Spread-out limit of the critical points for various statistical-mechanics models

@ Probability and Analysis on Random Structures and Related Topics

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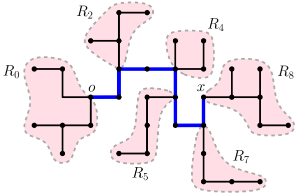
- Statistical-mechanics models exhiting phase transitions & critical behavior:
  - self-avoiding walk (SAW),
  - lattice trees (LT) & lattice animals (LA),
  - percolation,
  - oriented percolation (OP) and the contact process (CP).
- The 2-point function $\tau_p(x)$ & the critical point $p_c := \sup\{p \geq 0 : \sum_x \tau_p(x) < \infty\}$.

$$\text{E.g.,} \quad \tau_p^{\text{LT}}(x) = \sum_{T \in \mathcal{T}_{o,x}} \left(\frac{p}{|\Lambda|}\right)^{|E_T|},$$



where $\Lambda = \{x \in \mathbb{Z}^d : 0 < \|x\| \leq L\}$.

- $\lim_{L\uparrow\infty} p_c = 1$ for SAW, percolation, OP & CP; $\lim_{L\uparrow\infty} p_c = 1/e$ for LT & LA.
- Want to know: the spead of convergence to those mean-field values.

$$\text{E.g.,} \quad \lim_{L\uparrow\infty} \frac{p_c - 1/e}{L^{-d}} \overset{d>8}{=} \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o) - \begin{cases} 0 & \text{[LT]}, \\ \frac{1}{2e^2} \sum_{n=3}^{\infty} U^{*n}(o) & \text{[LA]}. \end{cases}$$

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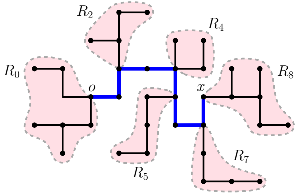

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E.g., $$\lim_{L \uparrow \infty} \frac{p_{\rm c} - 1/e}{L^{-d}} \overset{d>8}{=} \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o) - \begin{cases} 0 & \text{[LT]}, \\ \frac{1}{2e^2} \sum_{n=3}^{\infty} U^{*n}(o) & \text{[LA]}. \end{cases}$$

- The spread-out edges $\mathbb{B}_L^d := \{\{x,y\} \subset \mathbb{Z}^d : x-y \in \Lambda\}$.
- The 1-step distribution $D(x) := \frac{1}{|\Lambda|} \mathbb{1}_{\{x\in\Lambda\}}$. $\qquad \sigma^2 := \sum_{x\in\mathbb{Z}^d} \|x\|^2 D(x) = O(L^2)$.
- For each $E \subset \mathbb{B}_L^d$, the Boltzmann weight $W_p(E) := \prod_{\{x,y\}\in E} pD(x-y) = \Big(\frac{p}{|\Lambda|}\Big)^{|E|}$.
- The 2-point function $\tau_p(x) := \sum_{E\in\blacksquare} W_p(E)$, where
  - $\blacksquare = \Omega_{o,x}$: the set of spread-out random-walk (RW) paths between o and x;
  - $\blacksquare = \mathcal{S}_{o,x}$: the set of spread-out SAWs between o and x;
  - $\blacksquare = \mathcal{T}_{o,x}$: the set of spread-out LTs containing o and x;
  - $\blacksquare = \mathcal{A}_{o,x}$: the set of spread-out LAs containing o and x.

