10.2. If $\mathbf{A} = (3x^2 - 6yz)\mathbf{i} + (2y + 3xz)\mathbf{j} + (1 - 4xyz^2)\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from (0,1,1) to (1,1,1) along the following paths C:

(a) $x = t,\ y = t^2,\ z = t^3$

(b) The straight lines from (0, 0, 0) to (0, 0, 1), then to (0, 1, 1), and then to (1, 1, 1)

(c) The straight line joining (0, 0, 0) and (1, 1, 1)

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_C \{(3x^2 - 6yz)\mathbf{i} + (2y + 3xz)\}\mathbf{j} + (1 - 4xyz^2)\mathbf{k}\}\cdot(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$
$$= \int_C \{(3x^2 - 6yz)dx + (2y + 3xz)dy + (1 - 4xyz^2)dz$$

(a) If $x = t$, $y = t^2$, $z = t^3$, points (0, 0, 0) and (1, 1, 1) correspond to $t = 0$ and $t = 1$, respectively. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} \{3t^2 - 6(t^2)(t^3)\}dt + \{2t^2 + 3(t)(t^3)\}d(t^2) + \{1 - 4(t)(t^2)(t^3)^2\}d(t^3)$$
$$= \int_{t=0}^{1} \{3t^2 - 6t^5\}dt + (4t^3 + 6t^5)\}dt + (3t^2 - 12t^{11})dt = 2$$

Another method: Along C, $\mathbf{A} = (3t^2 - 6t^5)\,\mathbf{i} + (2t^2 + 3t^4)\mathbf{j} + (1 - 4t^9)\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})dt$. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_0^1 (3t^2 - 6t^5)dt + (4t^3 + 6t^5)dt +.(3t^2 - 12t^{11})dt = 2$$

(b) Along the straight line from (0, 0, 0) to (0, 1, 1), $x = 0$, $y = 0$, $dx = 0$, $dy = 0$, while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1} \{3(0)^2 - 6(0)(z)\}0 + \{2(0) + 3(0)(z)0 + \{1 - 4(0)(0)(z^2)\}dz = \int_{z=0}^{1} dz = 1$$

Along the straight line from (0, 0, 1) to (0, 1, 1), $x = 0$, $z = 1$, $dx = 0$, $dz = 0$, while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1} \{3(0)^2 - 6(y)(1)\}0 + \{2y + 3(0)(1)\}dy + \{1 - 4(0)(y)(1)^2\}0 = \int_{y=0}^{1} 2y\,dy = 1$$

Along the straight line from (0, 1, 1), to (1, 1, 1), $v = 1$, $z = 1$, $dy = 0$, $dz = 0$, while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1} \{3x^2 - 6(1)(1)\}dx + \{2(1) + 3x(1)\}0 + \{1 - 4x(1)(1)^2\}0 = \int_{x=0}^{1} (3x^2 - 6)dx = -5$$

Adding,

$$\int_C \mathbf{A}\cdot dx = 1 + 1 - 5 = -3$$

(c) The straight line joining (0, 0, 0) and (1, 1, 1) is given in parametric form by $x = t$, $y = t$, $z = t$. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2 - 6t^2)\,dt + (2t + 3t^2)\,dt + (1 - 4t^4)\,dt = 6/5$$

10.3. Find the work done in moving a particle once around an ellipse C in the xy plane, if the ellipse has its center at the origin with semimajor and semiminor axes 4 and 3, respectively, as indicated in Figure 10.7, and if the force field is given by

$$\mathbf{F} = (3x - 4y + 2z)\mathbf{i} + (4x + 2y - 3z^2)\mathbf{j} + (2xz - 4y^2 + z^3)\mathbf{k}$$

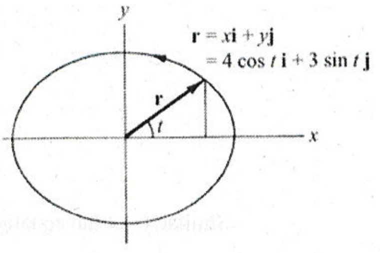

Figure 10.7

In the plane $z = 0$, $\mathbf{F} = (3x - 4y)\mathbf{i} + (4x + 2y)\mathbf{j} - 4y^2\mathbf{k}$, and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$, so that the work done is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_C \{(3x - 4y)\mathbf{i} + (4x + 2y)\mathbf{j} - 4y^2\mathbf{k}\}\cdot(dx\mathbf{i} + dy\mathbf{j})$$
$$= \oint_C (3x - 4y)\,dx + (4x + 2y)\,dy$$

Choose the parametric equations of the ellipse as $x = 4\cos t$, $y = 3\sin t$, where t varies from 0 to 2π (see Figure 10.7). Then the line integral equals

$$\int_{t=0}^{2\pi} \{3(4\cos t) - 4(3\sin t)\}\{-4\sin t\}dt + \{4(4\cos t) + 2(3\sin t)\}\{3\cos t\}dt$$

$$= \int_{t=0}^{2\pi} (48 - 30\sin t\cos t)dt = (48t - 15\sin^2 t)\Big|_0^{2\pi} = 96\pi$$

In traversing C we have chosen the counterclockwise direction indicated in Figure 10.7. We call this the *positive* direction or say that C has been traversed in the *positive sense*. If C were traversed in the clockwise (negative) direction, the value of the integral would be -96π.

10.4. Evaluate $\int_C y\,ds$ along the curve C given by $y = 2\sqrt{x}$ from $x = 3$ to $x = 24$.

Since $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y')^2}\,dx = \sqrt{1 + 1/x\,dx}$, we have

$$\int_C y\,ds = \int_2^{24} 2\sqrt{x}\sqrt{1 + 1/x}\,dx = 2\int_3^{24} \sqrt{x + 1}\,dx = \frac{4}{3}(x + 1)^{3/2}\Big|_3^{24} = 156$$

Green's theorem in the plane

10.5. Prove Green's theorem in the plane if C is a closed curve which has the property that any straight line parallel to the coordinate axes cuts C in, at most, two points.

