Free Probability Theory

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What is free probability?

- Free probability theory is probability theory based on free independence on a noncommutative probability space.
- Free probability was proposed in 1980s to attack problems in operator algebras (pure mathematical problems).
- In 1991 a connection between eigenvalues of random matrices and free probability was discovered. Then people began to use free probability in applied fields such as:
  - Wireless communications (Verdú, Tulino, Debbah, Benaych-Georges, etc)
  - Quantum information theory (Collins, Nechita, Fukuda, Belinschi, etc)

Organization of talk

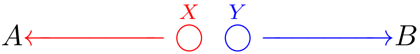
1. Noncommutative probability space
2. Free independence and free probability
3. Results

Measure-Theoretical Probability

- $(\Omega, \mathcal{F}, \mathbb{P})$: Probability space
  Ω: set,
  $\mathcal{F}$: σ-algebra,
  $\mathbb{P}$: probability measure
- $X : \Omega \to \mathbb{C}$: random variable
- $\mathbb{E}[X] = \int_{\mathbb{R}} X(\omega)\mathbb{P}(d\omega)$: Expectation
- Linearity:
  $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$,
  $\mathbb{E}[\alpha X] = \alpha\mathbb{E}[X]$.
- Positivity: $\mathbb{E}[|X|^2] \geq 0$.
- $\mathbb{E}[1] = 1$.

Measure-Theoretic Probability fails: Bell's inequality

A pair of particles X, Y is born and they go to converse directions.



Each particle has a spin. The spin can be measured along the direction with any angle θ and takes values either 1 or -1.
We measure the spin of particle X in θ direction at a point A. We get a value $x_1(\theta)$. Similarly $y_1(\phi)$. Repeat this experiment to get data $x_j(\theta), y_j(\phi)$. We may compute

$$\rho(\theta, \phi) := \frac{1}{N} \sum_{j=1}^{N} x_j(\theta) y_j(\phi) \quad \text{for large } N.$$

It is natural to model the spins as random variables. However,

Experiments show that:

It is impossible to realize a family of random variables $X(\theta), Y(\theta) : \Omega \to \{\pm 1\}$ *such that* $\mathbb{E}[X(\theta)Y(\phi)] = \rho(\theta, \phi)$ $(\forall \theta, \phi)$.

Experiments show that:

It is impossible to realize a family of random variables $X(\theta), Y(\theta) : \Omega \to \{\pm 1\}$ *such that* $\mathbb{E}[X(\theta)Y(\phi)] = \rho(\theta, \phi)$ $(\forall \theta, \phi)$.

Proof (Ref: Wikipedia "Bell's inequality").

Suppose that four random variables $X(\theta), X(\theta'), Y(\phi), Y(\phi') : \Omega \to \{-1, 1\}$ exist in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then

$$\begin{aligned} C &:= |\rho(\theta, \phi) + \rho(\theta', \phi) - \rho(\theta, \phi') + \rho(\theta', \phi')| \\ &= |\mathbb{E}[X(\theta)Y(\phi) + X(\theta')Y(\phi) - X(\theta)Y(\phi') + X(\theta')Y(\phi')]| \\ &= |\mathbb{E}[(X(\theta) + X(\theta'))Y(\phi) + (X(\theta') - X(\theta))Y(\phi')]| \\ &\leq \mathbb{E}[|X(\theta) + X(\theta')| + |X(\theta') - X(\theta)|] \\ &\leq 2. \text{ (Bell's inequality)} \end{aligned}$$

However, Aspect's experiment (1982) shows that for some $\theta, \theta', \phi, \phi'$ we can achieve $C \sim 2.8$. □

Noncommutative probability space

The idea of noncom. prob. sp. appeared in the book of von Neumann “Mathematical Foundations of Quantum Mechanics (1932)”, one year earlier than the Kolmogorov’s book “Foundations of the Theory of Probability (1933)”.

- $\mathcal{A}$: $*$-algebra with an involution $*$ (typically matrix algebra).
- φ: $\mathcal{A} \to \mathbb{C}$, linear, $\varphi(X^*X) \geq 0$, $\varphi(1) = 1$.

$(\mathcal{A}, \varphi)$ is called a noncommutative probability space. φ is called a state or expectation. $X \in \mathcal{A}$ is called a random variable.

