Critical points for various spread-out statistical-mechanics models in high dimensions

@ The 3rd Taiwan-Japan Joint Workshop on Applied Mathematics

Akira Sakai

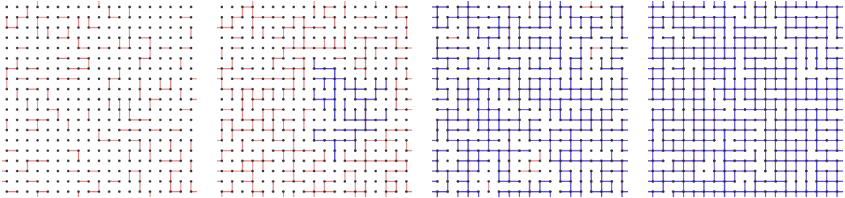
Department of Mathematics
Hokkaido University

May 1, 2025

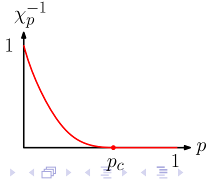
- Want to understand phase transitions & critical phenomena.

  E.g., percolation on $\mathbb{Z}^2$, with bond-occupation probability $p = 0.2,\ 0.4,\ 0.6,\ 0.8$.

  (https://www.ams.org/publicoutreach/feature-column/fcarc-percolation)



$$\chi_p = \underbrace{\sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(o \longleftrightarrow x)}_{\text{expected \# of connected sites to } o} \begin{cases} < \infty & \text{if } p < p_c, \\ = \infty & \text{if } p \geq p_c, \end{cases} \qquad \chi_p \underset{p\uparrow p_c}{\approx} \begin{cases} (\frac{1}{2} - p)^{-43/18} & \text{on } \mathbb{Z}^2, \\ (p_c - p)^{-1} & \text{on } \mathbb{Z}^{d\gg 6}. \end{cases}$$



- Important to know the exact value of the critical point p_c.

- Self-avoiding walk (SAW): a statistical-mechanics model for linear polymers.
- $\mathcal{G}_L^d = (\mathbb{Z}^d, E_L)$: the spread-out lattice with spread-out parameter $L \in [1, \infty)$.

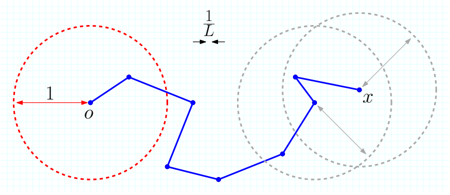
$$E_L = \Big\{\{x,y\} \subset \mathbb{Z}^d : 0 < \|x-y\| \le L\Big\}, \qquad \Delta = \deg(\mathcal{G}_L^d) = O(L^d), \qquad D(x,y) = \frac{\mathbb{1}_{\{\{x,y\}\in E_L\}}}{\Delta}.$$

- SAW 2-point function $G_p^{\text{SAW}}(o,x)$: the generating function for SAWs from o to x.

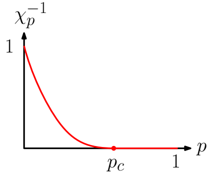
$$G_p^{\text{SAW}}(o,x) = \sum_{n=0}^{\infty} p^n \sum_{\substack{\omega=(\omega_0,\dots,\omega_n):\\ \omega_0=o,\ \omega_n=x}} \prod_{j=1}^{n} D(\omega_{j-1},\omega_j) \underbrace{\prod_{0\le s<t\le n} \mathbb{1}_{\{\omega_s\ne\omega_t\}}}_{\text{self-avoidance}} \equiv \sum_{\substack{\omega:o\to x\\ (\text{SAW})}} \Big(\frac{p}{\Delta}\Big)^{|\omega|}.$$

- Critical point p_{c}: the radius of convergence of the susceptibility χ_p.

$$p_{\mathrm{c}} = \sup\Big\{p \ge 0 : \chi_p = \sum_{x\in\mathbb{Z}^d} G_p(o,x) < \infty\Big\}.$$



