as, for each $R>0$,

$$\begin{aligned}\int_{\mathbb{R}} \left|\frac{z^2}{(z-x)^2}-1\right| \mu(dx) &\le \int_{[-R,R]} \left|\frac{z^2}{(z-x)^2}-1\right| \mu(dx) + (2+\gamma^{-2})\mu(\mathbb{R}\setminus[-R,R]) \\ &\le \int_{[-R,R]} \frac{x^2+2|zx|}{(\Im z)^2}\mu(dx) + (2+\gamma^{-2})\mu(\mathbb{R}\setminus[-R,R]) \\ &\le \frac{R^2+2R\sqrt{1+\gamma^{-2}}\Im z}{(\Im z)^2} + (2+\gamma^{-2})\mu(\mathbb{R}\setminus[-R,R]),\end{aligned}$$

and then

$$\limsup_{\substack{z\to\infty \\ z\in\nabla_\gamma}} \int_{\mathbb{R}} \left|\frac{z^2}{(z-x)^2}-1\right| \mu(dx) \le (2+\gamma^{-2})\mu(\mathbb{R}\setminus[-R,R]).$$

Letting R tend to infinity here yields (B.2), and hence (B.1).

By (B.1) we can find $\delta>0$ so large that

$$\Re[F_\mu'(z)] \ge \frac{1}{2}, \qquad z \in \overline{\nabla_{\gamma,\delta}},$$

and by Lemma B.1, F_μ is injective on $\overline{\nabla_{\gamma,\delta}}$.

The inclusion $\nabla_{\gamma+\varepsilon,(1+\varepsilon)\delta} \subseteq F_\mu(\nabla_{\gamma,\delta})$. For $z \in \partial\nabla_{\gamma,\delta}$ with $\gamma|\Re z| = \Im z$, one can see that

$$d(z,\nabla_{\gamma+\varepsilon,(1+\varepsilon)\delta}) \ge \frac{\varepsilon|z|}{\sqrt{(1+\gamma^2)(1+(\gamma+\varepsilon)^2)}} =: c_1|z|.$$

Here d denotes the Euclidean distance. For $z\in\partial\nabla_{\gamma,\delta}$ with $\Im z=\delta$, we have

$$d(z,\nabla_{\gamma+\varepsilon,(1+\varepsilon)\delta}) \ge \varepsilon\delta \ge \frac{\varepsilon\gamma|z|}{\sqrt{1+\gamma^2}} =: c_2|z|.$$

Therefore, if one takes $\delta_0>0$ so that $\sup_{z\in\nabla_{\gamma,\delta_0}}|F_\mu(z)-z|/|z| < \min\{c_1,c_2\}$, then the simple closed curve $\{\,F_\mu(z) : z\in\partial\nabla_{\gamma,\delta}\cup\{\infty\}\,\} \subseteq \mathbb{C}\cup\{\infty\}$ surrounds each point of $\nabla_{\gamma+\varepsilon,(1+\varepsilon)\delta}$ exactly once as soon as $\delta\ge\delta_0$, and hence by the argument principle we obtain $F_\mu(\nabla_{\gamma,\delta}) \supseteq \nabla_{\gamma+\varepsilon,(1+\varepsilon)\delta}$ as desired. The other inclusion can be proved analogously. ☐

The previous proposition allows us to define an injective function $F_\mu^{-1}\colon \nabla_{\gamma+\varepsilon,(1+\varepsilon)\delta} \to \mathbb{C}^+$ which satisfies $F_\mu\circ F_\mu^{-1} = \mathrm{id}$ on $\nabla_{\gamma+\varepsilon,(1+\varepsilon)\delta}$. Fixing any $0<\varepsilon<\gamma<1$ and large $\delta>0$, we set $\gamma' := \gamma+\varepsilon$ and $\delta' =: (1+\varepsilon)\delta$ for notational simplicity.

Proposition B.3. *Let μ be a probability measures on $\mathbb{R}$ and $n\in\mathbb{N}$. Let $F_\mu^{-1}\colon \nabla_{\gamma',\delta'} \to \mathbb{C}^+$ be defined as above. Then the following are equivalent.*

(1) $\displaystyle\int_{\mathbb{R}} x^{2n}\,\mu(dx) < +\infty.$

(2) *There exist $c_1,c_2,\dots,c_{2n}\in\mathbb{R}$ such that*

$$F_\mu^{-1}(z) = z + c_1 + \frac{c_2}{z} + \cdots + \frac{c_{2n}}{z^{2n-1}} + o(|z|^{-(2n-1)}), \qquad z\to\infty, z\in\nabla_{\gamma',\delta'}. \tag{B.3}$$

Proof. (1) $\Longrightarrow$ (2). We only consider the case $n=2$, which shall be enough to see how to handle the general n.