Let the equations of the curves AEB and AFB (see Figure 10.8) be $y = Y_1(x)$ and $y = Y_2(x)$, respectively. If $\Re$ is the region bounded by C, we have

$$\iint_{\Re} \frac{\partial P}{\partial y} dx\,dy = \int_{x=a}^{b} \left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial P}{\partial y} dy \right] dx$$

$$= \int_{x=a}^{b} P(x, y)\Big|_{y=Y_1(x)}^{Y_2(x)} dx = \int_a^b [P(x, Y_2) - P(x, Y_1)]dx$$

$$= -\int_a^b P(x, Y_1)dx - \int_b^a P(x, Y_2)dx = -\oint_C P\,dx$$

Then

$$\oint_C P\,dx = -\iint_{\Re} \frac{\partial P}{\partial y} dx\,dy \qquad (1)$$

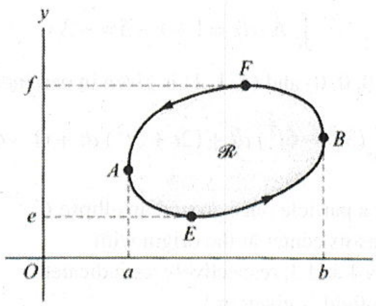

Figure 10.8

Similarly, let the equations of curves EAF and EBF be $x = X_1(y)$ and $x = X_2(y)$, respectively. Then

$$\iint_{\Re} \frac{\partial Q}{\partial x} dx\,dy = \int_{y=c}^{f} \left[\int_{x=X_1(y)}^{X_2(y)} \frac{\partial Q}{\partial x} dx \right] dy = \int_c^f [Q(X_2, y) - Q(X_1, y)]dy$$

$$= \int_f^c Q(X_1, y)dy + \int_c^f Q(X_2, y)dy = \oint_C Q\,dy$$

Then

$$\oint_C Q\,dy = \iint_{\Re} \frac{\partial Q}{\partial x}\,dx\,dy \tag{2}$$

Adding Equations (1) and (2),

$$\oint_C P\,dx + Q\,dy = \iint_{\Re} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

10.6. Verify Green's theorem in the plane for

$$\oint_C (2xy - x^2)dx + (x + y^2)dy$$

where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

The plane curve $y = x^2$ and $y^2 = x$ intersect at (0, 0) and (1, 1). The positive direction in traversing C is as shown in Figure 10.9.

Along $y = x^2$, the line integral equals

$$\int_{x=0}^{1} \{(2x)(x^2) - x^2\}dx + \{x + (x^2)^2\}d(x^2) = \int_0^1 (2x^3 + x^2 + 2x^5)dx = 7/6$$

Along $y^2 = x$, the line integral equals

$$\int_{y=1}^{0} \{(2)(y^2)(y) - (y^2)^2\}\,d(y^2) + \{y^2 + y^2\}\,dy = \int_1^0 (4y^4 - 2y^5 + 2y^2)dy = -17/15$$

Then the required line integral = 7/6 – 17/15 = 1/30.

$$\iint_{\Re} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_{\Re} \left\{ \frac{\partial}{\partial x}(x + y^2) - \frac{\partial}{\partial y}(2xy - x^2) \right\} dx\,dy$$

$$= \iint_{\Re} (1 - 2x)dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (1 - 2x)dy\,dx$$

$$= \int_{x=0}^{1} (y - 2xy)\Big|_{y=x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 (x^{1/2} - 2x^{3/2} - x^2 + 2x^3)\,dx = 1/30$$

Hence, Green's theorem is verified.

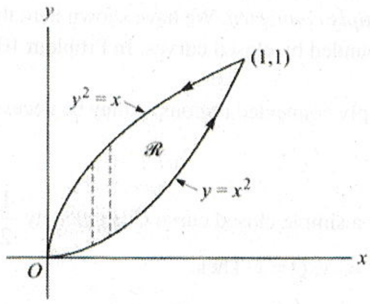

Figure 10.9

10.7. Extend the proof of Green's theorem in the plane given in Problem 10.5 to the curves C for which lines parallel to the coordinate axes may cut C in more than two points.

Consider a closed curve C such as is shown in Figure 10.10, in which lines parallel to the axes may meet C in more than two points. By constructing line ST, the region is divided into two regions $\Re_1$ and $\Re_2$, which are of the type considered in Problem 10.5 and for which Green's theorem applies, i.e.,

$$\int_{STUS} P\,dx + Q\,dy = \iint_{\Re_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \tag{1}$$

$$\int_{SVTS} P\,dx + Q\,dy = \iint_{\Re_1}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy \qquad (2)$$

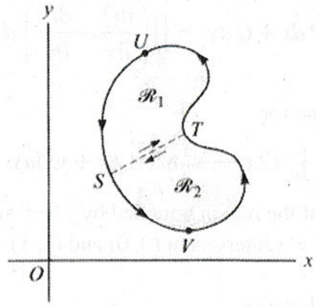

Figure 10.10

Adding the left-hand sides of Equations (1) and (2), and omitting the integrand $P\,dx + Q\,dy$ in each case, we have

$$\int_{STUS} + \int_{SVTS} = \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} = \int_{TUS} + \int_{SVT} = \int_{TUSVT}$$

using the fact that $\int_{ST} = -\int_{TS}$.

Adding the right-hand sides of Equations (1) and (2), omitting the integrand, $\iint_{\Re_1} + \iint_{\Re_2} = \iint_{\Re}$, where $\Re$ consists of regions $\Re_1$ and $\Re_2$.

Then $\int_{TUSVT} P\,dx + Q\,dy = \iint_{\Re}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$, and the theorem is proved.

A region $\Re$ such as is considered here and in Problem 10.5, for which any closed lying in $\Re$ can be continuously shrunk to a point without leaving $\Re$, is called a *simply connected region*. A region which is not simply connected is called *multiply connected*. We have shown here that Green's theorem in the plane applies to simply connected regions bounded by closed curves. In Problem 10.10 the theorem is extended to multiply connected regions.

For more complicated simply connected regions, it may be necessary to construct more lines, such as ST, to establish the theorem.

10.8. Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C x\,dy - y\,dx$.

In Green's theorem, put $P = -y$, $Q = x$. Then

$$\oint_C x\,dy - y\,dx = \iint_{\Re}\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)dx\,dy = 2\iint_{\Re} dx\,dy = 2A$$

where A is the required area. Thus, $A = \frac{1}{2}\oint_C x\,dy - y\,dx$.

10.9. Find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$.

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}(a\cos\theta)(b\cos\theta)d\theta - (b\sin\theta)(-a\sin\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)d\theta = \frac{1}{2}\int_0^{2\pi} ab\,d\theta = \pi ab$$

10.10. Show that Green's theorem in the plane is also valid for a multiply connected region $\Re$ such as is shown in Figure 10.11.

The shaded region $\Re$, shown in Figure 10.11, is multiply connected, since not every closed curve lying in $\Re$ can be shrunk to a point without leaving $\Re$, as is observed by considering a curve surrounding *DEFGD*, for example. The boundary of $\Re$, which consists of the exterior boundary *AHJKLA* and the interior boundary *DEFGD*, is to be traversed in the positive direction, so that a person traveling in this direction always has the region on his left. It is seen that the positive directions are those indicated Figure 10.11.