Scaled SAW on $\left(\frac{1}{L}\mathbb{Z}\right)^d$



- Removal of self-avoidance $\Rightarrow$ random walk (RW):

$$G_p^{\text{RW}}(o,x) = \sum_{n=0}^{\infty} p^n \sum_{\substack{\omega=(\omega_0,\dots,\omega_n):\\ \omega_0=o,\ \omega_n=x}} \prod_{j=1}^{n} D(\omega_{j-1},\omega_j), \qquad \chi_p^{\text{RW}} = \sum_{n=0}^{\infty} p^n = \begin{cases} (1-p)^{-1} & \text{if } p<1,\\ \infty & \text{if } p\ge 1. \end{cases}$$

- For percolation on $\mathcal{G}_L^d$ with bond-occupation probability $pD(x,y) = \frac{p}{\Delta}\mathbb{1}_{\{\{x,y\}\in E_L\}}$,

$$G_p^{\text{perc}}(o,x) = \mathbb{P}_p\Bigg(\bigcup_{\substack{\omega:o\to x\\ \text{(SAW)}}} \bigcap_{j=1}^{|\omega|} \{\{\omega_{j-1},\omega_j\} \text{ is occupied}\}\Bigg) \quad \Big(\le G_p^{\text{SAW}}(o,x)\Big).$$

Theorem 1 (with v.d.Hofstad, 2005)

Let $U(x) \propto \mathbb{1}_{\{\|x\|\le 1\}}$ be the uniform distribution on the Euclidean ball of radius 1, and let U^{*n} be its n-fold convolution in $\mathbb{R}^d$. $^\forall d > d_c$ (the model-dependent upper-critical dimension), $p_c = 1 + CL^{-d} + O(L^{-d-1})$ as $L\to\infty$, where

$$C = \begin{cases} \sum_{n=2}^{\infty} U^{*n}(o) & \text{for SAW \& the contact process on } \mathcal{G}_L^{d>4},\\ \frac{1}{2}\sum_{n=2}^{\infty} U^{*2n}(o) & \text{for oriented percolation on } \mathbb{Z}_+\times\mathcal{G}_L^{d>4},\\ U^{*2}(o) + \sum_{n=3}^{\infty} \frac{n+1}{2} U^{*n}(o) & \text{for percolation on } \mathcal{G}_L^{d>6}. \end{cases}$$

- Removal of self-avoidance $\Rightarrow$ random walk (RW):

$$G_p^{\text{RW}}(o,x)=\sum_{n=0}^{\infty}p^n\sum_{\substack{\omega=(\omega_0,\dots,\omega_n):\\ \omega_0=o,\ \omega_n=x}}\prod_{j=1}^{n}D(\omega_{j-1},\omega_j),\qquad \chi_p^{\text{RW}}=\sum_{n=0}^{\infty}p^n=\begin{cases}(1-p)^{-1} & \text{if } p<1,\\ \infty & \text{if } p\geq 1.\end{cases}$$

- For percolation on $\mathcal{G}_L^d$ with bond-occupation probability $pD(x,y)=\frac{p}{\Delta}\mathbb{1}_{\{\{x,y\}\in E_L\}}$,

$$G_p^{\text{perc}}(o,x)=\mathbb{P}_p\bigg(\bigcup_{\substack{\omega:o\to x\\ \text{(SAW)}}}\bigcap_{j=1}^{|\omega|}\Big\{\{\omega_{j-1},\omega_j\}\text{ is occupied}\Big\}\bigg)\quad\Big(\leq G_p^{\text{SAW}}(o,x)\Big).$$

Theorem 1 (with v.d.Hofstad, 2005)

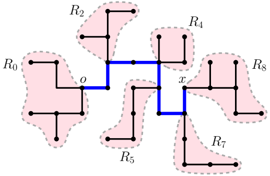
Let $U(x)\propto\mathbb{1}_{\{\|x\|\leq 1\}}$ be the uniform distribution on the Euclidean ball of radius 1, and let U^{*n} be its n-fold convolution in $\mathbb{R}^d$. $^\forall d>d_c$ (the model-dependent upper-critical dimension), $p_c=1+CL^{-d}+O(L^{-d-1})$ as $L\to\infty$, where

$$C=\begin{cases}\sum_{n=2}^{\infty}U^{*n}(o) & \text{for SAW \& the contact process on } \mathcal{G}_L^{d>4},\\ \frac{1}{2}\sum_{n=2}^{\infty}U^{*2n}(o) & \text{for oriented percolation on } \mathbb{Z}_+\times\mathcal{G}_L^{d>4},\\ U^{*2}(o)+\sum_{n=3}^{\infty}\frac{n+1}{2}U^{*n}(o) & \text{for percolation on } \mathcal{G}_L^{d>6}.\end{cases}$$