E.g., since $\sum_x \big|\{\omega = (\omega_0, \omega_1, \dots, \omega_n) \in \Omega_{o,x}\}\big| = |\Lambda|^n$,

$$p < 1 \quad \Rightarrow \quad \begin{cases} \chi_p^{\text{RW}} := \sum_x \tau_p^{\text{RW}}(x) = \sum_{n=0}^\infty p^n = (1-p)^{-1}, \\ \xi_p^{\text{RW}} := \sqrt{\sum_x \|x\|^2 \frac{\tau_p^{\text{RW}}(x)}{\chi_p^{\text{RW}}}} = \sqrt{\sigma^2 p}\,(1-p)^{-1/2}, \end{cases}$$

$$\tau_p^{\text{RW}}(x) = \delta_{o,x} + (pD * \tau_p^{\text{RW}})(x) := \delta_{o,x} + \sum_{y\in\mathbb{Z}^d} pD(y)\,\tau_p^{\text{RW}}(x-y).$$

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- Penrose (1994): $\exists C L^{-2d/7} \log L \geq p_c^{\mathrm{SAW}} - 1 \geq \begin{cases} 0 & [d \geq 3], \\ \exists c L^{-2} \log L & [d = 2], \\ \exists c' L^{-4/5} & [d = 1]. \end{cases}$

- Madras-Slade (1993): $d > 4 = d_c^{\mathrm{SAW}} \quad \Rightarrow \quad p_c^{\mathrm{SAW}} = 1 + O(L^{-d})$.

- Hara-Slade (1990) & v.d.Hofstad-Sakai (2005):

$$d > 6 = d_c^{\mathrm{perc}} \quad \Rightarrow \quad p_c^{\mathrm{perc}} = 1 + O(L^{-d}).$$

- v.d.Hofstad-Slade (2002, 2003) & v.d.Hofstad-Sakai (2004):

$$d > 4 = d_c^{\mathrm{OP}} = d_c^{\mathrm{CP}} \quad \Rightarrow \quad p_c^{\mathrm{OP}},\ p_c^{\mathrm{CP}} = 1 + O(L^{-d}).$$

- Bramson-Durrett-Swindle (1989): $p_c^{\mathrm{CP}} - 1 \asymp \lambda_L := \begin{cases} L^{-d} & [d \geq 3], \\ L^{-2} \log L^2 & [d = 2], \\ L^{-2/3} & [d = 1]. \end{cases}$

- Durrett-Perkins (1999): $\lim_{L \uparrow \infty} \frac{p_c^{\mathrm{CP}} - 1}{\lambda_L} = \begin{cases} \sum_{n=2}^{\infty} U^{*n}(o) & [d \geq 3], \\ \frac{3}{2\pi} & [d = 2], \end{cases}$

  where U is the uniform distribution on the d-dim. ball $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$.

- Penrose (1994): $p_c^{\mathrm{LT}},\ p_c^{\mathrm{LA}} = \frac{1}{e} + O(L^{-2d/7} \log L) \quad \forall d \geq 1$.

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- Clisby-Liang-Slade (2007):

$$p_c^{\text{SAW}} = 1 + \frac{1}{2d} + \frac{2}{(2d)^2} + \frac{6}{(2d)^3} + \frac{27}{(2d)^4} + \frac{157}{(2d)^5} + \cdots \text{(up to } O(d^{-10})\text{)}.$$

- Hara-Slade (1995): $p_c^{\text{perc}} = 1 + \frac{1}{2d} + \frac{7/2}{(2d)^2} + O(d^{-3})$.
- Cox-Durrett (1983): $p_c^{\text{OP}} = 1 + O(d^{-2})$.
- Liggett (1999): $p_c^{\text{CP}} = 1 + O(d^{-1})$.
- Miranda-Slade (2013):

$$p_c = \frac{1}{e} + \frac{3}{2e}\frac{1}{2d} + \begin{cases} \frac{115}{24e}\frac{1}{(2d)^2} + o(d^{-2}) & \text{[LT]}, \\ \left(\frac{115}{24e} - \frac{1}{2e^2}\right)\frac{1}{(2d)^2} + o(d^{-2}) & \text{[LA]}. \end{cases}$$

Theorem 1 (v.d.Hofstad-Sakai (2005))