- If $X = X^*$ (self-adjoint) then $\exists \mu_X = \mu_{\varphi,X}$: a probability measure on $\mathbb{R}$ (called the law of X) s.t.

$$\underbrace{\varphi(X^n)}_{n\text{th moment}} = \int_{\mathbb{R}} x^n \, d\mu_X(x), \qquad n \geq 0.$$

Examples

1. Matrix algebra.
$$\mathcal{A} = M_n(\mathbb{C}),$$
$\varphi_n(X) = \frac{1}{n}\mathrm{Tr}(X)$. $(\varphi_n(1) = 1, \varphi_n(X^*X) \geq 0)$.

   The law of X: $\mu_X = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j}$ (λ_j: Eigenvalues of X)

2. Random matrices.
$$\mathcal{A} = \{n \times n \text{ matrices with entries random variables}\}$$
$\varphi_n(X) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}(X_{jj})$, $X = (X_{ij})_{1\leq i,j\leq n}$.

   If X is a hermitian random variable then
$$\text{The law of } X : \mu_X = \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j}\right]$$
(Mean eigenvalue distribution')
λ_j: (random) eigenvalues of X

Examples

Measure-Theoretical probability $\subset$ Noncommutative probability

- $(\Omega, \mathcal{F}, \mathbb{P})$: Probability space
- $\mathcal{A} = \{X : \Omega \to \mathbb{C} \mid \mathbb{E}|X|^n < \infty, \forall n \geq 1\}$,
- $\varphi = \mathbb{E}$.

Intermediate Summary

- In quantum physics there is a family of random variables which cannot be modeled in the framework of Kolmogorov's measure-theoretic probability.
- We can solve this problem by introducing the concept "noncommutative probability space"

Independence

Classical independence. Suppose X, Y are bounded random variables.

$$\begin{aligned} X, Y \text{ are indep.} &\Leftrightarrow \mathbb{P}[X \in A, Y \in B] = \mathbb{P}[X \in A]\mathbb{P}[Y \in B] \ (\forall A, B) \\ &\Leftrightarrow \mathbb{E}[X^m Y^n] = \mathbb{E}[X^m]\mathbb{E}[Y^n] \ (\forall m, n). \end{aligned}$$

When $X, Y \in \mathcal{A}$ are noncomm., then

Definition

X, Y are indep. $\Leftrightarrow$
$\varphi(X^{m_1} Y^{n_1} X^{m_2} Y^{m_2} \cdots) = \varphi(X^{m_1+m_2+\cdots})\varphi(Y^{n_1+n_2+\cdots})$,
$\forall m_i, n_i \in \mathbb{N} \cup \{0\}$.

For mathematical interest, let's look at independence from a general point of view.

Definition

Independence of $X, Y \Leftrightarrow$ A universal rule for computing $\varphi(X^{m_1} Y^{n_1} X^{m_2} Y^{m_2} \cdots)$ only by $\varphi(X^m), \varphi(Y^n), m, n \geq 0$.

Free independence

Definition (Voiculescu Lect. Notes. Math. 85)

$X, Y \in \mathcal{A}$ are free independent $\overset{\text{def}}{\Leftrightarrow}$ for any finitely many polynomials $P_i(X), Q_i(Y)$ with $\varphi(P_i(X)) = \varphi(Q_i(Y)) = 0, \forall i$ we have

$$\varphi(\underbrace{\cdots P_1(X)Q_1(Y)P_2(X)Q_2(Y)\cdots}_{\text{finite product}}) = 0.$$

Example

X, Y: free independent.
Consider $P_1(X) = X - \varphi(X)1$, $Q_1(Y) = Y - \varphi(Y)1$, then $\varphi(P_1(X)Q_1(Y)) = 0 \Rightarrow \varphi(XY) = \varphi(X)\varphi(Y)$. Similarly

$$\varphi(XYX) = \varphi(X^2)\varphi(Y),$$
$$\varphi(XYXY) = \varphi(X^2)\varphi(Y)^2 + \varphi(X)^2\varphi(Y^2) - \varphi(X)^2\varphi(Y)^2.$$

Classical CLT

$X_1, X_2, X_3, \cdots$: indep. and identically distributed ($\mathbb{E}[X_i^n]$ does not depend on i), $\varphi(X_i) = 0,\ \varphi(X_i^2) = 1$, then

$$\text{The law of } \frac{X_1 + \cdots + X_N}{\sqrt{N}} \xrightarrow{N\to\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Free CLT