Best bounds before Theorem 1

- SAW:
  - Madras & Slade (1993): $d > 4 \quad\Rightarrow\quad p_c = 1 + O(L^{-d})$.
  - Penrose (1994): $^\exists CL^{-2d/7}\log L \ge p_c - 1 \ge \begin{cases} 0 & \text{if } d \ge 3,\\ ^\exists cL^{-2}\log L & \text{if } d = 2,\\ ^\exists cL^{-4/5} & \text{if } d = 1.\end{cases}$
- Percolation:
  - Hara & Slade (1990): $d > 6 \quad\Rightarrow\quad p_c = 1 + O(L^{-d})$.
- Oriented percolation:
  - v.d.Hofstad & Slade (2002): $d > 4 \quad\Rightarrow\quad p_c = 1 + O(L^{-d})$.
- The contact process:
  - Bramson & Durrett & Swindle (1989): $p_c - 1 \asymp f(L) \equiv \begin{cases} L^{-d} & \text{if } d \ge 3,\\ L^{-2}\log L^2 & \text{if } d = 2,\\ L^{-2/3} & \text{if } d = 1.\end{cases}$
  - Durrett & Perkins (1999): $\lim_{L\uparrow\infty} \dfrac{p_c - 1}{f(L)} = \begin{cases} \sum_{n=2}^\infty U^{*n}(o) & \text{if } d \ge 3,\\ \frac{3}{2\pi} & \text{if } d = 2,\end{cases}$
  - v.d.Hofstad & Sakai (2004): $d > 4 \quad\Rightarrow\quad p_c = 1 + O(L^{-d})$.

- Lattice animals (LA): a statistical-mechanics model for branched polymers.
- Lattice trees (LT): LA with no loops.
- LT 2-point function $G_p^{\mathrm{LT}}(o,x)$: the generating function for LTs containing o and x.



$$G_p^{\mathrm{LT}}(o,x) = \sum_{\substack{T\ni o,x \\ \text{(LT)}}} \left(\frac{p}{\Delta}\right)^{|E_T|}$$

$$= \sum_{n=0}^{\infty} p^n \sum_{\substack{\omega=(\omega_0,\dots,\omega_n): \\ \omega_0=o,\ \omega_n=x}} \prod_{j=1}^{n} D(\omega_{j-1},\omega_j) \underbrace{\prod_{k=0}^{n} \sum_{\substack{R_k\ni\omega_k \\ \text{(LT)}}} \left(\frac{p}{\Delta}\right)^{|E_{R_k}|} \prod_{0\le s<t\le n} \mathbb{1}_{\{R_s\cap R_t=\varnothing\}}}_{\text{difference from SAW}}.$$

- Removal of self-avoidance $\Rightarrow$ $p_c \sum_{\substack{R\ni o \\ \text{(LT)}}} \left(\frac{p_c}{\Delta}\right)^{|E_R|}$ "$\simeq$" 1. $\quad \therefore\ p_c \simeq \frac{1}{e}$.

$$\because\ I \equiv \sum_{\substack{R\ni o \\ \text{(LT)}}} \left(\frac{p_c}{\Delta}\right)^{|E_R|} \simeq 1 + \sum_{k=1}^{\Delta} \binom{\Delta}{k} \left(\frac{p_c I}{\Delta}\right)^k = \left(1+\frac{p_c I}{\Delta}\right)^{\Delta} \underset{L\uparrow\infty}{\sim} e^{p_c I} \simeq e.$$

Theorem 2 (with Kawamoto, 2024)

On $\mathbb{G}_L^{d>8}$, $p_{\mathrm{c}}^{\mathrm{LT/LA}} = \dfrac{1}{e} + CL^{-d} + O(L^{-d-1})$ *as* $L \to \infty$, *where*

$$C_{\mathrm{LT}} = \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o), \qquad C_{\mathrm{LA}} = C_{\mathrm{LT}} - \frac{1}{2e^2} \sum_{n=3}^{\infty} U^{*n}(o).$$

Best bounds before Theorem 2

- Penrose (1994): $d \geq 1 \quad \Rightarrow \quad p_{\mathrm{c}}^{\mathrm{LT/LA}} = \dfrac{1}{e} + O(L^{-2d/7} \log L)$.