$d > d_c \quad \Rightarrow \quad p_c^{SAW}, p_c^{perc}, p_c^{OP}, p_c^{CP} = 1 + CL^{-d} + O(L^{-d-1}),$ *where*

$$C = \begin{cases} \sum_{n=2}^{\infty} U^{*n}(o) & [\textit{SAW \& CP}], \\ \frac{1}{2}\sum_{n=2}^{\infty} U^{*2n}(o) & [\textit{OP}], \\ U^{*2}(o) + \sum_{n=3}^{\infty} \frac{n+1}{2} U^{*n}(o) & [\textit{percolation}]. \end{cases}$$

Theorem 2 (Kawamoto-Sakai (2022))

$d > 8 \quad \Rightarrow \quad p_c^{LT}, p_c^{LA} = \dfrac{1}{e} + CL^{-d} + O(L^{-d-1}),$ *where*

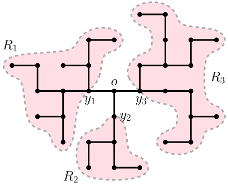
$$C_{LT} = \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o), \qquad C_{LA} = C_{LT} - \frac{1}{2e^2} \sum_{n=3}^{\infty} U^{*n}(o).$$

1. **Lace expansion** (e.g., Brydges-Spencer (1985), Hara-Slade (1990)):

$$\tau_p = \begin{cases} \delta_o + (pD + \pi_p) * \tau_p & \text{[SAW]}, \\ g_p\delta_o + \pi_p + (g_p\delta_o + \pi_p) * pD * \tau_p & \text{[LT]}, \\ g_p\delta_o + h_p + \pi_p + (g_p\delta_o + h_p + \pi_p) * pD * \tau_p & \text{[LA]}, \end{cases}$$

where

- $\pi_p = \sum_{n=1}^{\infty}(-1)^n\pi_p^{(n)}$: the model-dependent lace-expansion coefficients,
- $g_p := \sum_{T\in\mathcal{T}_o} W_p(T)$ or $\sum_{A\in\mathcal{A}_o} W_p(A)$: the 1-point function,



- $h_p(x) := (1-\delta_{o,x}) \sum_{A\in\mathcal{A}_{o,x}} W_p(A)\, \mathbb{1}_{\{o \Longleftrightarrow x \text{ in } A\}}$.

$$\therefore\ \chi_p = \hat{\tau}_p := \sum_x \tau_p(x) = \begin{cases} (1-p-\hat{\pi}_p)^{-1} & \text{[SAW]}, \\ \Big((g_p+\hat{\pi}_p)^{-1}-p\Big)^{-1} & \text{[LT]}, \\ \Big((g_p+\hat{h}_p+\hat{\pi}_p)^{-1}-p\Big)^{-1} & \text{[LA]}. \end{cases}$$

2. Hara-v.d.Hofstad-Slade (2003) (& Chen-Sakai (2015)):

$$\text{SAW on } \mathbb{Z}^{d>4}_{L\gg 1} \quad \Rightarrow \quad \begin{cases} \pi_p^{(1)}(x) = \delta_{o,x}\,(pD * \tau_p)(o), \\ \pi_p^{(n)}(x) = \dfrac{O(L^{-6})\,O(L^{-d})^{n-2}}{(\|x\|\vee L)^{3d-6}} & [{}^\forall n \geq 2, {}^\forall p \leq p_\mathrm{c}], \\ \tau_{p_\mathrm{c}}(x) = \dfrac{O(L^{-2})}{(\|x\|\vee L)^{d-2}} & [{}^\forall x \neq o], \\ \chi_p \asymp (p_\mathrm{c} - p)^{-1} & [p \uparrow p_\mathrm{c}]. \end{cases}$$