In order to establish the theorem, construct a line such as *AD*, called a *crosscut*, connecting the exterior and interior boundaries. The region bounded by *ADEFGDALKJHA* is simply connected, and so Green's theorem is valid. Then

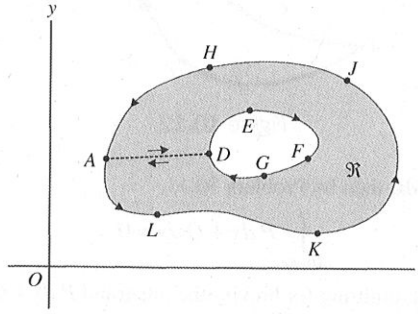

Figure 10.11

$$\oint_{ADEFGDALKJHA} P\,dx + Q\,dy = \iint_{\Re}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

But the integral on the left, leaving out the integrand, is equal to

$$\int_{AD} + \int_{DEFGD} + \int_{DA} + \int_{ALKJHA} = \int_{DEFGD} + \int_{ALKJHA}$$

since $\int_{AD} = -\int_{DA}$. Thus, if C_1 is the curve *ALKJHA*, C_2 is the curve *DEFGD*, and C is the boundary of $\Re$ consisting of C_1 and C_2 (traversed in the positive directions), then $\int_{C_1} + \int_{C_2} = \int_C$ and so

$$\oint_C P\,dx + Q\,dy = \iint_{\Re}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Independence of the path

10.11. Let $P(x, y)$ and $Q(x, y)$ be continuous and have continuous first partial derivatives at each point of a simply connected region $\Re$. Prove that a necessary and sufficient condition that $\oint_C P\,dx + Q\,dy = 0$ around every closed path C in $\Re$ is that $\partial P/\partial y = \partial Q/\partial x$ identically in $\Re$.

Sufficiency. Suppose $\partial P/\partial y = \partial Q/\partial x$. Then, by Green's theorem,

$$\oint_C P\,dx + Q\,dy = \iint_{\Re}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = 0$$

where $\Re$ is the region bounded by C.

Necessity. Suppose $\oint_C P\,dx + Q\,dy = 0$ around every closed path C in $\Re$ and that $\partial P/\partial y \neq \partial Q/\partial x$ at some point of $\Re$. In particular, suppose $\partial P/\partial y - \partial Q/\partial x > 0$ at the point (x_0, y_0).

By hypothesis, $\partial P/\partial y$ and ∂Q are continuous in $\Re$, so that there must be some region τ containing (x_0, y_0) as an interior point for which $\partial P/\partial y - \partial Q/\partial x > 0$. If Γ is the boundary of τ, then by Green's theorem,

$$\oint_C P\,dx + Q\,dy = \iint_{\tau}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy > 0$$

contradicting the hypothesis that $\oint_\Gamma P\,dx + Q\,dy = \iint_\tau \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy > 0$ for *all* closed curves in in $\Re$. Thus, $\partial Q/\partial x - \partial P/\partial y$ cannot be positive.

Similarly, we can show that $\partial Q/\partial x - \partial P/\partial y$ cannot be negative, and it follows that it must be identically zero; i.e., $\partial P/\partial y = \partial Q/\partial x$ identically in $\Re$.

10.12 Let P and Q be defined as in Problem 10.11. Prove that a necessary and sufficient condition that $\int_A^B P\,dx + Q\,dy$ be independent of the path in $\Re$ joining points A and B is that $\partial P/\partial y = \partial\, Q/\partial x$ identically in $\Re$.

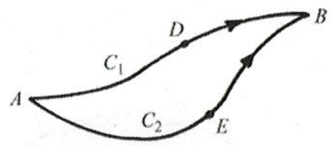

Figure 10.12

Sufficiency. If $\partial P/\partial y = \partial Q/\partial x$, then by Problem 10.11,

$$\int_{ADBEA} P\,dx + Q\,dy = 0$$

(See Figure 10.12.) From this, omitting for brevity the integrand $P\,dx + Q\,dy$, we have

$$\int_{ADB} + \int_{BEA} = 0, \quad \int_{ADB} = -\int_{BEA} = \int_{AEB} \quad \text{and so} \quad \int_{C_1} = \int_{C_2}$$

i.e., the integral is independent of the path.

Necessity. If the integral is independent of the path, then for all paths C_1 and C_2 in $\Re$ we have

$$\int_{C_1} = \int_{C_2}, \quad \int_{ADB} = \int_{AEB} \quad \text{and} \quad \int_{ADBEA} = 0$$

From this it follows that the line integral around any closed path in $\Re$ is zero, and, hence, by Problem 10.11 that $\partial P/\partial y = \partial Q/\partial x$.

10.13. Let P and Q be as in Problem 10.11. (a) Prove that a necessary and sufficient condition that $P\,dx + Q\,dy$ be an exact differential of a function $\varphi(x, y)$ is that $\partial P/\partial y = \partial Q/\partial x$. (b) Show that in such case $\int_A^B P\,dx + Q\,dy = \int_A^B d\phi = \phi(B) - \phi(A)$ where A and B are any two points.

(a) **Necessity.** If $P\,dx + Q\,dy = d\,\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$, an exact differential, then

$$\partial\phi/\partial x = P \tag{1}$$

$$\partial\phi/\partial y = 0 \tag{2}$$

Thus, by differentiating Equations (1) and (2) with respect to y and x, respectively, $\partial P/\partial y = \partial Q/\partial x$, since we are assuming continuity of the partial derivatives.

Sufficiency. By Problem 10.12, if $\partial P/\partial y = \partial Q/\partial x$, then $\int P\,dx + Q\,dy$ is independent of the path joining two points. In particular, let the two points be (a, b) and (x, y) and define

$$\phi(x, y) = \int_{(a,b)}^{(x,y)} P\,dx + Q\,dy$$

Then

$$\phi(x + \Delta x, y) - \phi(x, y) = \int_{(a,b)}^{(x+\Delta x,y)} P\,dx + Q\,dy - \int_{(a,b)}^{(x,y)} P\,dx + Q\,dy$$

$$= \int_{(x,y)}^{(x+\Delta x,y)} P\,dx + Q\,dy$$

Since the last integral is independent of the path joining (x, y) and $(x + \Delta x, y)$, we can choose the path to be a straight line joining these points (see Figure 10.13) so that $dy = 0$. Then, by the mean value theorem for integrals,

$$\frac{\phi(x+\Delta x, y) - \phi(x, y)}{\Delta x} = \frac{1}{\Delta x}\int_{(x,y)}^{(x+\Delta x, y)} P\,dx = P(x + \theta\,\Delta x, y) \qquad 0 < \theta < 1$$

Taking the limit as $\Delta x \to 0$, we have $\partial\phi/\partial x = P$.

Similarly, we can show that $\partial\phi/\partial y = Q$.

Thus, it follows that $P\,dx + Q\,dy = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = d\phi$.

