CHAPTER 9

Multiple Integrals

Much of the procedure for double and triple integrals may be thought of as a reversal of partial differentiation and otherwise is analogous to that for single integrals. However, one complexity that must be addressed relates to the domain of definition. With single integrals, the functions of one variable were defined on intervals of real numbers. Thus, the integrals only depended on the properties of the functions. The integrands of double and triple integrals are functions of two and three variables, respectively, and as such are defined on two- and three-dimensional regions. These regions have a flexibility in shape not possible in the single-variable cases. For example, with functions of two variables, and the corresponding double integrals, rectangular regions $a \leqq x \leqq b$, $c \leqq y \leqq d$ are common. However, in many problems the domains are regions bounded above and below by segments of plane curves. In the case of functions of three variables, and the corresponding triple integrals other than the regions $a \leqq x \leqq b, c \leqq y \leqq d, e \leqq z \leqq f$, there are those bounded above and below by portions of surfaces. In very special cases, double and triple integrals can be directly evaluated. However, the systematic technique of *iterated integration* is the usual procedure. It is here that the reversal of partial differentiation comes into play.

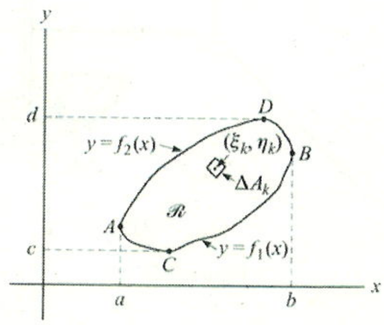

Figure 9.1

Definitions of double and triple integrals are given as follows. Also, the method of iterated integration is described.

Double Integrals

Let $F(x, y)$ be defined in a closed region $\mathfrak{R}$ of the xy plane (see Figure 9.1). Subdivide $\mathfrak{R}$ into n subregions $\Delta \mathfrak{R}_k$ of area ΔA_k, $k = 1, 2, \ldots, n$. Let (ξ_k, η_k) be some point of ΔA_k. Form the sum

$$\sum_{k=1}^{n} F(\xi_k, \eta_k)\, \Delta A_k \tag{1}$$

Consider

$$\lim_{n\to\infty} \sum_{k=1}^{n} F(\xi_k, \eta_k)\, \Delta A_k \tag{2}$$

where the limit is taken so that the number n of subdivisions increases without limit and such that the largest linear dimension of each ΔA_k approaches zero. See Figure 9.2(a). If this limit exists, it is denoted by

$$\iint_{\Re} F(x, y)\, dA \tag{3}$$

and is called the *double integral* of $F(x, y)$ over the region $\Re$.

It can be proved that the limit does exist if $F(x, y)$ is continuous (or sectionally continuous) in $\Re$.

The double integral has a great variety of interpretations with any individual one dependent on the form of the integrand. For example, if $F(x, y) = \rho(x, y)$ represents the variable density of a flat iron plate, then the double integral $\int_A \rho\, dA$ of this function over a same-shaped plane region A is the mass of the plate. In Figure 9.2(b) we assume that $F(x, y)$ is a height function [established by a portion of a surface $z = F(x, y)$] for a cylindrically shaped object. In this case the double integral represents a volume.

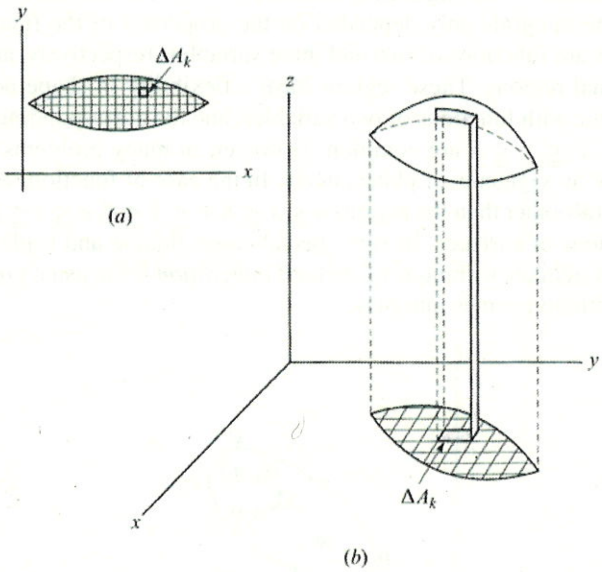

Figure 9.2

Iterated Integrals

If $\Re$ is such that any lines parallel to the y axis meet the boundary of $\Re$ in, at most, two points (as is true in Figure 9.1), then we can write the equations of the curves ACB and ADB bounding $\Re$ as $y = f_1(x)$ and $y = f_2(x)$, respectively, where $f_1(x)$ and $f_2(x)$ are single-valued and continuous in $a \leq x \leq b$. In this case we can evaluate the double integral (3) by choosing the regions $\Delta\Re_k$ as rectangles formed by constructing a grid of lines parallel to the x and y axes and ΔA_k as the corresponding areas. Then Equation (3) can be written

