where

$$M = \int_a^b \int_{f_1(x)}^{f_2(x)} \rho \, dy \, dx$$

Thus,

$$M\,\bar{x} = \int_0^1 \int_{x^2}^{\sqrt{2x-x^2}} x\, xy \, dy \, dx = \int_0^1 x^2 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{2-x^2}} dx = \int_0^1 x^2 \frac{1}{2}[2 - x^2 - x^4]dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{10} - \frac{x^7}{14}\right]_0^1 = = \frac{1}{3} - \frac{1}{10} \frac{1}{14} = \frac{17}{105}$$

$$M\bar{y} = \int_0^1 \int_{x^2}^{\sqrt{2x-5}} yx \, dy \, dx = -\frac{13}{120} + 4\frac{\sqrt{2}}{15}$$

9.4. Find the volume of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Required volume = 8 times volume of region shown in Figure 9.9

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} z \, dy \, dx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} = \sqrt{a^2 - x^2} \, dy \, dx$$

$$= 8 \int_{x=0}^{a} (a^2 - x^2) \, dx = \frac{16a^3}{3}$$

As an aid in setting up this integral, note that $z\,dy\,dx$ corresponds to the volume of a column such as shown darkly shaded in Figure 9.9. Keeping x constant and integrating with respect to y from $y = 0$ to $y = \sqrt{a^2 - x^2}$ corresponds to adding the volumes of all such columns in a slab parallel to the yz plane, thus giving the volume of this slab. Finally, integrating with respect to x from $x = 0$ to $x = a$ corresponds to adding the volumes of all such slabs in the region, thus giving the required volume.

9.5. Find the volume of the region bounded by $z = x + y$, $z = 6$, $x = 0$, $y = 0$, $z = 0$.

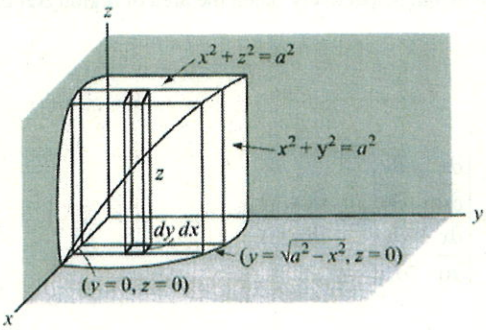

Figure 9.9

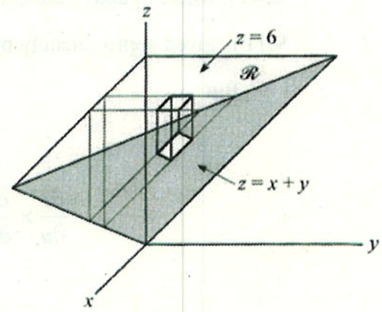

Figure 9.10

Required volume = volume of region shown in Figure 9.10

$$= \int_{x=0}^{6} \int_{y=0}^{6-x} \{6 - (x + y)\} dy \, dx$$

$$= \int_{x=0}^{6} (6 - x)\, y - \frac{1}{2} y^2 \Big|_{y=0}^{6-x} dx$$

$$= \int_{x=0}^{6} \frac{1}{2} (6 - x)^2 \, dx = 36$$

In this case the volume of a typical column (shown darkly shaded) corresponds to $\{6-(x+y)\}\,dy\,dx$. The limits of integration are then obtained by integrating over the region $\mathfrak{R}$ of Figure 9.10. Keeping x constant and integrating with respect to y from $y = 0$ to $y = 6 - x$ (obtained from $z = 6$ and $z = x + y$) corresponds to summing all columns in a slab parallel to the yz plane. Finally, integrating with respect to x from $x = 0$ to $x = 6$ corresponds to adding the volumes of all such slabs and gives the required volume.

Transformation of double integrals

9.6. Justify Equation (9), Page 225, for changing variables in a double integral.

In rectangular coordinates, the double integral of $F(x, y)$ over the region $\mathfrak{R}$ (shaded in Figure 9.11) is $\iint_{\mathfrak{R}} (F(x, y)\,dx\,dy$. We can also evaluate this double integral by considering a grid formed by a family of u and υ curvilinear coordinate curves constructed on the region $\mathfrak{R}$, as shown in Figure 9.11.