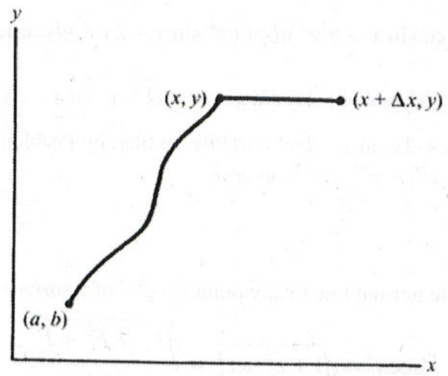

Figure 10.13

(b) Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. From (a),

$$\phi(x, y) = \int_{(a,b)}^{(x,y)} P\,dx + Q\,dy.$$

Then, omitting the integrand $P\,dx + Q\,dy$, we have

$$\int_A^B = \int_{(x_1,y_1)}^{(x_2,y_2)} = \int_{(a,b)}^{(x_2,y_2)} - \int_{(a,b)}^{(x_1,y_1)} = \phi(x_2, y_2) - \phi(x_1, y_1) = \phi(B) - \phi(A)$$

10.14. (a) Prove that $\int_{(1,2)}^{(3,4)} (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy\;dy$ is independent of the path joining (1, 2) and (3, 4). (b) Evaluate the integral in (a).

(a) P = 6xy2 – y3, Q = 6x2 y – 3xy2. Then ∂P/∂y = 12xy – 3y2 = ∂Q/∂x and, by Problem 10.12, the line integral is independent of the path.

(b) **Method 1:** Since the line integral is independent of the path, choose any path joining (1, 2) and (3, 4), for example, that consisting of lines from (1, 2) to (3, 2) (along which $y = 2$, $dy = 0$) and then (3, 2) to (3, 4) (along which $x = 3$, $dx = 0$). Then the required integral equals

$$\int_{x=1}^{3} (24x - 8)dx + \int_{y=2}^{4} (54y - 9y^2)dy = 80 + 156 = 236$$

Method 2: Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we must have

$$\frac{\partial\phi}{\partial y} = 6x^2y - 3xy^2 \qquad (1)$$

$$\frac{\partial\phi}{\partial y} = 6x^2y - 3xy^2 \qquad (2)$$

From Equation (1), $\phi = 3x^2y^2 - xy^3 + f(y)$. From Equation (2), $\phi = 3x^2y^2 - xy^3 + g(x)$. The only way in which these two expressions for ϕ are equal is if $f(y) = g(x) = c$, a constant. Hence, $\phi = 3x^2y^2 - xy^3 + c$. Then, by Problem 10.13.

$$\int_{(1,2)}^{(3,4)} (6xy^2 - y^3)dx + (6x^2y - 3xy^2)dy = \int_{(1,2)}^{(3,4)} d(3x^2y^2 - xy^3 + c)$$
$$= 3x^2y^2 - xy^3 + c\Big|_{(1,2)}^{(3,4)} = 236$$

Note that in this evaluation the arbitrary constant c can be omitted. See also Problem 6.16.

We could also have noted by inspection that

$$(6xy^2 - y^3)dx + (6x^2y - 3xy^2)dy = (6xy^2dx + 6x^2y\,dy) - (y^3dx + 3xy^2dy)$$
$$= d(3x^2y^2) - d(xy^3) = d(3x^2y^2 - xy^3)$$

from which it is clear that $\phi = 3x^2y^2 - xy^3 + c$.

10.15. Evaluate $\oint (x^2y\cos x + 2xy\sin x - y^2e^x)dx + (x^2\sin x - 2ye^x)dy$ around the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.

$$P = x^2y\cos x + 2xy\sin x - y^2e^x,\ Q = x^2\sin x - 2\,ye^x$$

Then $\partial P/\partial y = x^2\cos x + 2x\sin x - 2ye^x = \partial Q/\partial x$, so that, by Problem 10.11, the line integral around any closed path—in particular, $x^{2/3} + y^{2/3} = a^{2/3}$—is zero.

Surface integrals

10.16. If γ is the angle between the normal line to any point (x, y, z) of a surface S and the positive z axis, prove that

$$|\sec\gamma| = \sqrt{1 + z_x^2 + z_y^2} = \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}$$

according as the equation for S is $z = f(x, y)$ or $F(x, y, z) = 0$.

If the equation for S is $F(x, y, z) = 0$, a normal to S at (x, y, z) is $\nabla F = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$. Then

$$\nabla F \cdot \mathrm{K} = |\nabla F||\mathbf{k}|\cos\gamma \quad \text{or} \quad F_z = \sqrt{F_x^2 + F_y^2 + F_z^2}\cos\gamma$$

from which $|\sec\gamma| = \dfrac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}$ as required. as required.

In case the equation is $z = f(x, y)$, we can write $F(x, y) = 0$, from which $F_x = -z_x$, $F_y - z_y$, $F_z = 1$ and we find $|\sec\gamma| = \sqrt{1 + z_x^2 + z_y^2}$.

10.17. Evaluate $\iint_S U(x, y, z)dS$, where S is the surface of the paraboloid $z = 2 - (x^2 + y^2)$ above the xy plane and $U(x, y, z)$ is equal to (a) 1, (b) $x^2 + y^2$, y^2, and (c) $3z$. Give a physical interpretation in each case. (See Figure 10.14.)

The required integral is equal to

$$\iint_{\Re} U(x, y, z)\sqrt{1 + z_x^2 + z_y^2}\,dx\,dy. \qquad (1)$$

where $\Re$ is the projection of S on the xy plane given by $x^2 + y^2 = 2$, $z = 0$.

Since $z_x = -2x$, $z_y = -2y$, (1) can be written

$$\iint_{\Re} U(x, y, z)\sqrt{1 + 4x^2 + 4y^2}\,dx\,dy \qquad (2)$$

(a) If U(x, y, z) = 1, (2) becomes

$$\iint_{\Re} \sqrt{1 + 4x^2 + 4y^2}\,dx\,dy$$

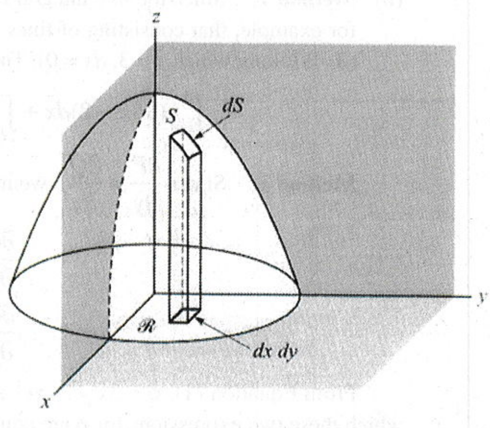

Figure 10.14

To evaluate this, transform to polar coordinates (ρ, ϕ). Then the integral becomes

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\sqrt{2}}\sqrt{1+4\rho^2}\;\rho\,d\rho\,d\phi=\int_{\phi=0}^{2\pi}\frac{1}{12}(1+4\rho^2)^{3/2}\Big|_{\rho=0}^{\sqrt{2}}d\phi=\frac{13\pi}{3}$$

Physically, this could represent the surface area of S or the mass of S assuming unit density.