$$\begin{aligned}\iint_{\Re} F(x, y)\, dx\, dy &= \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} F(x, y)\, dy\, dx \\ &= \int_{x=a}^{b} \left\{ \int_{y=f_1(x)}^{f_2(x)} F(x, y)\, dy \right\} dx\end{aligned} \tag{4}$$

where the integral in braces is to be evaluated first (keeping x constant) and finally integrating with respect to x from a to b. The result (4) indicates how a double integral can be evaluated by expressing it in terms of two single integrals called *iterated integrals*.

The process of iterated integration is visually illustrated in Figure 9.3(a) and (b) and further illustrated as follows.

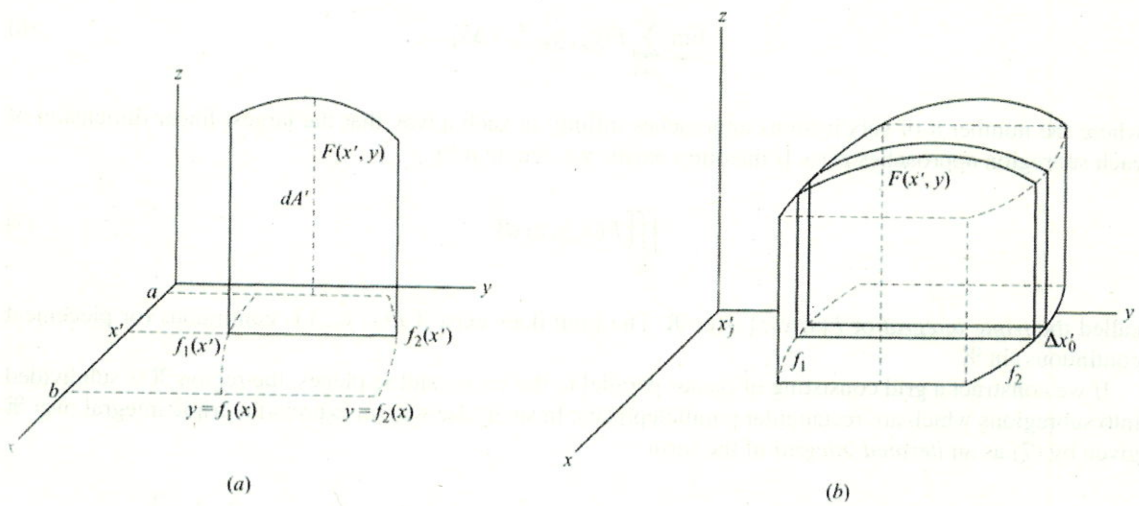

Figure 9.3

The general idea, as demonstrated with respect to a given three-space region, is to establish a plane section, integrate to determine its area, and then add up all the plane sections through an integration with respect to the remaining variable. For example, choose a value of x (say, $x = x'$). The intersection of the plane $x = x'$ with the solid establishes the plane section. In it, $z = F(x', y)$ is the height function, and if $y = f_1(x)$ and $y = f_2(x)$ for all z) are the bounding cylindrical surfaces of the solid, then the width is $f_2(x') - f_1(x')$, i.e., $y_2 - y_1$. Thus, the area of the section is $A = \int_{y_1}^{y_2} F(x', y)\, dy$. Now establish slabs $A_j\, \Delta x_j$, where, for each interval $\Delta x_j = x_j - x_{j-r}$, there is an intermediate value x_j'. Then sum these to get an approximation to the target volume. Adding the slabs and taking the limit yields

$$V = \lim_{x\to\infty} \sum_{j=1}^{n} A_j \Delta x_j = \int_a^b \left(\int_{y_1}^{y_2} F(x, y)\, dx \right) dx$$

In some cases the order of integration is dictated by the geometry. For example, if $\Re$ is such that any lines parallel to the x axis meet the boundary of $\Re$ in, at most, two points (as in Figure 9.1), then the equations of curves CAD and CBD can be written $x = g_1(y)$ and $x = g_2(y)$, respectively, and we find, similarly,

$$\iint_{\Re} F(x, y)\, dx\, dy = \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} F(x, y)\, dx\, dy$$
$$= \int_{y=c}^{d} \left\{ \int_{x=g_1(y)}^{g_2(y)} F(x, y)\, dx \right\} dy \qquad (5)$$

If the double integral exists, Equations (4) and (5) yield the same value. (See, however, Problem 9.21.) In writing a double integral, either of the forms (4) or (5), whichever is appropriate, may be used. We call one form an *interchange of the order of integration* with respect to the other form.