Key elements for the proof of Theorems 1 & 2

- Lace expansion $\Rightarrow$ a formal recursion equation for $G_p(o,x)$:

$$G_p^{\mathrm{SAW}}(o,x) = \delta_{o,x} + \sum_{y \in \mathbb{Z}^d} \Big(pD(o,y) + \pi_p(o,y) \Big) G_p^{\mathrm{SAW}}(y,x), \qquad \pi_p(o,y) = \sum_{j=1}^{\infty} (-1)^j \pi_p^{(j)}(o,y).$$

- Use translation invariance to identify p_{c} and bound the expansion coefficients $\{\pi_p^{(j)}\}$:

$$\chi_p^{\mathrm{SAW}} = 1 + (p + \|\pi_p\|_1)\chi_p^{\mathrm{SAW}} = \underbrace{\Big(1 - \|\pi_p\|_1 - p\Big)^{-1}}_{\therefore\, p_{\mathrm{c}} = 1 - \|\pi_{p_{\mathrm{c}}}\|_1} \underset{p \uparrow p_{\mathrm{c}}}{\sim} \frac{(p_{\mathrm{c}} - p)^{-1}}{1 + \|\partial_p \pi_{p_{\mathrm{c}}}\|_1}.$$

$\pi_p^{(1)}(y) =$ $\delta_{o,y} \leq (pD * G_p^{\mathrm{SAW}})(o)\,\delta_{o,y}, \qquad \pi_p^{(2)}(y) =$ $\leq G_p^{\mathrm{SAW}}(y)^3.$

Theorem 2 (with Kawamoto, 2024)

On $\mathbb{G}_L^{d>8}$, $p_c^{\mathrm{LT/LA}} = \frac{1}{e} + C L^{-d} + O(L^{-d-1})$ *as* $L \to \infty$, *where*

$$C_{\mathrm{LT}} = \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o), \qquad C_{\mathrm{LA}} = C_{\mathrm{LT}} - \frac{1}{2e^2} \sum_{n=3}^{\infty} U^{*n}(o).$$

Best bounds before Theorem 2

- Penrose (1994): $d \ge 1 \quad \Rightarrow \quad p_c^{\mathrm{LT/LA}} = \frac{1}{e} + O(L^{-2d/7} \log L)$.

Key elements for the proof of Theorems 1 & 2

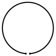
- Lace expansion $\Rightarrow$ a formal recursion equation for $G_p(o,x)$:

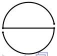
$$G_p^{\mathrm{SAW}}(o,x) = \delta_{o,x} + \sum_{y \in \mathbb{Z}^d} \Big(pD(o,y) + \pi_p(o,y) \Big) G_p^{\mathrm{SAW}}(y,x), \qquad \pi_p(o,y) = \sum_{j=1}^{\infty} (-1)^j \pi_p^{(j)}(o,y).$$

- Use translation invariance to identify p_c and bound the expansion coefficients $\{\pi_p^{(j)}\}$:

$$\chi_p^{\mathrm{SAW}} = 1 + (p + \|\pi_p\|_1)\chi_p^{\mathrm{SAW}} = \underbrace{\Big(1 - \|\pi_p\|_1 - p\Big)^{-1}}_{\therefore\ p_c = 1 - \|\pi_{p_c}\|_1} \underset{p \uparrow p_c}{\sim} \frac{(p_c - p)^{-1}}{1 + \|\partial_p \pi_{p_c}\|_1}.$$

$$\pi_p^{(1)}(y) = \text{[diagram]}\ \delta_{o,y} \le (pD * G_p^{\mathrm{SAW}})(o)\, \delta_{o,y}, \qquad \pi_p^{(2)}(y) = o\ \text{[diagram]}\ y \le G_p^{\mathrm{SAW}}(y)^3.$$

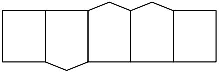




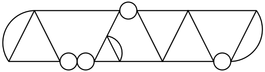
History of the lace expansion for the spread-out models

- Brydges & Spencer (1985), Hara & Slade (1992) for SAW in $d > 4$.
- Hara & Slade (1990) for percolation in $d > 6$.