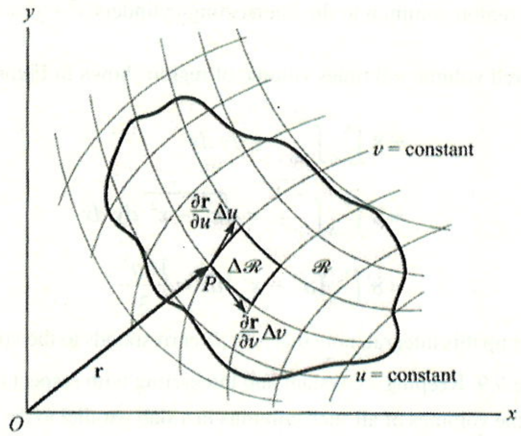

Figure 9.11

Let P be any point with coordinates (x, y) or (u, υ), where $x = f(u, \upsilon)$ and $y = g(u, \upsilon)$. Then the vector $\mathbf{r}$ from O to p is given by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = f(u, \upsilon)\mathbf{i} + g(u, \upsilon)\mathbf{j}$. The tangent vectors to the coordinate curves $u = c_1$ and $\upsilon = c_2$, where c_1 and c_2 are constants, are $\partial\mathbf{r}/\partial\upsilon$ and $\partial\mathbf{r}/\partial u$, respectively. Then the area of region $\Delta\mathfrak{R}$ of Figure 9.11 is given approximately by $\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial \upsilon}\right| \Delta u\, \Delta\upsilon$.

But

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial \upsilon} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial \upsilon} & \frac{\partial y}{\partial \upsilon} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial \upsilon} & \frac{\partial y}{\partial \upsilon} \end{vmatrix} \mathbf{k} = \frac{\partial(x, y)}{\partial(u, \upsilon)} \mathbf{k}$$

so that

$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial \upsilon}\right| \Delta u\, \Delta\upsilon = \left|\frac{\partial(x, y)}{\partial(u, \upsilon)}\right| \Delta u\, \Delta\upsilon$$

The double integral is the limit of the sum

$$\sum F\{f(u,\upsilon), g(u,\upsilon)\} \left|\frac{\partial(x, y)}{\partial(u, \upsilon)}\right| \Delta u\, \Delta\upsilon$$

taken over the entire region $\mathfrak{R}$. An investigation reveals that this limit is

$$\iint_{\mathfrak{R}} F\{f(u,\upsilon), g(u,\upsilon)\} = \left|\frac{\partial(x, y)}{\partial(u, \upsilon)}\right| du\, d\upsilon$$

where $\Re'$ is the region in the $u\upsilon$ plane into which the region $\Re$ is mapped under the transformation $x = f(u, \upsilon)$, $y = g(u, \upsilon)$.

Another method of justifying this method of change of variables makes use of line integrals and Green's theorem in the plane (see Problem 10.32).

9.7. If $u = x^2 - y^2$ and $\upsilon = 2xy$, find $\partial(u, \upsilon)$ in terms of u and υ.

$$\frac{\partial(u,\upsilon)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ \upsilon_x & \upsilon_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2)$$

From the identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$, we have

$$(x^2+y^2)^2 = u^2 + \upsilon^2 \qquad \text{and} \qquad x^2 + y^2 = \sqrt{u^2+\upsilon^2}$$

Then, by Problem 6.43,

$$\frac{\partial(x,y)}{\partial(u,\upsilon)} = \frac{1}{\partial(u,\upsilon)/\partial(x,y)} = \frac{1}{4(x^2+y^2)} = \frac{1}{4\sqrt{u^2+\upsilon^2}}$$

Another method: Solve the given equations for x and y in terms of u and υ and find the Jacobian directly.

9.8. Find the polar moment of inertia of the region in the xy plane bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$, $xy = 4$, assuming unit density.

Under the transformation $x^2 - y^2 = u$, $2xy = \upsilon$, the required region $\Re$ in the xy plane, shaded in Figure 9.12(a), is mapped into region $\Re'$ of the $u\upsilon$ plane, shaded in Figure 9.12(b). Then:

$$\text{Required polar moment of inertia} = \iint_{\Re} (x^2+y^2)\,dx\,dy = \iint_{\Re'} (x^2+y^2) \left|\frac{\partial(x,y)}{\partial(u,\upsilon)}\right| du\,d\upsilon$$

$$= \iint_{\Re'} \sqrt{u^2+\upsilon^2}\,\frac{du\,d\upsilon}{4\sqrt{u^2+\upsilon^2}} = \frac{1}{4}\int_{u=1}^{9}\int_{\upsilon=4}^{8} du\,d\upsilon = 8$$

where we have used the results of Problem 9.7.