In case $\Re$ is not of the type shown in Figure 9.3, it can generally be subdivided into regions $\Re_1, \Re_2, \ldots,$ which are of this type. Then the double integral over $\Re$ is found by taking the sum of the double integrals over $\Re_1, \Re_2, \ldots.$

Triple Integrals

These results are easily generalized to closed regions in three dimensions. For example, consider a function $F(x, y, z)$ defined in a closed three-dimensional region $\Re$. Subdivide the region into n subregions of volume ΔV_k, $k = 1, 2, \ldots, n$. Letting (ξ_k, η_k, ζ_k) be some point in each subregion, we form

$$\lim_{x \to \infty} \sum_{k=1}^{n} F(\xi_k, \eta_k, \xi_k)\, \Delta V_k \tag{6}$$

where the number n of subdivisions approaches infinity in such a way that the largest linear dimension of each subregion approaches zero. If this limit exists, we denote it by

$$\iiint_{\Re} F(x, y, z)\, dV \tag{7}$$

called the *triple integral* of $F(x, y, z)$ over $\Re$. The limit does exist if $F(x, y, z)$ is continuous (or piecemeal continuous) in $\Re$.

If we construct a grid consisting of planes parallel to the xy, yz, and xz planes, the region $\Re$ is subdivided into subregions which are rectangular parallelepipeds. In such case we can express the triple integral over $\Re$ given by (7) as an *iterated integral* of the form

$$\int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(a)} \int_{z=f_1(x,y)}^{f_2(x,y)} F(x, y, z) dx\, dy\, dz = \int_{x=a}^{b} \left[\int_{y=g_1(x)}^{g_2(x)} \left\{ \int_{z=f_1(x,y)}^{f_2(x,y)} F(x, y, z)\, dz \right\} dy \right] dx \tag{8}$$

(where the innermost integral is to be evaluated first) or the sum of such integrals. The integration can also be performed in any other order to give an equivalent result.

The interated triple integral is a sequence of integrations, first from surface portion to surface portion, then from curve segment to curve segment, and finally from point to point. (See Figure 9.4.)

Extensions to higher dimensions are also possible.

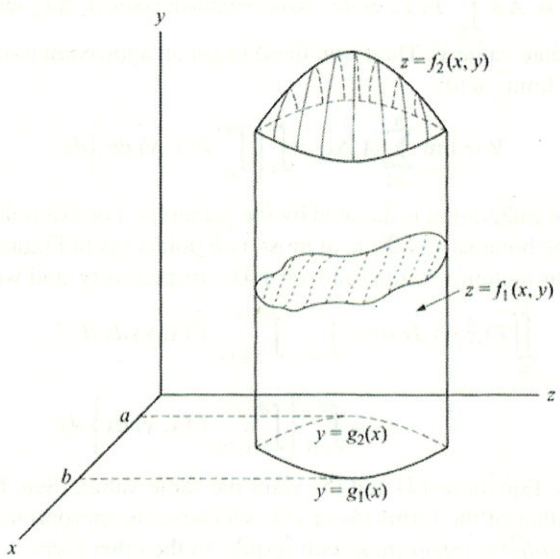

Figure 9.4

Transformations of Multiple Integrals

In evaluating a multiple integral over a region $\Re$, it is often convenient to use coordinates other than rectangular, such as the curvilinear coordinates considered in Chapters 6 and 7.

If we let (u, υ) be curvilinear coordinates of points in a plane, there will be a set of transformation equations $x = f(u, \upsilon)$, $y = g(u, \upsilon)$ mapping points (x, y) of the xy plane into points (u, υ) of the $u\upsilon$ plane.