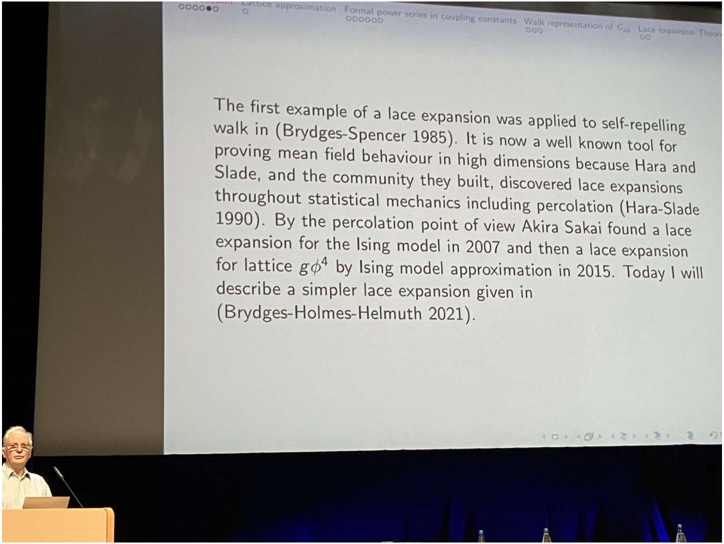
- Hara & Slade (1990) for LT/LA in $d > 8$.



- Nguyen & Yang (1993) for oriented percolation in spatial dimensions $d > 4$.
- Sakai (2001) for the contact process in $d > 4$.
- Sakai (2007 & 2022), Kamijima & Sakai (2025) for the classical Ising model in $d > 4$.



- Sakai (2015 & 2022), Brydges & Helmuth & Holmes (2021) for the $|\varphi|^4$ model in $d > 4$.
- Kamijima & Sakai (202?) for the quantum (or tranverse-field) Ising model in $d \geq 4$.



David Brydges was awarded Henri Poincaré Prize @ ICMP2024 in Strasbourg.

Best bounds for the nearest-neighbor models, as $d \uparrow \infty$:

- Clisby & Liang & Slade (2007):

$$p_{\mathrm{c}}^{\mathrm{SAW}} = 1 + \frac{1}{2d} + \frac{2}{(2d)^2} + \frac{6}{(2d)^3} + \frac{27}{(2d)^4} + \frac{157}{(2d)^5} + \cdots \text{(up to } O(d^{-10})\text{)}.$$

- Hara & Slade (1995):

$$p_{\mathrm{c}}^{\mathrm{perc}} = 1 + \frac{1}{2d} + \frac{7/2}{(2d)^2} + O(d^{-3}).$$

- Cox & Durrett (1983):

$$p_{\mathrm{c}}^{\mathrm{OP}} = 1 + O(d^{-2}).$$

- Liggett (1999):

$$p_{\mathrm{c}}^{\mathrm{CP}} = 1 + O(d^{-1}).$$

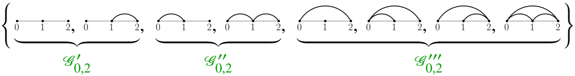
- Miranda & Slade (2013):

$$p_{\mathrm{c}} = \frac{1}{e} + \frac{3}{2e}\frac{1}{2d} + \begin{cases} \frac{115}{24e}\frac{1}{(2d)^2} + o(d^{-2}) & \text{for LT,} \\ \left(\frac{115}{24e} - \frac{1}{2e^2}\right)\frac{1}{(2d)^2} + o(d^{-2}) & \text{for LA.} \end{cases}$$

1. Derivation of the recursion equation.

- Naive expansion of the self-avoidance constraint.

$$\mathbb{1}_{\{\omega \text{ is SAW}\}} = \prod_{0\le i<j\le|\omega|} (1-\delta_{\omega_i,\omega_j}) = \sum_{\Gamma\in\mathscr{G}_{0,|\omega|}} \prod_{ij\in\Gamma} (-\delta_{\omega_i,\omega_j}).$$

E.g., $\mathscr{G}_{0,2} = \Big\{$  $\Big\}$.