(b) If U(x, y, z) = x2 + y2, (2) becomes $\iint_{\Re}(x^2+y^2)\sqrt{1+4x^2+4y^2}\,dx\,dy$ or, in polar coordinates,

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\sqrt{2}}\rho^3\sqrt{1+4\rho^2}\;d\rho\,d\phi=\frac{149\pi}{30}$$

where the integration with respect to ρ is accomplished by the substitution $\sqrt{1+4\rho^2}=u$.

Physically, this could represent the moment of inertia of S about the z axis assuming unit density, or the mass of S assuming a density = x^2+y^2.

(c) If U(x, y, z) = 3z, (2) becomes

$$\iint_{\Re}3z\sqrt{1+4x^2+4y^2}\,dx\,dy=\iint_{\Re}3\{2-(x^2+y^2)\}\sqrt{1+4x^2+4y^2}\,dx\,dy$$

or, in polar coordinates,

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\sqrt{2}}3\rho(2-\rho^2)\sqrt{1+4\rho^2}\,d\rho\,d\phi=\frac{111\pi}{10}$$

Physically, this could represent the mass of S assuming a density = $3z$, or three times the first moment of S about the xy plane.

10.18. Find the surface area of a hemisphere of radius a cut off by a cylinder having this radius as diameter.

Equations for the hemisphere and cylinder (see Figure 10.15) are given, respectively, by $x^2+y^2+z^2=a^2$ (or $z\ \sqrt{a^2-x^2-y^2}$) and $(x-a/2)^2+y^2=a^2/4$ (or $x^2+y^2=ax$).

Since

$$z_x=\frac{-x}{\sqrt{a^2-x^2-y^2}}\quad\text{and}\quad z_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$$

we have

$$\text{Required surface area}=2\iint_{\Re}\sqrt{1+z_x^2+z_y^2}\,dx\,dy=2\iint_{\Re}\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

Two methods of evaluation are possible.

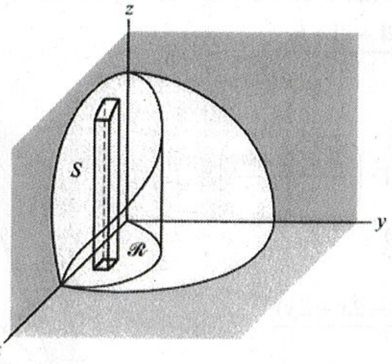

Figure 10.15

Method 1: Using polar coordinates.

Since $x^2 + y^2 = ax$ in polar coordinates is $\rho = a\cos\phi$, the integral becomes

$$2\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{a\cos\phi}\frac{a}{\sqrt{a^2-\rho^2}}\rho\,d\rho\,d\phi = 2a\int_{\phi=0}^{\pi/2}-\sqrt{a^2-\rho^2}\Big|_{\rho=0}^{a\cos\phi}d\phi$$

$$= 2a^2\int_0^{\pi/2}(1-\sin\phi)d\phi = (\pi-2)a^2$$

Method 2: The integral is equal to

$$2\int_{x=0}^{a}\int_{y=0}^{\sqrt{ax-x^2}}\frac{a}{\sqrt{a^2-x^2-y^2}}dx\,dy = 2a\int_{x=0}^{a}\sin^{-1}\frac{y}{\sqrt{ax-x^2}}\Bigg|_{y=0}^{\sqrt{ax-x^2}}dx$$

$$= 2a\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}dx$$

Letting $x = a\tan^2\theta$, this integral becomes

$$4a^2\int_0^{\pi/4}\theta\tan\theta\sec^2\theta d\theta = 4a^2\left\{\frac{1}{2}\theta\tan^2\theta\,\Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\tan^2\theta d\theta\right\}$$

$$= 2a^2\left\{\theta\tan^2\theta\,\Big|_0^{\pi/4} - \int_0^{\pi/4}(\sec^2\theta - 1)d\theta\right\}$$

$$= 2a^2\left\{\pi/4 - (\tan\theta - \theta)\,\Big|_0^{\pi/4}\right\} = (\pi-2)a^2$$

Note that these integrals are actually *improper* and should be treated by appropriate limiting procedures (see Problem 5.74 and Chapter 12).

10.19. Find the centroid of the surface in Problem 10.17.

By symmetry, $\bar{x} = \bar{y} = 0$ and $\bar{z} = \dfrac{\iint_S z\,dS}{\iint_S z\,dS} = \dfrac{\iint_{\Re} z\sqrt{1+4x^2+4y^2}\,dx\,dy}{\iint_{\Re}\sqrt{1+4x^2+4y^2}\,dx\,dy}$

The numerator and denominator can be obtained from the results of Problems 10.17(c) and 10.17(a), respectively, and we thus have $\bar{z} = \dfrac{37\pi/10}{13\pi/3} = \dfrac{111}{130}$.

10.20. Evalute $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$, where $\mathbf{A} = xy\mathbf{i} - x^2\mathbf{j} + (x+z)\mathbf{k}$, S is that portion of the plane $2x + 2y + z = 6$ included in the first octant, and $\mathbf{n}$ is a unit normal to S. (See Figure 10.16.)

A normal to S is $\nabla(2x+2y+z-6) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, and so

$$\mathbf{n} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{2^2+2^2+1^2}} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{3}$$

Then

$$\mathbf{A}\cdot\mathbf{n} = \{xy\mathbf{i} - x^2\mathbf{j} + (x+z)\mathbf{k}\}\cdot\left(\frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{3}\right)$$

$$= \frac{2xy - 2x^2 + (x+z)}{3}$$

$$= \frac{2xy - 2x^2 + (x+6-2x-2y)}{3}$$

$$= \frac{2xy - 2x^2 - x - 2y + 6}{3}$$

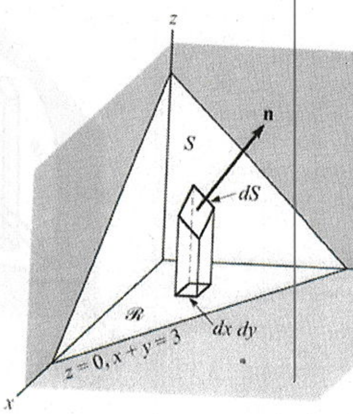

Figure 10.16

The required surface integral is, therefore,

$$\iint_S \left(\frac{2xy - 2x^2 - x - 2y + 6}{3}\right) dS = \iint_{\Re} \left(\frac{2xy - 2x^2 - x - 2y + 6}{3}\right) \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy$$
$$= \iint_{\Re} \left(\frac{2xy - 2x^2 - x - 2y + 6}{3}\right) \sqrt{1^2 + 2^2 + 2^2}\, dx\, dy$$
$$= \int_{x=0}^{3} \int_{y=0}^{3-x} (2xy - 2x^2 - x - 2y + 6) dy\, dx$$
$$= \int_{x=0}^{3} (xy^2 - 2x^2 y - xy - y^2 + 6y)\Big|_0^{3-x} dx = 27/4$$

10.21. In dealing with surface integrals we have restricted ourselves to surface which are two-sided. Give an example of a surface which is not two-sided.