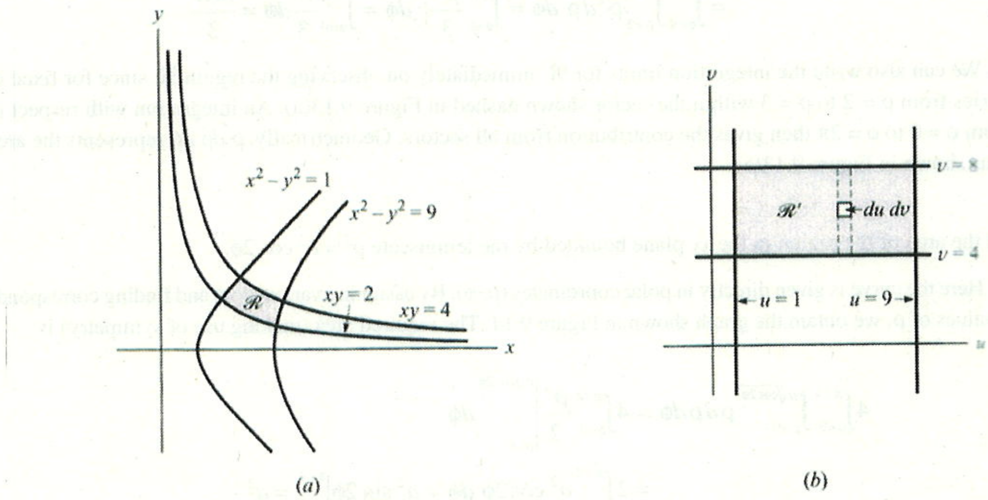

Figure 9.12

Note that the limits of integration for the region $\Re'$ can be constructed directly from the region $\Re$ in the xy plane without actually constructing the region $\Re'$. In such case we use a grid, as in Problem 9.6. The coordinates (u, υ) are curvilinear coordinates, in this case called *hyperbolic coordinates*.

9.9 Evaluate $\iint_{\Re} \sqrt{x^2 + y^2}\, dx\, dy$, where $\Re$ is the region in the xy plane bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

The presence of $x^2 + y^2$ suggests the use of polar coordinates (ρ, ϕ), where $x = \rho \cos \phi$, $y = \rho \sin \phi$ (see Problem 6.39). Under this transformation the region $\Re$ [Figure 9.13(*a*)] is mapped into the region $\Re'$ [Figure 9.13(*b*)].

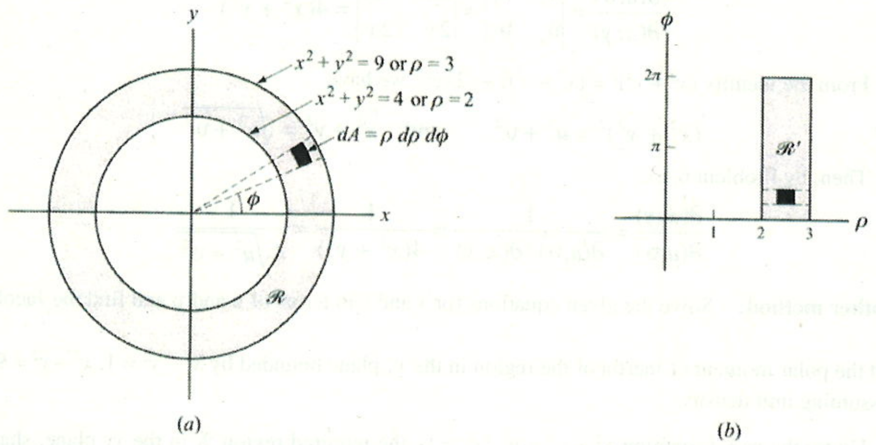

Figure 9.13

Since $\dfrac{\partial(x, y)}{\partial(\rho, \phi)} = \rho$, it follows that

$$\iint_{\Re} \sqrt{x^2 + y^2}\; dx\, dy = \iint_{\Re'} \sqrt{x^2 + y^2} \left|\frac{\partial(x, y)}{\partial(\rho,\phi)}\right| d\rho\; d\phi = \iint_{\Re'} \rho \cdot \rho d\; d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\rho=2}^{3} \rho^2 d\rho\; d\phi = \int_{\phi=0}^{2\pi} \frac{\rho^3}{3}\Big|_2^3 d\phi = \int_{\phi=0}^{2\pi} \frac{19}{3} d\phi = \frac{38\pi}{3}$$

We can also write the integration limits for $\Re'$ immediately on observing the region $\Re$, since for fixed ϕ. ρ varies from $\rho = 2$ to $\rho = 3$ within the sector shown dashed in Figure 9.13(a). An integration with respect to ϕ from $\phi = 0$ to $\phi = 2\pi$ then gives the contribution from all sectors. Geometrically, $\rho\; d\rho\; d\phi$ represents the area dA, as shown in Figure 9.13(a).

9.10. Find the area of the region in the xy plane bounded by the lemniscate $\rho^2 = a^2 \cos 2\phi$.

Here the curve is given directly in polar coordinates (ρ, ϕ). By assigning various to ϕ and finding corresponding values of ρ, we obtain the graph shown in Figure 9.14. The required area (making use of symmetry) is

$$4\int_{\phi=0}^{\pi/4} \int_{\rho=0}^{a\sqrt{\cos 2\phi}} \rho d\rho d\phi = 4\int_{\phi=0}^{\pi/4} \frac{\rho^3}{2}\Bigg|_{\rho=0}^{a\sqrt{\cos 2\phi}} d\phi$$

$$= 2\int_{\phi=0}^{\pi/4} a^2 \cos 2\phi\; d\phi = a^2 \sin 2\phi\Big|_{\phi=0}^{\pi/4} = a^2$$