In such case the region $\Re$ of the xy plane is mapped into a region $\Re'$ of the uv plane. We then have

$$\iint F(x, y)\, dx\, dy = \iint G(u,\upsilon) \left|\frac{\partial(x, y)}{\partial(u,\upsilon)}\right| du\, d\upsilon \tag{9}$$

where $G(u, \upsilon), \equiv F\{f(u, \upsilon), g(u, \upsilon)\}$ and

$$\frac{\partial(x, y)}{\partial(u,\upsilon)} \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial \upsilon} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial \upsilon} \end{vmatrix} \tag{10}$$

is the *Jacobian* of x and y with respect to u and υ (see Chapter 6).

Similarly, if (u, υ, w) are curvilinear coordinates in three dimensions, there will be a set of transformation equations $x = f(u, \upsilon, w)$, $y = g(u, \upsilon, w)$, $z = h(u, \upsilon, w)$ and we can write

$$\iiint_{\Re} F(x, y, z)\, dx\, dy\, dz = \iiint_{\Re} G(u,\upsilon, w) \left|\frac{\partial(x, y, z)}{\partial(u,\upsilon,w)}\right| du\, dv\, dw \tag{11}$$

where $G(u, \upsilon, w) \equiv F\{(f(u, \upsilon, w), g(u, \upsilon, w), h(u, \upsilon, w)\}$ and

$$\frac{\partial(x, y, z)}{\partial(u,\upsilon, w)} \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial \upsilon} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial \upsilon} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial \upsilon} & \dfrac{\partial z}{\partial w} \end{vmatrix} \tag{12}$$

is the Jacobian of x, y, and z with respect to u, υ, and w.

The results (9) and (11) correspond to change of variables for double and triple integrals. Generalizations to higher dimensions are easily made.

The Differential Element of Area in Polar Coordinates, Differential Elements of Area in Cylindral and Spherical Coordinates

Of special interest is the differential element of area dA for polar coordinates in the plane, and the differential elements of volume dV for cylindrical and spherical coordinates in three-space. With these in hand, the double and triple integrals as expressed in these systems are seen to take the following forms. (See Figure 9.5.)

The transformation equations relating cylindrical coordinates to rectangular Cartesian ones appear in Chapter 7, in particular,

$$x = \rho \cos \phi,\ y = \rho \sin \phi,\ z = z$$

The coordinate surfaces are circular cylinders, planes, and planes. (See Figure 9.5.)

At any point of the space (other than the origin), the set of vectors $\left\{\dfrac{\partial \mathbf{r}}{\partial \rho}, \dfrac{\partial \mathbf{r}}{\partial \phi}, \dfrac{\partial \mathbf{r}}{\partial z}\right\}$ constitutes an orthogonal basis.

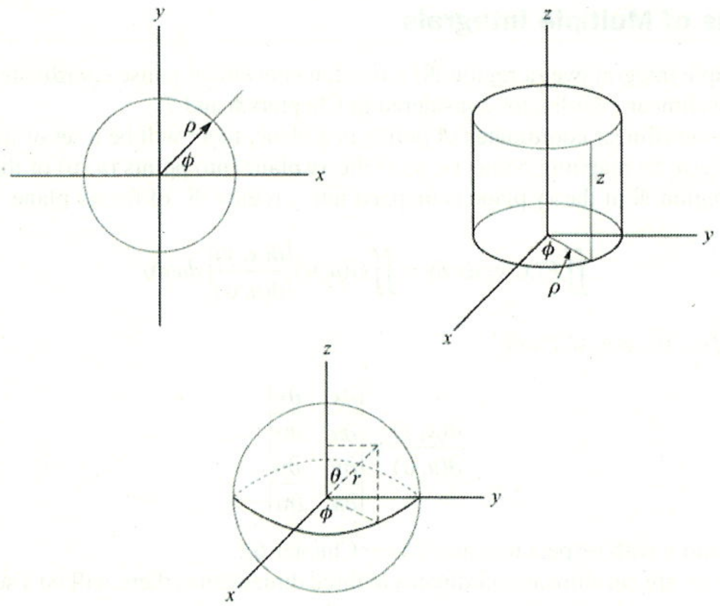

Figure 9.5

In the cylindrical case, $\mathbf{r} = \rho \cos \phi \mathbf{i} + \rho \sin \phi \mathbf{j} + z\mathbf{k}$ and the set is

$$\frac{\partial \mathbf{r}}{\partial \rho} = \cos\phi \mathbf{i} + \sin\phi \mathbf{j}, \quad \frac{\partial \mathbf{r}}{\partial \rho} = -\rho \sin\phi \mathbf{i} + \rho \cos\phi \mathbf{j}, \quad \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}$$

Therefore, $\frac{\partial \mathbf{r}}{\partial \rho} \cdot \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial z} = \rho$.