If noncom. selfadjoint random variables $X_1, X_2, X_3, \cdots$ are free indep. and identically distributed ($\varphi(X_i^n)$ does not depend on i) and $\varphi(X_i) = 0,\ \varphi(X_i^2) = 1$, then

$$\text{The law of } \frac{X_1 + \cdots + X_N}{\sqrt{N}} \xrightarrow{N \to \infty} ??$$

Theorem (Voiculescu J. Funct. Anal. 86)

The limit law is given by

$$\frac{1}{2\pi}\sqrt{4 - x^2}\, dx, \quad -2 < x < 2. \quad \textit{(Wigner's semicircle law)}$$

Eigenvalues of random matrices

In 1950s eigenvalues of random matrices were proposed to describe energy levels of nucleons of a nucleus. Wigner proved the following.

$X(n) = (X_{ij}(n))_{1\leq i,j\leq n}$: GUE, that is:

- $X(n)$ is an $n \times n$ hermitian random matrix.
- $(X_{ij}(n))_{i\geq j}$: indep. Gaussians.
- $\mathbb{E}[X_{ij}(n)] = 0, \quad \mathbb{E}[|X_{ij}(n)|^2] = \begin{cases} \frac{1}{2n}, & i \neq j \\ \frac{1}{n}, & i = j. \end{cases}$

The law of X: $\mu_{X(n)} = \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}]$.

Theorem (Wigner, Ann. Math. 55)

$$\mu_{X(n)} \xrightarrow{n\to\infty} \frac{1}{2\pi}\sqrt{4-x^2}dx.$$

Is there a connection between random matrices and free independence?

Random matrices and free independence

Theorem (Voiculescu, Invent. Math. 91)

- *$X_i(n)$: $n \times n$ GUE ($n \geq 1$),*
- *For each n, $X_1(n), X_2(n), X_3(n), \ldots$ are indep. (as vector-valued random variables)*
- *$\varphi_n(A) := \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}[A_{jj}]$.*

Then $X_1(n), X_2(n), X_3(n), \ldots$ are asymptotically free independent as $n \to \infty$.

Corollary

Thm of Voiculescu '91 + free CLT $\Rightarrow$ Wigner's thm '55

Free convolution

$X = X^*$, $Y = Y^*$: free indep.

- μ_{X+Y} is called the free convolution of μ_X, μ_Y and is denoted by $\mu_X \boxplus \mu_Y$.
- If $X \geq 0$ then $\mu_{\sqrt{X}Y\sqrt{X}}$ is called the free multiplicative convolution and is denoted by $\mu_X \boxtimes \mu_Y$.

Computing free convolutions

- Cauchy transform:

$$G_X(z) = G_{\mu_X}(z) = \int_{\mathbb{R}} \frac{1}{z-x}\mu_X(dx) = \varphi((z-X)^{-1}), \qquad z \in \mathbb{C}^+$$

- Free cumulant transform

$$\mathcal{C}_X^{\boxplus}(z) := G_X^{-1}(z) - \frac{1}{z}.$$

Theorem (Bercovici-Voiculescu, Indiana Univ Math J 93)

If X, Y are free indep. then $\mathcal{C}_{X+Y}^{\boxplus}(z) = \mathcal{C}_X^{\boxplus}(z) + \mathcal{C}_Y^{\boxplus}(z)$ in their common domain.

$\mathcal{C}_X^{\boxplus}$ corresponds to $\mathcal{C}_X^{*}(z) = \log \mathbb{E}[e^{zX}]$: if X, Y are indep. then $\mathcal{C}_{X+Y}^{*} = \mathcal{C}_X^{*} + \mathcal{C}_Y^{*}$.

Examples

$$\mathcal{C}_X^{\boxplus}(z) := G_X^{-1}(z) - \frac{1}{z}.$$

- Semicircle law $\mathbf{w}$:

$$\mathbf{w}(dx) = \frac{\sqrt{4-x^2}}{2\pi}dx, \quad |x| < 2.$$

  $\mathcal{C}_{\mathbf{w}}^{\boxplus}(z) = z^2$
  $\mathbf{w}$ appears as the eigenvalue distribution of large GUE.

