# Dichotomy of global capacity density

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Let $\varphi$ be an outer measure in $\mathbb{R}^n$, $n \ge 2$, such that $0 < \varphi(B(x,r)) < \infty$ for every open ball $B(x,r)$ with center at $x$ and radius $r$. For $E \subset \mathbb{R}^n$ and $r > 0$ define

$$\mathcal{D}(\varphi, E, r) = \inf_{x \in \mathbb{R}^n} \frac{\varphi(E \cap B(x,r))}{\varphi(B(x,r))}.$$

By definition $0 \le \mathcal{D}(\varphi, E, r) \le 1$. We are interested in the limit of $\mathcal{D}(\varphi, E, r)$ as $r \to \infty$.

A typical example of $\varphi$ is the $n$-dimensional Lebesgue outer measure $m$. For each $c \in (0,1)$, it is easy to construct a closed set $E_c$ such that $\lim_{r\to\infty} \mathcal{D}(m, E_c, r) = c$. If $\varphi$ is a capacity, then the situation is completely different. For many capacities the limit must be either 0 or 1. By $C_\ell(E)$ we denote the logarithmic capacity of $E \subset \mathbb{R}^2$. In connection with BMOA, Stegenga proved

$$\lim_{r\to\infty} \mathcal{D}(C_\ell, E, r) = \begin{cases} 0 & \text{if } \mathcal{D}(C_\ell, E, r) = 0 \text{ for all } r > 0, \\ 1 & \text{if } \mathcal{D}(C_\ell, E, r) > 0 \text{ for some } r > 0. \end{cases}$$

In this talk we show that this *dichotomy* is enjoyed by many capacities such as Riesz capacities, weighted $L^p$-capacities in the Euclidean space and variational capacities in a metric measure space. On the other hand, there is a capacity that fails the dichotomy. The case when $\varphi$ is Newtonian capacity is of particular interest. We observe that the dichotomy relates to the estimate of the principal frequency and intrinsic ultracontractivity.

Some parts of the talk are joint works with Tsubasa Itoh, Anders Björn, Jana Björn, and Nageswari Shanmugalingam.