That the geometric interpretation of $\frac{\partial \mathbf{r}}{\partial \rho} \cdot \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial z} \, d\rho \, d\phi \, dz$ is an infinitesimal rectangular parallelepiped suggests that the differential element of volume in cylindrical coordinates is

$$dV = \rho \, d\rho \, d\phi \, dz$$

Thus, for an integrable but otherwise arbitrary function $F(\rho, \phi, z)$ of cylindrical coordinates, the iterated triple integral takes the form

$$\int_{z_1}^{z_2} \int_{g_1(z)}^{g_2(z)} \int_{f_1(\phi,z)}^{f_2(\phi,z)} F(\rho,\phi,z)\rho \, d\rho \, d\phi \, dz$$

The differential element of area for polar coordinates in the plane results by suppressing the z coordinate. It is

$$dA = \left| \frac{\partial \mathbf{r}}{\partial \rho} \times \frac{\partial \mathbf{r}}{\partial \phi} \right| d\rho \, d\phi$$

and the iterated form of the double integral is

$$\int_{\rho_1}^{\rho_2} \int_{\phi_1(\rho)}^{\phi_2(\rho)} F(\rho,\phi)\rho \, d\rho \, d\phi$$

The transformation equations relating spherical and rectangular Cartesian coordinates are

$$x = r \sin \theta \cos \phi, \; y = r \sin \theta \sin \phi, \; z = r \cos \theta$$

In this case the coordinate surfaces are spheres, cones, and planes. (See Figure 9.5.)

Following the same pattern as with cylindrical coordinates we discover that

$$dV = r^2 \sin \theta \, dr \, d\theta \, d\phi$$

and the iterated triple integral of $F(r, \theta, \phi)$ has the spherical representation

$$\int_{r_1}^{r_2} \int_{\theta_1(\phi)}^{\theta_2(\phi)} \int_{\phi_1(r,\theta)}^{\phi_2(r,\theta)} F(r,\theta,\phi)\, r^2 \sin\theta \; dr\, d\theta\, d\phi$$

Of course, the order of these integrations may be adapted to the geometry.

The coordinate surfaces in spherical coordinates are spheres, cones, and planes. If r is held constant—say, $r = a$—then we obtain the differential element of surface area

$$dA = a^2 \sin\theta \; d\theta \; d\phi$$

The first octant surface area of a sphere of radius a is

$$\int_0^{\pi/2} \int_0^{\pi/2} a^2 \sin\theta \; d\theta \; d\phi = \int_0^{\pi/2} a^2 \left(-\cos\theta\right)\Big|_0^{\frac{\pi}{2}} d\phi = \int_0^{\pi/2} a^2 \; d\phi = a^2 \frac{\pi}{2}$$

Thus, the surface area of the sphere is $4\pi a^2$.

SOLVED PROBLEMS

Double integrals

9.1. (a) Sketch the region $\Re$ in the xy plane bounded by $y = x^2$, $x = 2$, $y = 1$. (b) Give a physical interpretation to $\iint_{\Re} (x^2 + y^2)\, dx\, dy$. (c) Evaluate the double integral in (b).

(a) The required region $\Re$ is shown shaded in Figure 9.6.

(b) Since $x^2 + y^2$ is the square of the distance from any point (x, y) to $(0, 0)$, we can consider the double integral as representing the *polar moment of inertia* (i.e., moment of intertia with respect to the origin) of the region $\Re$ (assuming unit density).

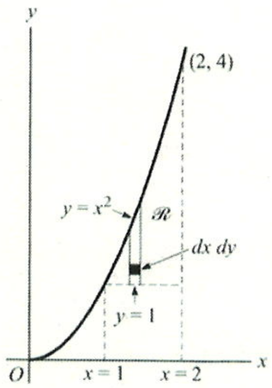

Figure 9.6

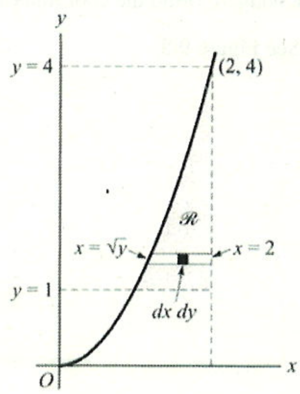

Figure 9.7

We can also consider the double integral as representing the *mass* of the region $\Re$, assuming a density varying as $x^2 + y^2$.