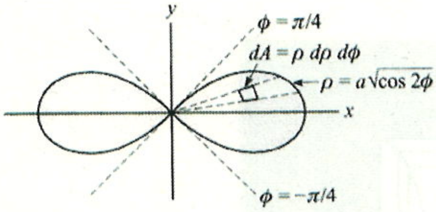

Figure 9.14

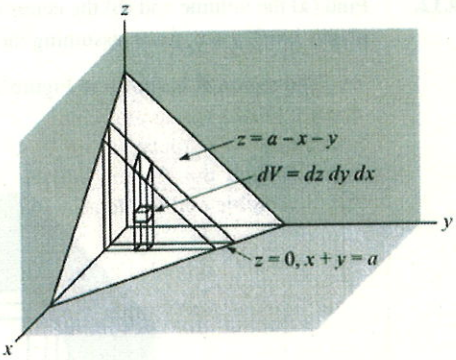

Figure 9.15

Triple integrals

9.11. (a) Sketch the three-dimensional region $\Re$ bounded by $x + y + z = a$ $(a > 0)$, $x = 0$, $y = 0$, $z = 0$. (b) Give a physical interpretation to

$$\iiint_{\Re} (x^2 + y^2 + z^2)dx\,dy\,dz$$

(c) Evaluate the triple integral in (b).

(a) The required region $\Re$ is shown in Figure 9.15.

(b) Since $x^2 + y^2 + z^2$ is the square of the distance from any point (x, y, z) to $(0, 0, 0)$, we can consider the triple integral as representing the *polar moment of inertia* (i.e., moment of inertia with respect to the origin) of the region $\Re$ (assuming unit density).

We can also consider the triple integral as representing the *mass* of the region if the density varies as $x^2 + y^2 + z^2$.

(c) The triple integral can be expressed as the iterated integral

$$\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x^2 + y^2 + z^2)dz\,dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x} x^2z + y^2z + \frac{z^3}{3}\Bigg|_{z=0}^{a-x-y} dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left\{x^2(a-x) - x^2y + (a-x)y^2 - y^3 + \frac{(a-x-y)^3}{3}\right\}dy\,dx$$

$$= \int_{x=0}^{a} x^2(a-x)y - \frac{x^2y^2}{2} + \frac{(a-x)y^3}{3} - \frac{y^4}{4} - \frac{(a-x-y)^4}{12}\Bigg|_{y=0}^{a-x} dx$$

$$= \int_0^a\left\{x^2(a-x)^2 - \frac{x^2(a-x)^2}{2} + \frac{(a-x)^4}{3} - \frac{(a-x)^4}{4} + \frac{(a-x)^4}{12}\right\}dx$$

$$= \int_0^a\left\{\frac{x^2(a-x)^2}{2} + \frac{(a-x)^4}{6}\right\}dx = \frac{a^5}{20}$$

The integration with respect to z (keeping x and y constant) from $z = 0$ to $z = a - x - y$ corresponds to summing the polar moments of inertia (or masses) corresponding to each cube in a vertical column. The subsequent integration with respect to y from $y = 0$ to $y = a - x$ (keeping x constant) corresponds to addition of contributions from all vertical columns contained in a slab parallel to the yz plane. Finally, integration with respect to x from $x = 0$ to $x = a$ adds up contributions from all slabs parallel to the yz plane.

Although this integration has been accomplished in the order z, y, x, any other order is is clearly possible and the final answer should be the same.

9.12. Find (a) the volume and (b) the centroid of the region $\mathfrak{R}$ bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 6$, $z = 0$, assuming the density to be a constant σ.

The region $\mathfrak{R}$ is shown in Figure 9.16.

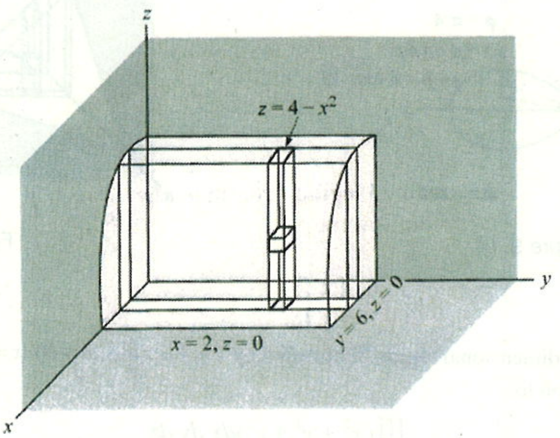

Figure 9.16

(a) Required volume $= \iiint dx\, dy\, dz$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} dz\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{6} (4 - x^2) dy\, dx$$

$$= \int_{x=0}^{2} (4 - x^2) y\Big|_{y=0}^{6} dx$$

$$= \int_{x=0}^{2} (2 - 6x^2)\, dx = 32$$

(b) Total mass $= \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} \sigma\, dz\, dy\, dx = 32\sigma$ by (a), since σ is constant. Then