Recall that, by definition, $F_\mu^{-1}(\nabla_{\gamma',\delta'})$ is contained in some $\nabla_{\gamma,\delta}$. By Proposition 4.43, for some real constants b_1,b_2,b_3,b_4 we have

$$F_\mu(w) = w - b_1 - \frac{b_2}{w} - \frac{b_3}{w^2} - \frac{b_4}{w^3} + o(|w|^{-3}), \qquad w\to\infty, w\in\nabla_{\gamma,\delta}. \tag{B.4}$$

We set $w := F_\mu^{-1}(z), z\in\nabla_{\gamma',\delta'}$. By the construction of the inverse function and the fact $F_\mu(w) = w+o(w)$ we can see that $w\to\infty$ whenever $z\to\infty$. Putting w into (B.4) we get

$$z = F_\mu^{-1}(z) + O(1),$$

where $O(1)$ is a bounded function. In particular we get

$$w = F_\mu^{-1}(z) = z + o(z). \tag{B.5}$$

Next we substitute (B.5) into (B.4) to obtain

$$z = F_\mu^{-1}(z) - b_1 - \frac{b_2}{z+o(z)} + o(z^{-1}),$$

which amounts to

$$w = F_\mu^{-1}(z) = z + b_1 + \frac{b_2}{z} + o(z^{-1}). \tag{B.6}$$

Finally, we substitute (B.6) into (B.4) to obtain

$$z = F_\mu^{-1}(z) - b_1 - \frac{b_2}{z + b_1 + b_2/z + o(z^{-1})} - \frac{b_3}{(z + b_1 + o(1))^2} - \frac{b_4}{(z + o(z))^3} + o(|z|^3).$$

Using the geometric series expansion $1/(1-\zeta) = 1 + \zeta + \zeta^2 + \cdots$ and recollecting terms, we obtain

$$F_\mu^{-1}(z) = z + b_1 + \frac{b_2}{z} + \frac{b_3 - b_1 b_2}{z^2} + \frac{b_4 - 2b_1 b_3 - b_2^2 + b_1^2 b_2}{z^3} + o(z^3)$$

as desired.

(2) $\implies$ (1) is very similar to the proof of (1) $\implies$ (2). In this case we use the formula $F_\mu^{-1}(F_\mu(w)) = w$ instead, which holds on a subdomain $\nabla_{\tilde{\gamma},\tilde{\delta}} \subseteq \nabla_{\gamma,\delta}$ selected so that $F_\mu(\nabla_{\tilde{\gamma},\tilde{\delta}}) \subseteq \nabla_{\gamma',\delta'}$, e.g. $\tilde{\gamma} = \gamma' + \varepsilon, \tilde{\delta} = \delta'/(1-\varepsilon)$. The remaining argument is exactly the same. $\square$

Completing the proof of Proposition 5.14. It remains to prove $\int_{\mathbb{R}} t^{2n}\,\mu(dt) < +\infty$, assuming that $\int_{\mathbb{R}} t^{2n}\,\lambda(dt) < +\infty$ and $\int_{\mathbb{R}} t^{2n}\,\nu(dt) < +\infty$, where $\lambda := \mu \rhd \nu$. Take a domain $\nabla_{\gamma',\delta'}$ on which F_ν^{-1} can be defined, and set $w = F_\nu^{-1}(iy), y > \delta'$. Then $F_\lambda(w) = F_\mu(F_\nu(w))$ yields $F_\mu(iy) = F_\lambda(F_\nu^{-1}(iy))$. By Propositions 4.43 and B.3, we can expand $F_\lambda(\zeta)$ and $F_\nu^{-1}(iy)$ into truncated Laurent series. Following the lines for computing (5.11), we obtain a truncated Laurent series of $F_\mu(iy) = F_\lambda(F_\nu^{-1}(iy))$, from which we can conclude that $\int_{\mathbb{R}} t^{2n}\,\mu(dt) < +\infty$. $\square$

B.1. **Notes.** Proposition A.6 is well known and can be found in the literature, e.g. in [43, Theorem 4.5.5]. Lemma B.1 was proved by Noshiro and Warschawski [121, 149]. Proposition B.2 was proved by Bercovici and Voiculescu [31]. Proposition B.3 is due to Benaych-Georges [28, Theorem 1.3]. The coefficients $c_1, c_2, c_3, ...$ in Proposition B.3 are called the free cumulants of μ. There are combinatorial formulas relating $(c_n)_{n\ge1}$, the moment sequence $(m_n(\mu))_{n\ge1}$ and the Boolean cumulants $(b_n)_{n\ge1}$ and monotone cumulants $(\kappa_n(\mu))_{n\ge1}$; see [12] and references therein.

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