$$\begin{aligned} \therefore\ p_\mathrm{c} = 1 - \hat{\pi}_{p_\mathrm{c}} &= 1 + \hat{\pi}^{(1)}_{p_\mathrm{c}} + O(L^{-2d}) \\ &= 1 + \hat{\pi}^{(1)}_1 + \underbrace{(p_\mathrm{c} - 1)}_{O(L^{-d})}\ \underbrace{\partial_p \hat{\pi}^{(1)}_{p*}}_{O(L^{-d})} + O(L^{-2d}) \\ &= 1 + (D * \tau_1)(o) + O(L^{-2d}) \\ &= 1 + \sum_{n=2}^{\infty} D^{*n}(o) - \underbrace{\sum_x D(x) \sum_{\omega \in \Omega_{o,x} \setminus S_{o,x}} W_1(\omega)}_{O(L^{-2d})} + O(L^{-2d}). \end{aligned}$$

N.b., $\sum_{n=2}^{\infty} D^{*n}(o) = L^{-d} \sum_{n=2}^{\infty} U^{*n}(o) + O(L^{-d-1})$.

3. Hara-v.d.Hofstad-Slade (2003) & Liang (2022):

$$\text{LT \& LA on } \mathbb{Z}^{d>8}_{L\gg 1} \quad \Rightarrow \quad \begin{cases} 1 \le g_p \le 4 & [{}^\forall p \le p_{\rm c}], \\ \pi_p^{(n)}(x) = \dfrac{O(L^{-6})\,O(L^{-d})^{n-1}}{(\|x\|\vee L)^{2d-6}} & [{}^\forall n \ge 1, {}^\forall p \le p_{\rm c}], \\ h_p(x) \le (1-\delta_{o,x})\,\tau_p(x)^2, & \\ \tau_{p_{\rm c}}(x) = \dfrac{O(L^{-2})}{(\|x\|\vee L)^{d-2}} & [{}^\forall x \ne o], \\ \chi_p \asymp (p_{\rm c}-p)^{-1/2} & [p \uparrow p_{\rm c}]. \end{cases}$$

$$\therefore\ p_{\rm c} = \big(g_{p_{\rm c}} + \hat{h}_{p_{\rm c}} + \hat{\pi}_{p_{\rm c}}\big)^{-1} = \frac{1}{g_{p_{\rm c}}}\bigg(1 - \frac{\hat{h}_{p_{\rm c}} + \hat{\pi}_{p_{\rm c}}}{g_{p_{\rm c}}}\bigg) + O(L^{-2d}).$$

4. Inductive approach: let $p_1 g_{p_1} = 1$, $\ell_1 = 1$, $\ell_{j+1} = 1 + \frac{\ell_j}{2}$ (so that $\ell_j \uparrow 2$).

- $0 \le 1 - \frac{p_1}{p_{\rm c}} = -\underbrace{\frac{g_{p_{\rm c}} - g_{p_1}}{g_{p_1}}}_{\ge 0} - \frac{\hat{h}_{p_{\rm c}} + \hat{\pi}_{p_{\rm c}}}{g_{p_1}}. \qquad \therefore\ p_{\rm c} - p_1 = O(L^{-d\ell_1}).$
- $g_{p_{\rm c}} - g_{p_1} = \Big(1 - \frac{p_1}{p_{\rm c}}\Big)\underbrace{\sum_{T\in\mathcal{T}_o}\sum_{n=1}^{|E_T|}\Big(\frac{p_{\rm c}}{p_1}\Big)^n W_{p_1}(T)}_{=:F} = \cdots = -\frac{\hat{h}_{p_{\rm c}} + \hat{\pi}_{p_{\rm c}}}{g_{p_1} + F}F.$

$$\therefore\ p_{\rm c} - p_1 = p_1 \frac{\hat{\pi}^{(1)}_{p_{\rm c}} - \hat{h}_{p_{\rm c}}}{g_{p_1} + F} + O(L^{-2d}). \quad \therefore\ p_{\rm c} - p_1 = O(L^{-d\ell_j}) \overset{\because F \ge \frac{1}{2}\chi_{p_1}}{\Longrightarrow} p_{\rm c} - p_1 = O(L^{-d\ell_{j+1}}).$$