(c) **Method 1:** The double integral can be expressed as the iterated integral

$$\int_{x=1}^{2} \int_{y=1}^{x^2} (x^2 + y^2)\, dy\, dx = \int_{x=1}^{2} \left\{ \int_{y=1}^{x^2} (x^2 + y^2)\, dy \right\} dx = \int_{x=1}^{2} x^2 y + \frac{y^2}{3} \Bigg|_{y=1}^{x^2}$$

$$dx = \int_{x=1}^{2} \left(x^4 + \frac{x^6}{3} - x^2 - \frac{1}{3} \right) dx = \frac{1006}{105}$$

The integration with respect to y (keeping x constant) from $y = 1$ to $y = x^2$ corresponds formally to summing in a vertical column (see Figure 9.6). The subsequent integration with respect to x from $x = 1$ to $x = 2$ corresponds to addition of contributions from all such vertical columns between $x = 1$ and $x = 2$.

Method 2: The double integral can also be expressed as the iterated integral

$$\int_{y=1}^{4}\int_{x=\sqrt{y}}^{2}(x^2+y^2)\,dx\,dy = \int_{y=1}^{4}\left\{\int_{x=\sqrt{y}}^{2}(x^2+y^2)\,dx\right\}dy = \int_{y=1}^{4}\frac{x^3}{3}+xy^2\Big|_{x=\sqrt{y}}^{2}\,dy$$

$$= \int_{x=1}^{2}\left(x^4+\frac{x^6}{3}-x^2-\frac{1}{3}\right)dx = \frac{1006}{105}$$

In this case the vertical column of region $\Re$ in Figure 9.6 is replaced by a horizontal column, as in Figure 9.7. Then the integration with respect to x (keeping y constant) from $x = \sqrt{y}$ to $x = 2$ corresponds to summing in this horizontal column. Subsequent integration with respect to y from $y = 1$ to $y = 4$ corresponds to addition of contributions for all such horizontal columns between $y = 1$ and $y = 4$.

9.2. Find the volume of the region bounded by the elliptic paraboloid $z = 4 - x^2 - \frac{1}{4}y^2$ and the plane $z = 0$.

Because of the symmetry of the elliptic paraboloid, the result can be obtained by multiplying the first octant volume by 4.

Letting $z = 0$ yields $4x^2 + y^2 = 16$. The limits of integration are determined from this equation. The required volume is

$$4\int_0^2\int_0^{2\sqrt{4-x^2}}\left(4-x^2-\frac{1}{4}y^2\right)dy\,dx = 4\int_0^2\left(4y-x^2y-\frac{1}{4}\frac{y^3}{3}\right)\Big|_0^{2\sqrt{4-x^2}}dx = 16\Pi$$

Hint: Use trigonometric substitutions to complete the integrations.

9.3. The geometric model of a material body is a plane region R bounded by $y = x^2$ and $y = \sqrt{2-x^2}$ on the interval $0 \leqq x \leqq 1$, and with a density function $\rho = xy$. (a) Draw the graph of the region. (b) Find the mass of the body. (c) Find the coordinates of the center of mass.

(a) See Figure 9.8.

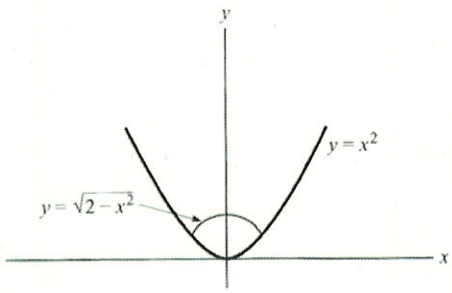

Figure 9.8

(b) $$M = \int_a^b\int_{f_1}^{f_2}\rho\,dy\,dx = \int_0^1\int_{x^2}^{\sqrt{2-x^2}} yx\;dy\,dx = \int_0^1\left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{2-x^2}} x\,dx$$

$$= \int_0^1\frac{1}{2}x(2-x^2-x^4)\,dx = \left[\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{12}\right]_0^1 = \frac{7}{24}$$

(c) The coordinates of the center of mass are defined to be

$$\bar{x} = \frac{1}{M}\int_a^b\int_{f_1(x)}^{f_2(x)} x\,\rho\,dy\,dx \quad\text{and}\quad \bar{y} = \frac{1}{M}\int_a^b\int_{f_1(x)}^{f_2(x)} y\,\rho\,dy\,dx$$