- $\mathscr{C}_{0,n}$ $(\subset \mathscr{G}_{0,n})$: the set of connected graphs from 0 to n.

$$\Gamma\in\mathscr{C}_{0,n} \quad\Leftrightarrow\quad \text{(i) } {}^\exists 0t,\ sn\in\Gamma, \quad \text{(ii) } 0<{}^\forall k<n,\ {}^\exists st\in\Gamma \text{ such that } s<k<t.$$

E.g.,
$$\sum_{\Gamma\in\mathscr{G}'_{0,2}} \prod_{ij\in\Gamma} (-\delta_{\omega_i,\omega_j}) = \prod_{1\le i<j\le 2} (1-\delta_{\omega_i,\omega_j}),$$
$$\sum_{\Gamma\in\mathscr{G}''_{0,2}} \prod_{ij\in\Gamma} (-\delta_{\omega_i,\omega_j}) = \sum_{\Gamma\in\mathscr{C}_{0,1}} \prod_{ij\in\Gamma} (-\delta_{\omega_i,\omega_j}) \prod_{1\le i<j\le 2} (1-\delta_{\omega_i,\omega_j}),$$
$$\sum_{\Gamma\in\mathscr{G}'''_{0,2}} \prod_{ij\in\Gamma} (-\delta_{\omega_i,\omega_j}) = \sum_{\Gamma\in\mathscr{C}_{0,2}} \prod_{ij\in\Gamma} (-\delta_{\omega_i,\omega_j}).$$

$$\therefore \prod_{0\le i<j\le 2} (1-\delta_{\omega_i,\omega_j}) = \prod_{1\le i<j\le 2} (1-\delta_{\omega_i,\omega_j}) + \sum_{n=1}^{2} \sum_{\Gamma\in\mathscr{C}_{0,n}} \prod_{ij\in\Gamma} (-\delta_{\omega_i,\omega_j}) \prod_{n\le i<j\le 2} (1-\delta_{\omega_i,\omega_j}).$$

$$\therefore G_p(o,x)-\delta_{o,x} = \sum_{y\in\mathbb{Z}^d} pD(o,y)\,G_p(y,x) + \sum_{y\in\mathbb{Z}^d} \pi_p(o,y)\,G_p(y,x).$$

2 Lace expansion for the connected part $\sum_{\Gamma \in \mathscr{C}_{0,n}} \prod_{ij \in \Gamma} (-\delta_{\omega_i,\omega_j})$.

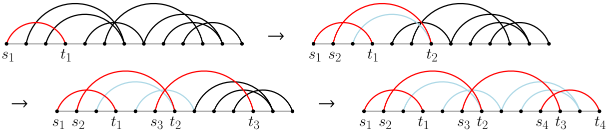
- $\mathscr{L}_{0,n}$: the set of laces (= minimally connected graphs) from 0 to n.

  Construction of a lace $\mathsf{L}_\Gamma = \{s_j t_j\} \in \mathscr{L}_{0,n}$ from $\Gamma \in \mathscr{C}_{0,n}$.

$$t_1 = \max\{t : 0t \in \Gamma\}, \qquad s_1 = 0,$$

$$t_{j+1} = \max\{t : {}^{\exists}s < t_j,\ st \in \Gamma\}, \qquad s_{j+1} = \min\{s : st_{j+1} \in \Gamma\}.$$

E.g.,



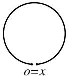
- Resummation over compatible edges $\Delta(L)$ (to partially restore self-avoidance).

$$\sum_{\Gamma \in \mathscr{C}_{0,n}} \prod_{ij \in \Gamma} (-\delta_{\omega_i,\omega_j}) = \sum_{L \in \mathscr{L}_{0,n}} \prod_{st \in L} (-\delta_{\omega_s,\omega_t}) \sum_{\substack{\Gamma \in \mathscr{C}_{0,n}: \\ \mathsf{L}_\Gamma = L}} \prod_{ij \in \Gamma \setminus L} (-\delta_{\omega_i,\omega_j})$$

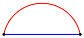
$$= \sum_{L \in \mathscr{L}_{0,n}} \prod_{st \in L} (-\delta_{\omega_s,\omega_t}) \underbrace{\prod_{ij \in \Delta(L)} (1 - \delta_{\omega_i,\omega_j})}_{\text{restored self-avoidance}}.$$