$$\bar{x} = \frac{\text{Total moment about } yz \text{ plane}}{\text{Total mass}} = \frac{\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} \sigma x\, dz\, dy\, dx}{\text{Total mass}} = \frac{24}{32\sigma} = \frac{3}{4}$$

$$\bar{y} = \frac{\text{Total moment about } xz \text{ plane}}{\text{Total mass}} = \frac{\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} \sigma y\, dz\, dy\, dx}{\text{Total mass}} = \frac{96\sigma}{32\sigma} = 3$$

$$\bar{z} = \frac{\text{Total moment about } xy \text{ plane}}{\text{Total mass}} = \frac{\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} \sigma z\, dz\, dy\, dx}{\text{Total mass}} = \frac{256\sigma/5}{32\sigma} = \frac{8}{5}$$

Thus, the centroid has coordinates (3/4, 3, 8/5).

Note that the value for $\bar{y}$ could have been predicted because of symmetry.

Transformation of triple integrals

9.13. Justify Equation (11), Page 225, for changing variables in a triple integral.

By analogy with Problem 9.6, we construct a grid of curvilinear coordinate surfaces which subdivide the region $\mathfrak{R}$ into subregions, a typical one of which is $\Delta\mathfrak{R}$ (see Figure 9.17).

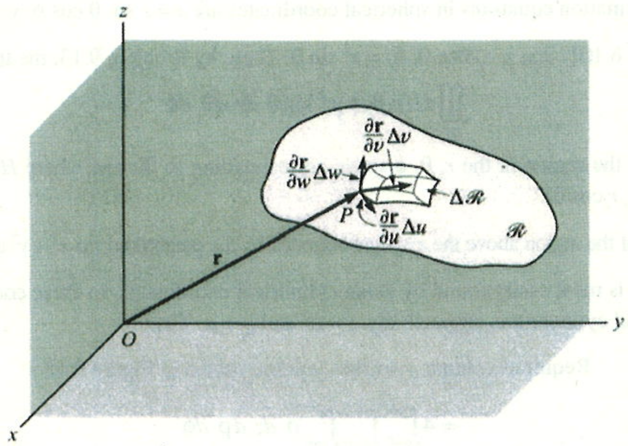

Figure 9.17

The vector **r** from the origin O to point P is

$$r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = f(u, \upsilon, w)\mathbf{i} + g(u, \upsilon, w)\mathbf{j} + h(u, \upsilon, w)\mathbf{k}$$

assuming that the transformation equations are $x = f(u, \upsilon, w)$, $y = g(u, \upsilon, w)$, and $z = h(u, \upsilon, w)$.

Tangent vectors to the coordinate curves corresponding to the intersection of pairs of coordinate surfaces are given by $\partial\mathbf{r}/\partial u$, $\partial\mathbf{r}/\partial\upsilon$, $\partial\mathbf{r}/\partial w$. Then the volume of the region $\Delta\Re$ of Figure 9.17 is given approximately by

$$\left|\frac{\partial \mathbf{r}}{\partial u}\cdot\frac{\partial \mathbf{r}}{\partial \upsilon}\times\frac{\partial \mathbf{r}}{\partial w}\right|\Delta u\,\Delta\upsilon\,\Delta w = \left|\frac{\partial(x, y, z)}{\partial(u, \upsilon, w)}\right|\Delta u\,\Delta\upsilon\,\Delta w$$

The triple integral of $F(x, y, z)$ over the region is the limit of the sum

$$\sum F\{f(u,\upsilon, w), g(u,\upsilon, w), h(u,\upsilon, w)\}\left|\frac{\partial(x, y, z)}{\partial(u,\upsilon, w)}\right|\Delta u\Delta\upsilon\Delta w$$

An investigation reveals that this limit is

$$\iiint F\{f(u,\upsilon, w), g(u,\upsilon, w), h(u,\upsilon, w)\}\left|\frac{\partial(x, y, z)}{\partial(u,\upsilon, w)}\right| du\,d\upsilon\,dw$$

where $\Re'$ is the region in the $u\upsilon w$ space into which the region $\Re$ is mapped under the transformation.

Another method for justifying this change of variables in triple integrals makes use of Stokes's theorem (see Problem 10.84).

9.14. What is the mass of a circular cylindrical body represented by the region $0 \leqq \rho \leqq c$, $0 \leqq \phi \leqq 2\pi$, $0 \leqq z \leqq h$, and with the density function $\mu = z\sin^2\phi$?

$$M = \int_0^h \int_0^{2\pi} \int_0^c z\sin^2\phi\rho\; d\rho\; d\phi\; dz = \pi$$

9.15. Use spherical coordinates to calculate the volume of a sphere of radius a.

$$V = 8\int_0^a \int_0^{\pi/2} \int_0^{\pi/2} a^2 \sin\theta\, dr\; d\theta\; d\phi = \frac{4}{3}\pi a^3$$

9.16. Express $\iiint_{\Re'} F(x, y, z)\; dx\; dy\; dz$ in (a) cylindrical and (b) spherical coordinates.

(a) The transformation equations in cylindrical coordinates are $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$.