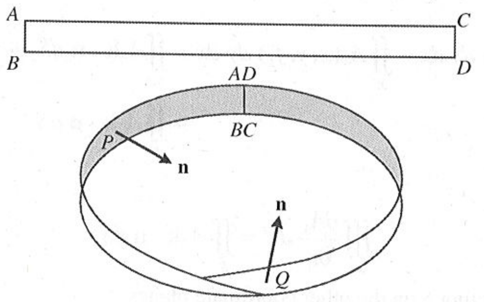

Figure 10.17

Take a strip of paper such as *ABCD*, as shown in Figure 10.17. Twist the strip so that points *A* and *B* fall on *D* and *C*, respectively, as in the figure. If **n** is the positive normal at point *P* of the surface, we find that as **n** moves around the surface, it reverses its original direction when it reaches *P* again. If we tried to color only one side of the surface, we would find the whole thing colored. This surface, called a *Möbius strip*, is an example of a one-sided surface. This is sometimes called a *nonorientable* surface. A two-sided surface is *orientable*.

The divergence theorem

10.22. Prove the divergence theorem. (See Figure 10.18.)

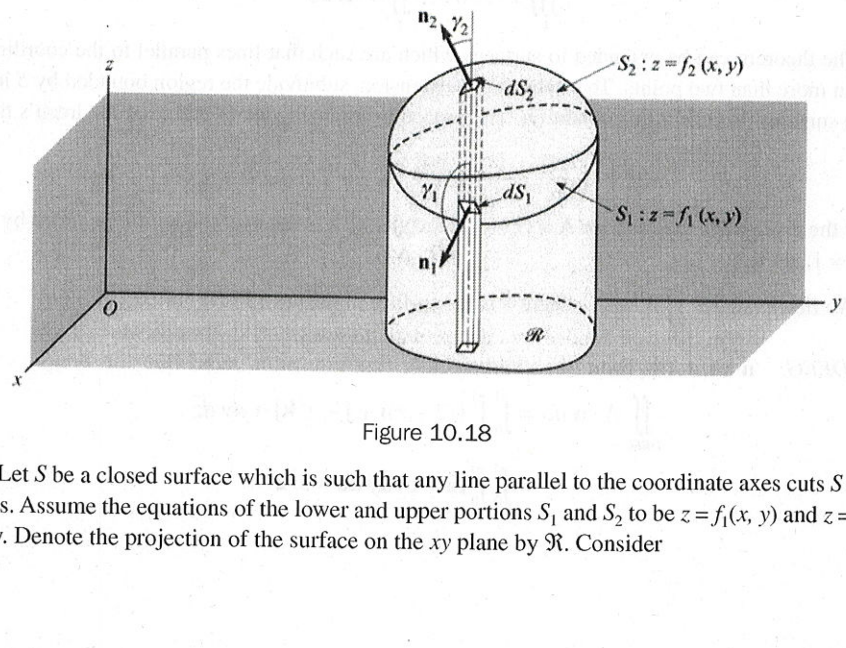

Figure 10.18

Let S be a closed surface which is such that any line parallel to the coordinate axes cuts S in, at most, two points. Assume the equations of the lower and upper portions S_1 and S_2 to be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. Denote the projection of the surface on the xy plane by $\Re$. Consider

$$\iiint_V \frac{\partial A_3}{\partial z} dV = \iiint_V \frac{\partial A_3}{\partial z} dz\, dy\, dx = \iint_{\Re} \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial A_3}{\partial z} dz \right] dy\, dx$$

$$= \iint_{\Re} A_3(x,y,z)\Big|_{z=f_1}^{f_2} dy\, dx = \iint_{\Re} [A_3(x,y,f_2) - A_3(x,y,f_1)]\, dy\, dx$$

For the upper portion S_2, $dy\, dx = \cos\gamma_2\, dS_2 = \mathbf{k} \cdot \mathbf{n}_2\, dS_2$ since the normal $\mathbf{n}_2$ to S_2 makes an acute angle γ_2 with $\mathbf{k}$.

For the lower portion S_1, $dy\, dx = -\cos\gamma_1\, dS_1 = -\mathbf{k} \cdot \mathbf{n}_1\, dS_1$ since the normal $\mathbf{n}_1$ to S_1 makes an obtuse angle γ_1 with $\mathbf{k}$.

Then

$$\iint_{\Re} A_3(x,y,f_2)\, dy\, dx = \iint_{S_2} A_3\mathbf{k} \cdot \mathbf{n}_2\, dS_2$$

$$\iint_{\Re} A_3(x,y,f_1)\, dy\, dx = -\iint_{S_1} A_3\mathbf{k} \cdot \mathbf{n}_1\, dS_1$$

and

$$\iint_{\Re} A_3(x,y,f_2)\, dy\, dx - \iint_{\Re} A_3(x,y,f_1)\, dy\, dx = \iint_{S_2} A_3\mathbf{k} \cdot \mathbf{n}_2 dS_2 + \iint_{S_1} A_3\mathbf{k} \cdot \mathbf{n}_1 dS_1$$

$$= \iint_S A_3\mathbf{k} \cdot \mathbf{n}\, dS$$

so that

$$\iiint_V \frac{\partial A_3}{\partial z}\, dV = \iint_S A_3\mathbf{k} \cdot \mathbf{n}\, dS \tag{1}$$

Similarly, by projecting S on the other coordinate planes,

$$\iiint_V \frac{\partial A_1}{\partial x}\, dV = \iint_S A_1\mathbf{i} \cdot \mathbf{n}\, dS \tag{2}$$

$$\iiint_V \frac{\partial A_2}{\partial y}\, dV = \iint_S A_2\mathbf{j} \cdot \mathbf{n}\, dS \tag{3}$$

Adding Equations (1), (2), and (3),

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV = \iint_S (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\, \mathbf{n}\, dS$$

or

$$\iiint_V \nabla \cdot \mathbf{A}\, dV = \iint_S \mathbf{A} \cdot \mathbf{n}\, dS$$

The theorem can be extended to surfaces which are such that lines parallel to the coordinate axes meet them in more than two points. To establish this extension, subdivide the region bounded by S into subregions whose surfaces do satisfy this condition. The procedure is analogous to that used in Green's theorem for the plane.

10.23. Verify the divergence theorem for $\mathbf{A} = (2x - z)\mathbf{i} + x^2y\mathbf{j} - xz^2\mathbf{k}$ taken over the region bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

We first evaluate $\iint_S \mathbf{A} \cdot \mathbf{n} dS$, where S is the surface of the cube in Figure 10.19.

