QUENCHED TAIL ESTIMATE FOR THE RANDOM WALK IN RANDOM SCENERY

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Let $(\{S_t\}_{t\geq 0}, \{P_x\}_{\mathbb{Z}^d})$ be the continuous time simple random walk on $\mathbb{Z}^d$ and $(\{z(x)\}_{x\in\mathbb{Z}^d}, \mathbb{P})$ non-negative IID random variable with a power law tail

$$\mathbb{P}(z(x) > r) = r^{-\alpha+o(1)}, \quad r \to \infty \tag{1}$$

for some $\alpha > 0$. Random walk in random scenery is a process defined as

$$A_t = \int_0^t z(S_u)\mathrm{d}u. \tag{2}$$

This process was first appeared in the independent works by Borodin and Kesten-Spitzer in 1979, who aimed at constructing a new class of self similar processes as scaling limits. Physically, it can be interpreted as a diffusing particle in a random shear flow and also its Laplace transform $E_x[e^{A_t}]$ represents a solution of the so-called parabolic Anderson model.

The scaling limits of this type of processes have been studied a lot since Borodin and Kesten-Spitzer. On the other hand, the tail estimates, including large deviation principle, have become active recently. Partly this is related to the recent development on the tail estimates for the self-intersection local time. However, the most of the works focuses on the *annealed* measure $\mathbb{P}\otimes P_x$ and there the most natural assumption turns out to be

$$\mathbb{P}(z(x) > r) = \exp\left\{-r^{\alpha+o(1)}\right\} \quad (\alpha > 0). \tag{3}$$

We studied the classical power law tail case under the *quenched* measure and obtained the following result.

Theorem 1. *For $\rho > 0$, $\mathbb{P}$-almost surely,*

$$P_0\left(A_t \geq t^{\rho}\right) = \exp\left\{-t^{p(\alpha,\rho)+o(1)}\right\} \tag{4}$$

as $t \to \infty$, where for $d = 1$,

$$p(\alpha,\rho) = \begin{cases} \frac{2\alpha\rho}{\alpha+1} - 1, & \rho \in \left(\frac{\alpha+1}{2\alpha} \vee 1, \frac{\alpha+1}{\alpha}\right], \\ \alpha(\rho-1), & \rho > \frac{\alpha+1}{\alpha} \end{cases} \tag{5}$$

and for $d \geq 2$,

$$p(\alpha,\rho) = \begin{cases} \frac{2\alpha\rho-d}{2\alpha+d}, & \rho \in \left(\frac{d}{2\alpha} \vee 1, \frac{\alpha+d}{\alpha}\right], \\ \frac{\alpha(\rho-1)}{d}, & \rho > \frac{\alpha+d}{\alpha}. \end{cases} \tag{6}$$

Moreover, when $\alpha > 1$, $\rho = 1$ and $c > \mathbb{E}[z(0)]$, the standard large deviation type probability $P_0\left(A_t \geq ct\right)$ decays as $\exp\{-t^{p(\alpha,1)+o(1)}\}$.