As in Problem 6.39, $\partial(x, y, z)/\partial(\rho, \phi, z) = \rho$. Then, by Problem 9.13, the triple integral becomes

$$\iiint_{\Re'} G(\rho,\phi, z)\rho\, d\rho\; d\phi\; dz$$

where $\Re'$ is the region in the ρ, ϕ, z space corresponding to $\Re$ and where $G(\rho, \phi, z \equiv F(\rho\cos\phi, \rho\sin\phi, z)$.

(b) The transformation equations in spherical coordinates are $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$.

By Problem 6.101, $\partial(x, y, z)/\partial(r, \theta, \phi) = r^2 \sin\theta$. Then, by Problem 9.13, the triple integral becomes

$$\iiint_{\Re'} H(r,\theta,\phi) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

where $\Re'$ is the region in the r, θ, ϕ space corresponding to $\Re$, and where $H(r, \theta, \phi) \equiv F(r \sin\theta \cos\phi, r\sin\theta\sin\phi, r\cos\theta)$.

9.17. Find the volume of the region above the xy plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$.

The volume is most easily found by using cylindrical coordinates. In these coordinates the equations for the paraboloid and cylinder are, respectively, $z = \rho^2$ and $\rho = a$. Then

Required volume = 4 times volume shown in Figure 9.18

$$= 4\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{a}\int_{z=0}^{\rho^2} \rho \, dz \, d\rho \, d\phi$$

$$= 4\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{a} \rho^3 d\rho \, d\phi$$

$$= 4\int_{hi=0}^{\pi/2} \frac{\rho^4}{4}\Big|_{=0}^{a} d\phi = \frac{\pi}{2}a^4$$

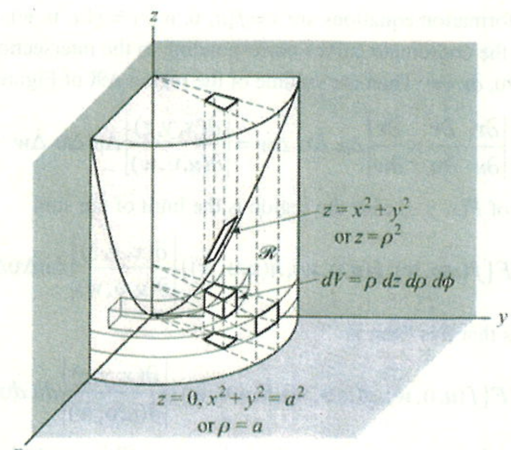

Figure 9.18

The integration with respect to z (keeping ρ and ϕ constant) from $z = 0$ to $z = \rho^2$ corresponds to summing the cubical volumes (indicated by dV) in a vertical column extending from the xy plane to the paraboloid. The subsequent integration with respect to ρ (keeping ϕ constant) from $\rho = 0$ to $\rho = a$ corresponds to addition of volumes of all columns in the wedge-shaped region. Finally, integration with respect to ϕ corresponds to adding volumes of all such wedge-shaped regions.

The integration can also be performed in other orders to yield the same result.

We can also set up the integral by determining the region $\Re'$ in ρ, ϕ, z space into which $\Re$ is mapped by the cylindrical coordinate transformation.

9.18. (a) Find the moment of inertia about the z axis of the region in Problem 9.17, assuming that the density is the constant σ. (b) Find the radius of gyration.

(a) The moment of inertia about the z axis is

$$I_z = 4\int_{\phi_0}^{\pi/2}\int_{\rho=0}^{a}\int_{z=0}^{\rho^2} \rho^2 \sigma\rho \, dz \, d\rho \, d\phi$$

$$= 4\sigma\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{a} \rho^5 d\rho \, d\phi = 4\sigma\int_{\phi=0}^{\pi/2} \frac{\rho^6}{6}\Big|_{\rho=0}^{a} d\phi = \frac{\pi a^6 \sigma}{3}$$

The result can be expressed in terms of the mass M of the region, since, by Problem 9.17,

$$M = \text{volume} \times \text{desnity} = \frac{\pi}{2} a^4 \sigma \quad \text{so that} \quad I_z = \frac{\pi a^6 \sigma}{3} = \frac{\pi a^6}{3} \cdot \frac{2M}{\pi a^4} = \frac{2}{3} M a^2$$

Note that in setting up the integral for I_z we can think of $\sigma\, \rho\, dz\, d\rho\, d\phi\, dz\, d\rho\, d\phi$ as being the mass of the cubical volume element, $\rho^2 \sigma\, \rho\, dz\, d\rho\, d\phi$ as the moment of inertia of this mass with respect to the z axis, and $\iiint_{\Re'} \rho^2 \sigma \rho\, dz\, d\rho\, d\phi$ as the total moment of inertia about the z axis. The limits of integration are determined as in Problem 9.17.