Face DEFG: $\mathbf{n} = \mathbf{i}$, $x = 1$. Then

$$\iint_{DEFG} \mathbf{A} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 \{(2 - z)\mathbf{i} + \mathbf{j} - z^2\mathbf{k}\} \cdot \mathbf{i}\, dy\, dz$$

$$= \int_0^1 \int_0^1 (2 - z)\, dy\, dz = 3/2$$

Face ABCO: $\mathbf{n} = -\mathbf{i}$, $x = 0$. Then

$$\iint_{ABCO} \mathbf{A}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 (-z\mathbf{i})\cdot(-\mathbf{i})\, dy\, dz$$

$$= \int_0^1\int_0^1 z\, dy\, dz = 1/2$$

Face ABEF: $\mathbf{n} = \mathbf{j}$, $y = 1$. Then

$$\iint_{ABEF} \mathbf{A}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 \{(2-z)\mathbf{i} + x^2\mathbf{j} - xz^2\mathbf{k}\}\cdot\mathbf{j}\, dx\, dz = \int_0^1\int_0^1 x^2\, dx\, dz = 1/3$$

Face OGDC: $\mathbf{n} = -\mathbf{j}$, $y = 0$. Then

$$\iint_{OGDC} \mathbf{A}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 \{(2x-z)\mathbf{i} - xz^2\mathbf{k}\}\cdot(-\mathbf{j})\, dx\, dz = 0$$

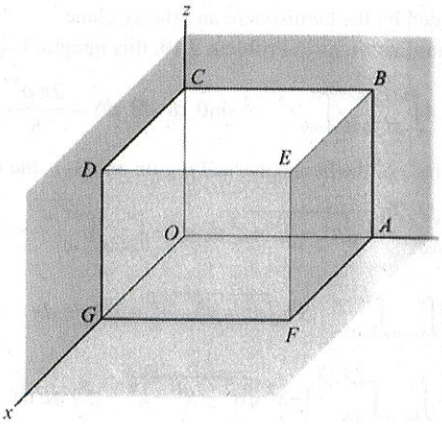

Figure 10.19

Face BCDE: $\mathbf{n} = \mathbf{k}$, $z = 1$. Then

$$\iint_{BCDE} \mathbf{A}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 \{(2x-1)\mathbf{i} + x^2y\mathbf{j} - x\mathbf{k}\}\cdot\mathbf{k}\, dx\, dy = \int_0^1\int_0^1 -x\, dx\, dy - 1/2$$

Face AFGO: $\mathbf{n} = -\mathbf{k}$, $z = 0$. Then

$$\iint_{AFGO} \mathbf{A}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 \{2x\mathbf{i} - x^2y\mathbf{j}\}\cdot(-\mathbf{k})\, dx\, dy = 0$$

Adding, $\iint_S \mathbf{A}\cdot\mathbf{n}\, dS = \frac{3}{2} + \frac{1}{2} + \frac{1}{3} + 0 - \frac{1}{2} + 0 = \frac{11}{6}$. Since

$$\iiint_V \nabla\cdot\mathbf{A}\, dV = \int_0^1\int_0^1\int_0^1 (2 + x^2 - 2xz)dx\, dy\, dz = \frac{11}{6}$$

the divergence theorem is verified in this case.

10.24. Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\, ds$, where S is a closed surface.

By the divergence theorem,

$$\iint_S \mathbf{r}\cdot\mathbf{n}\, dS = \iiint_V \nabla\cdot\mathbf{r}\, dV$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})dV$$

$$= \iiint_V \left(\frac{\partial x}{\partial x}\mathbf{i} + \frac{\partial y}{\partial y}\mathbf{j} + \frac{\partial z}{\partial z}\mathbf{k}\right)dV = 3\iiint_V dV = 3V$$

where V is the volume enclosed by S.

10.25. Evaluate $\iint_S xz^2 dy\, dz + (x^2y - z^3)dz\, dx + (2xy + y^2z)dx\, dy$, where S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$ (a) by the divergence theorem (Green's theorem in space) and (b) directly.

(a) Since $dy\, dz = dS \cos\alpha$, $dz\, dx = dS\, dS \cos\beta$, and $dx\, dy = dS \cos\gamma$, the integral can be written

$$\iint_S \{xz^2 \cos\alpha + (x^2y - z^3)\cos\beta + (2xy + y^2z)\cos\gamma\} dS = \iint_S \mathbf{A}\cdot\mathbf{n}\, dS$$

where $\mathbf{A} = xz^2\mathbf{i} + (x^2y - z^3)\,\mathbf{j} + (2xy + y^2z)\mathbf{k}$ and $\mathbf{n} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$, the outward drawn unit normal. Then, by the divergence theorem the integral equals

$$\iiint_V \nabla\cdot\mathbf{A}\, dV = \iiint_V \left\{\frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(x^2y - z^3) + \frac{\partial}{\partial z}(2xy + y^2z)\right\} dV = \iiint_V (x^2 + y^2 + z^2) dV$$

where V is the region bounded by the hemisphere and the xy plane.

By use of spherical coordinates, as in Problem 9.19, this integral is equal to

$$4\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\alpha} r^2 \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{2\pi a^5}{5}$$

(b) If S_1 is the convex surface of the hemispherical region and S_2 is the base ($z = 0$), then

$$\iint_{S_1} xz^2\, dy\, dz = \int_{y=-a}^{a}\int_{z=0}^{\sqrt{a^2-y^2}} z^2\sqrt{a^2 - y^2 - z^2}\, dz\, dy - \int_{y=-a}^{a}\int_{z=0}^{\sqrt{a^2-x^2}} -z^2\sqrt{a^2 - y^2 - z^2}\, dz\, dx$$

$$\iint_{S_1} (x^2y - z^3)\, dy\, dx = \int_{x=-a}^{a}\int_{x=0}^{\sqrt{a^2-x^2}} \{x^2\sqrt{a^2 - y^2 - z^2} - z^3\}\, dz\, dx$$

$$-\int_{x=-a}^{a}\int_{z=0}^{\sqrt{a^2-x^2}} \{-x^2\sqrt{a^2 - x^2 - z^2} - z^3\}\, dz\, dx$$

$$\iint_{S_1} (2xy - y^2z)\, dx\, dy = \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \{2xy + y^2\sqrt{a^2 - y^2 - z^2}\}\, dy\, dx$$

$$\iint_{S_2} xz^2 dy\, dz = 0, \qquad \iint_{S_2} (x^2y - z^3)\, dz\, dx = 0,$$

$$\iint_{S_2} (2xy - y^2z)\, dx\, dy = \iint_{S_2} \{2xy - y^2(0)\}\, dx\, dy = \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2xy\, dy\, dx = 0$$

By addition of the preceding, we obtain

$$4\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} z^2\sqrt{a^2 - y^2 - z^2}\, dz\, dy + 4\int_{x=0}^{a}\int_{z=0}^{\sqrt{a^2-x^2}} x^2\sqrt{a^2 - x^2 - z^2}\, dz\, dx$$

$$+4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y^2\sqrt{a^2 - x^2 - y^2}\, dy\, dx$$

Since by symmetry all these integrals are equal, the result, on using polar coordinates, is

$$12\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y^2\sqrt{a^2 - x^2 - y^2}\, dy\, dx = 12\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{a} \rho^2\sin^2\phi\sqrt{a^2 - \rho^2}\, \rho\, d\rho\, d\phi = \frac{2\pi a^5}{5}$$

Stokes's theorem

10.26. Prove Stokes's theorem.

Let S be a surface which is such that its projections on the xy, yz, and xz planes are are regions bounded by simple closed curves, as indicated in Figure 10.20. Assume S to have representation $z = f(x, y)$ or $x = g(y, z)$ or $y = h(x, z)$, where f, g, and h are single-valued, continuous, and differentiable functions. We must show that

$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S [\nabla\times(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k})]\cdot\mathbf{n}\,dS$$
$$= \int_C A\cdot d\mathbf{r}$$

where C is the boundary of S.