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$$\therefore\ p_c = \Big(g_{p_c} + \hat{h}_{p_c} + \hat{\pi}_{p_c}\Big)^{-1} = \frac{1}{g_{p_c}}\bigg(1 - \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_c}}\bigg) + O(L^{-2d}).$$

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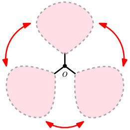
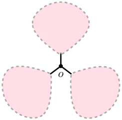
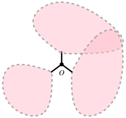
- $g_{p_c} - g_{p_1} = \Big(1 - \frac{p_1}{p_c}\Big)\underbrace{\sum_{T\in\mathcal{T}_o}\sum_{n=1}^{|E_T|}\Big(\frac{p_c}{p_1}\Big)^n W_{p_1}(T)}_{=:F} = \cdots = -\frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1} + F}F.$

$$\therefore\ p_c - p_1 = p_1\frac{\hat{\pi}^{(1)}_{p_c} - \hat{h}_{p_c}}{g_{p_1} + F} + O(L^{-2d}). \quad \therefore\ p_c - p_1 = O(L^{-d\ell_j}) \overset{\because F \ge \frac{1}{2}\chi_{p_1}}{\Longrightarrow} p_c - p_1 = O(L^{-d\ell_{j+1}}).$$

5. Identifying the dominant contribution to g_{p_1} for LT:

g_{p_1} =  =  − ,

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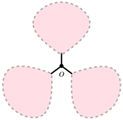


$$= \sum_{Y\subset\Lambda}\Big(\frac{p_1}{|\Lambda|}\Big)^{|Y|}\prod_{y\in Y}\sum_{R_y\in\mathcal{T}_y\setminus\mathcal{T}_o}\Big(\frac{p_1}{|\Lambda|}\Big)^{|E_{R_y}|}$$

$$= \sum_{Y\subset\Lambda}\Big(\frac{p_1}{|\Lambda|}\Big)^{|Y|}\prod_{y\in Y}\Bigg(\underbrace{\sum_{R_y\in\mathcal{T}_y}\Big(\frac{p_1}{|\Lambda|}\Big)^{|E_{R_y}|}}_{g_{p_1}}-\underbrace{\sum_{R_y\in\mathcal{T}_{o,y}}\Big(\frac{p_1}{|\Lambda|}\Big)^{|E_{R_y}|}}_{\tau_{p_1}(y)}\Bigg)$$

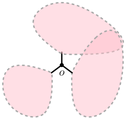
$$= \sum_{Y\subset\Lambda}\Big(\frac{1}{|\Lambda|}\Big)^{|Y|}\prod_{y\in Y}\Big(1-\frac{\tau_{p_1}(y)}{g_{p_1}}\Big).$$

N.b., $\sum_{Y\subset\Lambda}\big(\frac{1}{|\Lambda|}\big)^{|Y|}=\big(1+\frac{1}{|\Lambda|}\big)^{|\Lambda|}=e\big(1-\frac{1}{2|\Lambda|}\big)+O(|\Lambda|^{-2})$.