(b) The radius of gyration is the value K such that $MK^2 = \frac{2}{3} Ma^2$; i.e., $K^2 = \frac{2}{3} a^2$ or $K = a\sqrt{2/3}$.

The physical significance of K is that if all the mass M were concentrated in a thin cylindrical shell of radius K, then the moment of inertia of this shell about the axis of the cylinder would be I_z.

9.19. (a) Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $z^2 \sin^2 \alpha = (x^2 + y^2) \cos^2 \alpha$, where α is a constant such that $0 \leqq \alpha \leqq \pi$. (b) From the result in (a), find the volume of a sphere of radius a.

In spherical coordinates the equation of the sphere is $r = a$ and that of the cone is $\theta = \alpha$. This can be seen directly or by using the transformation equations $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$. For example, $z^2 \sin^2 \alpha = (x^2 + y^2) \cos^2 \alpha$ becomes, on using these equations, $r^2 \cos^2\theta \sin^2\alpha = (r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi) \cos^2\alpha$, i.e., $r^2 \cos^2\theta \sin^2\alpha = r^2 \sin^2\theta \cos^2\alpha$, from which $\tan\theta = \pm \tan\alpha$ and so $\theta = \alpha$ or $\theta = \pi - \alpha$. It is sufficient to consider one of these—say, $\theta = \alpha$.

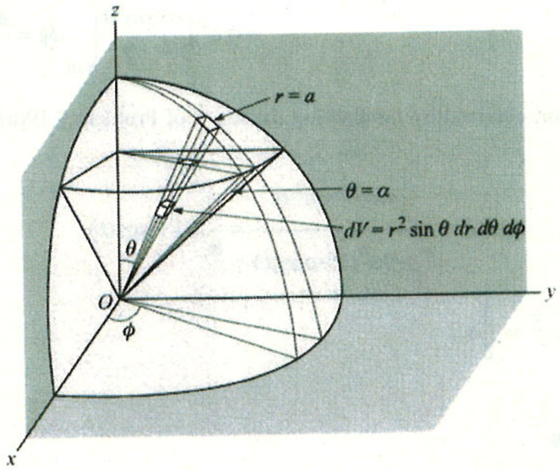

Figure 9.19

(a) Required volume = 4 times volume (shaded) in Figure 9.19

$$= 4\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\alpha} \int_{r=0}^{\rho^2} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 4\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\alpha} \frac{r^3}{3} \sin\theta \Big|_{r=0}^{\alpha} d\theta\, d\phi$$

$$= \frac{4a^3}{3} \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\alpha} \sin\theta\, d\theta\, d\phi$$

$$= \frac{4a^3}{3} \int_{\phi=0}^{\pi/2} -\cos\theta \Big|_{\theta=0}^{\alpha} d\phi$$

$$= \frac{2\pi a^3}{3} (1 - \cos\alpha)$$

The integration with respect to r (keeping θ and ϕ constant) from $r = 0$ to $r = a$ corresponds to summing the volumes of all cubical elements (such as indicated by dV) in a column extending from $r = 0$ to $r = a$. The subsequent integration with respect to θ (keeping ϕ constant) from $\theta = 0$ to $\theta = \pi/4$ corresponds to summing the volumes of all columns in the wedge-shaped region. Finally, integration with respect to ϕ corresponds to adding volumes of all such wedge-shaped regions.

(b) Letting $\alpha = \pi$, the volume of the sphere thus obtained is

$$\frac{2\pi a^3}{3}(1-\cos\pi) = \frac{4}{3}\pi a^3$$

9.20. (a) Find the centroid of the region in Problem 9.19. (b) Use the result in (a) to find the centroid of a hemisphere.

(a) The centroid ($\bar{x}$, $\bar{y}$, $\bar{z}$) is, due to symmetry, given by $\bar{x} = \bar{y} = 0$ and

$$\bar{z} = \frac{\text{Total moment about } xy \text{ plane}}{\text{Total mass}} = \frac{\iiint z\sigma\, dV}{\iiint \sigma\, dV}$$

Since $z = r\cos\theta$ and σ is constant, the numerator is

$$4\sigma\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\alpha}\int_{r=0}^{\rho^2} r\cos\theta\cdot r^2\sin\theta\, dr\, d\theta\, d\phi = 4\sigma\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\alpha}\left.\frac{r^4}{4}\right|_{r=0}^{a}\sin\theta\cos\theta\, d\theta\, d\phi$$

$$= \sigma a^4\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\alpha}\sin\theta\cos\theta\, d\theta\, d\phi$$

$$= \sigma a^4\int_{\phi=0}^{\pi/2}\left.\frac{sin^2\theta}{2}\right|_{\theta=0}^{a} d\phi = \frac{\pi\sigma a^4\sin^2\alpha}{4}$$

The denominator, obtained by multiplying the result of Problem 9.19(a) by σ, is $\frac{2}{3}\pi\sigma a^3$ $(1 - \cos\alpha)$. Then

$$\bar{z} = \frac{\frac{1}{4}\pi\sigma a^4\sin^2\alpha}{\frac{2}{3}\pi\sigma a^3(1-\cos\alpha)} = \frac{3}{8}a(1+\cos\alpha).$$

(b) Letting $\alpha = \pi/2$, $\bar{z} = \frac{3}{8}$a.