- Free Poisson law $\boldsymbol{\pi}$:

$$\boldsymbol{\pi}(dx) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}dx, \quad 0 < x < 4.$$

  $\mathcal{C}_{\boldsymbol{\pi}}^{\boxplus}(z) = \frac{z}{1-z}$
  $\boldsymbol{\pi}$ appears as the eigenvalue distribution of large Wishart matrices (=square of GUE).

Generalization of CLT: stable laws

If $X_1, X_2, X_3, \ldots$: i.i.d. random variables and if

$$\text{Law of } \frac{X_1 + \cdots + X_N}{N^{1/\alpha}} \xrightarrow{N\to\infty} \mu \neq \text{Dirac delta}$$

then, μ is called a stable law. Always $\alpha \in (0, 2]$.

Theorem (See e.g. Zolotarev's book '86)

Let μ be a stable law. Then $\mathcal{C}_\mu^*(z) = -re^{i\alpha\rho\pi} z^\alpha, z \in i\mathbb{R}_+, r > 0$ ($\Leftrightarrow \int_{\mathbb{R}} e^{zx}\mu(dx) = e^{-re^{i\alpha\rho\pi} z^\alpha}$). We write $\mu = \mathrm{n}_{r,\rho,\alpha}$.

$\mathcal{C}_X^*(z) = \log \mathbb{E}[e^{zX}]$

Free stable laws

If $X_1, X_2, X_3, \cdots$: free indep. i.d. random variables and if

$$\text{Law of } \frac{X_1 + \cdots + X_N}{N^{1/\alpha}} \xrightarrow{N\to\infty} \mu \neq \text{Dirac delta}$$

then, μ is called a free stable law. Always $\alpha \in (0, 2]$.

Theorem (Bercovici, Pata, Biane, Ann. Math. '99)

If μ is free stable then $\mathcal{C}_\mu^{\boxplus}(z) = -re^{i\alpha\rho\pi} z^\alpha, r > 0$. We write $\mu = \mathrm{f}_{r,\rho,\alpha}$.

$\mathcal{C}_\mu^{\boxplus}(z) := G_\mu^{-1}(z) - \frac{1}{z}$.

Theorem (See e.g. Zolotarev's book '86)

Suppose $\alpha \in (0,1), \rho \in [0,1]$. *The classical stable law* $\mathrm{n}_{1,\rho,\alpha}$ *has the density*

$$\frac{1}{\pi}\sum_{n\geq 1}(-1)^{n-1}\frac{\Gamma(1+\alpha n)}{n!}\sin(n\alpha\rho\pi)x^{-\alpha n-1}, \quad x>0,$$

$$\frac{1}{\pi}\sum_{n\geq 1}(-1)^{n-1}\frac{\Gamma\left(1+n/\alpha\right)}{n!}\sin(n\rho\pi)x^{n-1}, \quad x\to 0^{+} \quad (\textit{asymptotic exp.})$$

Theorem (Kuznetsov-H. Elect. Comm. Probab. '14)

Suppose $\alpha \in (0,1), \rho \in [0,1]$, $x^* := \alpha(1-\alpha)^{1/\alpha-1}$. *The free stable law* $\mathrm{f}_{1,\rho,\alpha}$ *has the density*

$$\frac{1}{\pi}\sum_{n\geq 1}(-1)^{n-1}\frac{\Gamma(1+\alpha n)}{n!\Gamma(2+(\alpha-1)n)}\sin(n\alpha\rho\pi)x^{-\alpha n-1}, \quad x\geq x^*,$$

$$\frac{1}{\pi}\sum_{n\geq 1}(-1)^{n-1}\frac{\Gamma\left(1+n/\alpha\right)}{n!\Gamma\left(2+\left(1/\alpha-1\right)n\right)}\sin(n\rho\pi)x^{n-1}, \quad 0\leq x\leq x^*.$$

Proof.

Let

$$M_\mu(s) := \int_0^\infty x^s \, \mu(dx) \quad \text{(Mellin transform)}$$

First we compute

$$M_{\mathrm{f}_{1,\rho,\alpha}}(s) = \frac{1}{\pi} \sin(\pi\rho s) \frac{\Gamma(s)\Gamma\left(1 - s/\alpha\right)}{\Gamma\left(2 + s - s/\alpha\right)}.$$

Then the density function of $f_{1,\rho,\alpha}$ is obtained from the inversion formula:

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} x^{-s-1} M_{\mathrm{f}_{1,\rho,\alpha}}(s) ds.$$

□