3 Diagrammatic bounds on $\pi_p^{(j)}(x)$ (assuming translation invariance).

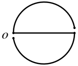
$$\pi_p^{(1)}(x) = \sum_{\substack{\omega:o\to x\\(|\omega|\ge 1)}} p^{|\omega|} \prod_{n=1}^{|\omega|} D(\omega_{n-1},\omega_n)\; \underbrace{\delta_{\omega_0,\omega_{|\omega|}} \prod_{i j \ne 0|\omega|} (1-\delta_{\omega_i,\omega_j})} =$$



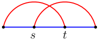
$$\delta_{o,x} \le (pD * G_p)(o)\,\delta_{o,x},$$



$$\pi_p^{(2)}(x) = \sum_{\substack{\omega:o\to x\\(|\omega|\ge 1)}} p^{|\omega|} \prod_{n=1}^{|\omega|} D(\omega_{n-1},\omega_n)\; \underbrace{\sum_{0<s<t<|\omega|} \delta_{\omega_0,\omega_t}\delta_{\omega_s,\omega_{|\omega|}} \prod_{i j\ne 0t,s|\omega|} (1-\delta_{\omega_i,\omega_j})} =$$

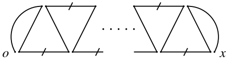


$$\le G_p(x)^3.$$



For $j \ge 3$, if $\sup_x (|x| \vee 1)^{d-2} G_p(x)$ is bounded uniformly in $p < p_c$,
eventually proven by continuity & bootstrapping

$$\pi_p^{(j)}(x) =$$



$$\lesssim B_p^{(j-3)\vee 0} \frac{O(j^{2+3(d-2)})}{(|x| \vee 1)^{3(d-2)}},$$

where $B_p = \sum_{x\ne o} G_p(x)^2$.

4 Why $d > 4$.

$$B_p \lesssim \sum_{x\ne o} G_1^{\mathrm{RW}}(x)^2 = \int_{[-\pi,\pi]^d} \left(\frac{\hat{D}(k)}{1-\hat{D}(k)}\right)^2 \frac{\mathrm{d}^d k}{(2\pi)^d} = O\!\left(\tfrac{1}{d-4}\right),$$

where $\hat{D}(k) = \frac{1}{d}\sum_{j=1}^{d} \cos k_j$.

$$\therefore \sum_{x\ne o} |x|^2 \pi_p(x) \lesssim \sum_{x\ne o} |x|^{2-3(d-2)} = \sum_{x\ne o} |x|^{-d-2(d-4)} < \infty.$$

- For SAW, percolation, oriented percolation (OP), the contact process (CP) in $d > d_c$,

$$p_c \underset{L\uparrow\infty}{=} 1 + CL^{-d} + O(L^{-d-1}), \qquad C = \begin{cases} \sum_{n=2}^\infty U^{*n}(o) & \text{[SAW \& CP]}, \\ \frac{1}{2}\sum_{n=2}^\infty U^{*2n}(o) & \text{[OP]}, \\ U^{*2}(o) + \sum_{n=3}^\infty \frac{n+1}{2} U^{*n}(o) & \text{[percolation]}. \end{cases}$$

- For LT/LA in $d > 8$,

$$p_c \underset{L\uparrow\infty}{=} \frac{1}{e} + CL^{-d} + O(L^{-d-1}), \qquad C = \sum_{n=2}^\infty \frac{n+1}{2e} U^{*n}(o) + \begin{cases} 0 & \text{[LT]}, \\ -\frac{1}{2e^2}\sum_{n=3}^\infty U^{*n}(o) & \text{[LA]}. \end{cases}$$

- Key elements for the proof:
  - Lace expansion for the 2-point function.
  - Translation invariance for idntifying p_c and bounding the expansion coefficients.