Miscellaneous problems

9.21. Prove that (a) $\int_0^1\left\{\int_0^1\frac{x-y}{(x+y)^3}dy\right\}dx = \frac{1}{2}$ and (b) $\int_0^1\left\{\int_0^1\frac{x-y}{(x+y)^3}dx\right\}dy = -\frac{1}{2}$,

(a) $$\int_0^1\left\{\int_0^1\frac{x-y}{(x+y)^3}dy\right\}dx = \int_0^1\left\{\int_0^1\frac{2x-(x+y)}{(x+y)^3}dy\right\}dx$$

$$= \int_0^1\left\{\int_0^1\left(\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right)dy\right\}dx$$

$$= \int_0^1\left.\left(\frac{-x}{(x+y)^2} - \frac{1}{x+y}\right)\right|_{y=0}^{1} dx$$

$$= \int_0^1\frac{dx}{(x+y)^2} - \left.\frac{-1}{x+1}\right|_0^1 = \frac{1}{2}$$

(b) This follows at once on formally interchanging x and y in (a) to obtain

$$\iint_{\Re} \frac{x-y}{(x+y)^3}dx\,dy, \int_0^1\left\{\int_0^1 \frac{x-y}{(x+y)^3}dx\right\}dy = -\frac{1}{2}$$ and then multiplying both sides by –1.

This example shows that interchange in order of integration may not always produce equal results. A sufficient condition under which the order may be interchanged is that the double integral over the corresponding region exists. In this case $\iint_{\Re} \frac{x-y}{(x+y)^3}dx\,dy$, where $\Re$ is the region $0 \leqq x \leqq 1, 0 \leqq y \leqq 1$, fails to exist because of the discontinuity of the integrand at the origin. The integral is actually an *improper* double integral (see Chapter 12).

9.22. Prove that $\int_0^x\left\{\int_0^t F(u)du\right\}dt = \int_0^x (x-u)F(u)du$

Let $I(x) = \int_0^x\left\{\int_0^t F(u)du\right\}dt, \quad J(x) = \int_0^z (x-u)F(u)du.$ Then

$$I'(x) = \int_0^z F(u)du, \quad J'(x) = \int_0^z F(u)du$$

using Leibniz's rule, Page 198. Thus, $I'(x) = J'(x)$, and so $I(x) = J(x) = c$, where c is a constant. Since $I(0) = J(0) = 0$, $c = 0$, and so $I(x) = J(x)$.

The result is sometimes written in the form

$$\int_0^x\int_0^x F(x)dx^2 = \int_0^x (x-u)F(u)du$$

The result can be generalized to give (see Problem 9.58)

$$\int_0^x\int_0^x\cdots\int_0^x F(x)dx^n = \frac{1}{(n-1)!}\int_0^x (x-u)^{n-1}F(u)du$$

SUPPLEMENTARY PROBLEMS

Double integrals

9.23. (a) Sketch the region $\Re$ in the xy plane bounded by $y^2 = 2x$ and $y = x$. (b) Find the area of $\Re$. (c) Find the polar moment of inertia of $\Re$, assuming constant density σ.

Ans. (b) $\frac{2}{3}$ (c) $48\sigma/35 = 72M/35$, where M is the mass of $\Re$

9.24. Find the centroid of the region in problem 9.23.

Ans. $\bar{x} = \frac{4}{5}, \bar{y} = 1$

9.25. Given $\int_{y=0}^{3}\int_{x=1}^{\sqrt{4-y}} (x+y)dx\,dy$, (a) sketch the region and give a possible physical interpretation of the double integral, (b) interchange the order of integration, and (c) evaluate the double integral.

Ans. (b) $\int_{x=1}^{2}\int_{y=1}^{4-x^2} (x+y)dy\,dx$ (c) 241/60

9.26. Show that $\int_{x=1}^{2}\int_{y=\sqrt{x}}^{x} \sin\frac{\pi x}{2y}\,dx + \int_{x=2}^{4}\int_{y=\sqrt{x}}^{2} \sin\frac{\pi x}{2y}\,dy\,dx = \frac{4(\pi+2)}{\pi^3}$.

9.27. Find the volume of the tetrahedron bounded by $x/a + y/b + z/c = 1$ and the coordinate planes.

Ans. $abc/6$

9.28. Find the volume of the region bounded by $z = x^3 + y^2$, $z = 0$, $x = -a$, $x = -a$, $y = -a$, $y = a$.

Ans. $8a^4/3$