6. Identifying sub-dominant contributions to g_{p_1} for LT:

$$\sum_{Y\neq\varnothing}\Big(\frac{1}{|\Lambda|}\Big)^{|Y|}\bigg(\prod_{y\in Y}\Big(1-\frac{\tau_{p_1}(y)}{g_{p_1}}\Big)-1\bigg)=\sum_{Y\neq\varnothing}\Big(\frac{1}{|\Lambda|}\Big)^{|Y|}\sum_{\varnothing\neq Z\subset Y}\prod_{y\in Z}\frac{-\tau_{p_1}(y)}{g_{p_1}}$$
$$=\sum_{Z\neq\varnothing}\prod_{y\in Z}\frac{-\tau_{p_1}(y)}{g_{p_1}}\Big(\frac{1}{|\Lambda|}\Big)^{|Z|}\sum_{Y\supset Z}\Big(\frac{1}{|\Lambda|}\Big)^{|Y\setminus Z|}$$
$$=\sum_{Z\neq\varnothing}\prod_{y\in Z}\bigg(\frac{-\tau_{p_1}(y)}{g_{p_1}}D(y)\bigg)\Big(1+\frac{1}{|\Lambda|}\Big)^{|\Lambda\setminus Z|}$$
$$=\sum_{Z:|Z|=1}\cdots\cdots\cdots+\underbrace{\sum_{Z:|Z|\geq 2}\cdots\cdots\cdots}_{O(L^{-2d})}.$$

N.b., $\sum_{Z:|Z|=1}\cdots=-\Big(\frac{\tau_{p_1}}{g_{p_1}}*D\Big)(o)\,\Big(1+\frac{1}{|\Lambda|}\Big)^{|\Lambda|-1}=-e\,\sum_{n=2}^{\infty}D^{*n}(o)+O(L^{-2d})$.

Similarly, 

$=e\sum_{n=3}^{\infty}\frac{n-1}{2}D^{*n}(o)+O(L^{-2d})$.

- For SAW, percolation, OP, CP for $d > d_c$,

$$p_c = 1 + CL^{-d} + O(L^{-d-1}), \quad C = \begin{cases} \sum_{n=2}^\infty U^{*n}(o) & \text{[SAW \& CP]}, \\ \frac{1}{2}\sum_{n=2}^\infty U^{*2n}(o) & \text{[OP]}, \\ U^{*2}(o) + \sum_{n=3}^\infty \frac{n+1}{2} U^{*n}(o) & \text{[percolation]}. \end{cases}$$

- For LT & LA for $d > 8$,

$$p_c = \frac{1}{e} + CL^{-d} + O(L^{-d-1}), \quad C = \sum_{n=2}^\infty \frac{n+1}{2e} U^{*n}(o) + \begin{cases} 0 & \text{[LT]}, \\ -\frac{1}{2e^2}\sum_{n=3}^\infty U^{*n}(o) & \text{[LA]}. \end{cases}$$

- The critical point is a divergent point of $\chi_p = \sum_x \tau_p(x)$. E.g., for SAW,

Lace expansion $\Rightarrow$ $\tau_p = \delta_o + (pD + \pi_p) * \tau_p$ $\Rightarrow$ $\chi_p = (1 - p - \hat{\pi}_p)^{-1}$.

- $p = e^{-h-X_b}$ ($X = \{X_b\}_{b\in\mathbb{B}_L^d}$: i.i.d., integrable) $\Rightarrow$ SAW on random conductors.

Theorem 3 (Chino-Sakai (2016))

If the law of X is translation-invariant and ergodic, then the quenched critical point $\hat{h}_X(x)$ is a.s. independent of the starting point x of SAWs.

- For SAW, percolation, OP, CP for $d > d_c$,

$$p_c = 1 + CL^{-d} + O(L^{-d-1}), \quad C = \begin{cases} \sum_{n=2}^{\infty} U^{*n}(o) & \text{[SAW \& CP]}, \\ \frac{1}{2}\sum_{n=2}^{\infty} U^{*2n}(o) & \text{[OP]}, \\ U^{*2}(o) + \sum_{n=3}^{\infty} \frac{n+1}{2} U^{*n}(o) & \text{[percolation]}. \end{cases}$$

- For LT & LA for $d > 8$,

$$p_c = \frac{1}{e} + CL^{-d} + O(L^{-d-1}), \quad C = \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o) + \begin{cases} 0 & \text{[LT]}, \\ -\frac{1}{2e^2}\sum_{n=3}^{\infty} U^{*n}(o) & \text{[LA]}. \end{cases}$$

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