- Ongoing work with Chino & Fukushima-Kimura & Kitagawa on SAW on random conductors $X = \{X_b\}_{b\in E_L}$ (i.i.d. copies of X, nonnegative, mean 1, $\lambda_\beta = \mathbb{E}[e^{-\beta X}]$):

$$\hat\chi_{X,p}(x) = \sum_{\substack{\omega:x\to\bullet \\ \text{(SAW)}}} \prod_{j=1}^{|\omega|} p e^{-\beta X_{\omega_{j-1},\omega_j}} D(\omega_{j-1},\omega_j), \qquad \mathbb{E}[\hat\chi_{X,p}(x)] = \sum_{\substack{\omega:x\to\bullet \\ \text{(SAW)}}} \left(\frac{p\lambda_\beta}{\Delta}\right)^{|\omega|},$$

$$\hat p^X_{\text{quenched}}(x) = \sup\{p : \hat\chi_{X,p}(x) < \infty\}, \qquad p_{\text{annealed}} = \sup\{p : \mathbb{E}[\hat\chi_{X,p}(x)] < \infty\} = \frac{p_c}{\lambda_\beta}.$$

- Chino & Sakai (2016): $\hat p^X_{\text{quenched}}(x)$ is a.s. an x-independent constant $\in [p_{\text{annealed}}, p_c e^{\beta}]$.

Open problem: $\hat p^X_{\text{quenched}} = p_{\text{annealed}}$ in $d > 4$, or $\hat p^X_{\text{quenched}} \neq p_{\text{annealed}}$ as long as $d < \infty$?

- For SAW, percolation, oriented percolation (OP), the contact process (CP) in $d > d_c$,

$$p_c \underset{L\uparrow\infty}{=} 1 + CL^{-d} + O(L^{-d-1}), \qquad C = \begin{cases} \sum_{n=2}^{\infty} U^{*n}(o) & \text{[SAW \& CP]}, \\ \frac{1}{2}\sum_{n=2}^{\infty} U^{*2n}(o) & \text{[OP]}, \\ U^{*2}(o) + \sum_{n=3}^{\infty} \frac{n+1}{2} U^{*n}(o) & \text{[percolation]}. \end{cases}$$

- For LT/LA in $d > 8$,

$$p_c \underset{L\uparrow\infty}{=} \frac{1}{e} + CL^{-d} + O(L^{-d-1}), \qquad C = \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o) + \begin{cases} 0 & \text{[LT]}, \\ -\frac{1}{2e^2}\sum_{n=3}^{\infty} U^{*n}(o) & \text{[LA]}. \end{cases}$$

- Key elements for the proof:
  - Lace expansion for the 2-point function.
  - Translation invariance for idntifying p_c and bounding the expansion coefficients.
- Ongoing work with Chino & Fukushima-Kimura & Kitagawa on SAW on random conductors $\boldsymbol{X} = \{X_b\}_{b\in E_L}$ (i.i.d. copies of X, nonnegative, mean 1, $\lambda_\beta = \mathbb{E}[e^{-\beta X}]$):

$$\hat{\chi}_{\boldsymbol{X},p}(x) = \sum_{\substack{\omega:x\to\bullet \\ \text{(SAW)}}} \prod_{j=1}^{|\omega|} pe^{-\beta X_{\omega_{j-1},\omega_j}} D(\omega_{j-1},\omega_j), \qquad \mathbb{E}[\hat{\chi}_{\boldsymbol{X},p}(x)] = \sum_{\substack{\omega:x\to\bullet \\ \text{(SAW)}}} \left(\frac{p\lambda_\beta}{\Delta}\right)^{|\omega|},$$

$$\hat{p}^{\boldsymbol{X}}_{\text{quenched}}(x) = \sup\{p : \hat{\chi}_{\boldsymbol{X},p}(x) < \infty\}, \qquad p_{\text{annealed}} = \sup\{p : \mathbb{E}[\hat{\chi}_{\boldsymbol{X},p}(x)] < \infty\} = \frac{p_c}{\lambda_\beta}.$$

- Chino & Sakai (2016): $\hat{p}^{\boldsymbol{X}}_{\text{quenched}}(x)$ is a.s. an x-independent constant $\in [p_{\text{annealed}}, p_c e^{\beta}]$.

Open problem: $\hat{p}^{\boldsymbol{X}}_{\text{quenched}} = p_{\text{annealed}}$ in $d > 4$, or $\hat{p}^{\boldsymbol{X}}_{\text{quenched}} \neq p_{\text{annealed}}$ as long as $d < \infty$?