Consider first $\iint_S [\nabla\times(A_1\mathbf{i})]\cdot\mathbf{n}\,dS$.

Since $\nabla\times(A_1\mathbf{i}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & 0 & 0\end{vmatrix} = \frac{\partial A_1}{\partial z}\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{k}$,

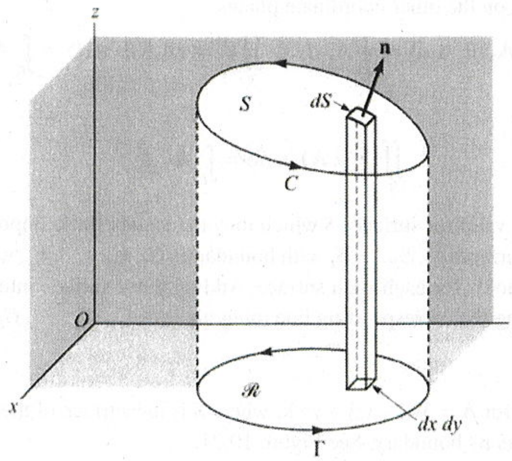

Figure 10.20

$$[\nabla\times(A_1\mathbf{i})]\cdot\mathbf{n}\,dS = \left(\frac{\partial A_1}{\partial z}\mathbf{n}\cdot\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{n}\cdot\mathbf{k}\right)dS \tag{1}$$

If $z = f(x, y)$ is taken as the equation of S, then the position vector to any point of S is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$ so that $\frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k} = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$. But $\frac{\partial \mathbf{r}}{\partial y}$ is a vector tangent to S and thus perpendicular to $\mathbf{n}$, so that

$$\mathbf{n}\cdot\frac{\partial \mathbf{r}}{\partial y} = \mathbf{n}\cdot\mathbf{j} + \frac{\partial z}{\partial y}\mathbf{n}\cdot\mathbf{k} = 0 \quad \text{or} \quad \mathbf{n}\cdot\mathbf{j} = -\frac{\partial z}{\partial y}\mathbf{n}\cdot\mathbf{k}$$

Substitute in Equation (1) to obtain

$$\left(\frac{\partial A_1}{\partial z}\mathbf{n}\cdot\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{n}\cdot\mathbf{k}\right)dS = \left(\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\mathbf{n}\cdot\mathbf{k} - \frac{\partial A_1}{\partial y}\mathbf{n}\cdot\mathbf{k}\right)dS$$

or

$$[\nabla\times(A_1\mathbf{i})]\cdot\mathbf{n}\,dS = -\left(\frac{\partial A_1}{\partial z} + \frac{\partial A_1}{\partial y}\frac{\partial z}{\partial y}\right)\mathbf{n}\cdot\mathbf{k}\,dS \tag{2}$$

Now on S, $A_1(x, y, z) = A_1[x, y, f(x, y)] = F(x, y)$; hence, $\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y}$ and Equation (2) becomes

$$[\nabla\times(A_1\mathbf{i})]\cdot\mathbf{n}\,dS = -\frac{\partial F}{\partial y}\mathbf{n}\cdot\mathbf{k}\,dS = -\frac{\partial F}{\partial y}dx\,dy$$

Then

$$\iint_S [\nabla \times (A_1\mathbf{i})] \cdot \mathbf{n}\, dS = \iint_{\Re} -\frac{\partial F}{\partial y} dx\, dy$$

where $\Re$ is the projection of S on the xy plane. By Green's theorem for the plane, the last integral equals $\oint_\Gamma F\, dx$ where Γ is the boundary of $\Re$. Since at each point (x, y) of Γ the value of F is the same as the value of A_1 at each point (x, y, z) of C, and since dx is the same for both curves, we must have

$$\oint_\Gamma F\, dx = \oint_C A_1\, dx$$

or

$$\iint_S [\nabla \times (A_1\mathbf{i})] \cdot \mathbf{n}\, dS = \oint_C A_1\, dx$$

Similarly, by projections on the other coordinate planes,

$$\iint_S [\nabla \times (A_2\mathbf{j})] \cdot \mathbf{n}\, dS = \oint_C A_2\, dy, \quad \iint_S [\nabla \times (A_3\mathbf{k})] \cdot \mathbf{n}\, dS = \oint_C A_3\, dz$$

Thus, by addition,

$$\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{A} \cdot d\mathbf{r}$$

The theorem is also valid for surfaces S which may not satisfy these imposed restrictions. Assume that S can be subdivided into surfaces $S_1, S_2, \ldots S_k$ with boundaries $C_1, C_2, \ldots, C_k$, which do satisfy the restrictions. Then Stokes's theorem holds for each such surface. Adding these surface integrals, the total surface integral over S is obtained. Adding the corresponding line integrals over $C_1, C_2 \ldots, C_k$, the line integral over C is obtained.

10.27. Verify Stokes's theorem for $\mathbf{A} = 3y\,\mathbf{i} - xz\mathbf{j} + yz^2\mathbf{k}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary. See Figure 10.21.

The boundary C of S is a circle with equations $x^2 + y^2 = 4$, $z = 2$ and parametric equations $x = 2\cos t$, $y = 2\sin t$, $z = 2$, where $0 \leqq t < 2\pi$. Then

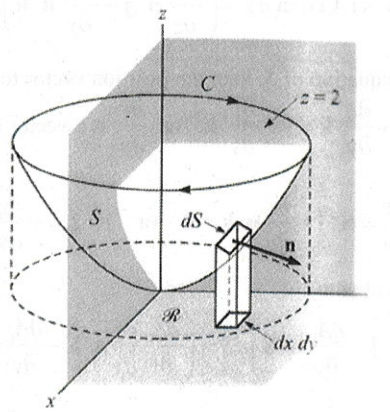

Figure 10.21

$$\begin{aligned}\oint_C \mathbf{A} \cdot d\mathbf{r} &= \oint_C 3y\, dx - xz\, dy + yz^2 dz \\ &= \int_{2\pi}^{0} 3(2\sin t)(-2\sin t)dt - (2\cos t)(2)(2\cos t)dt \\ &= \int_0^{2\pi} (12\sin^2 t + 8\cos^2 t)dt = 20\pi\end{aligned}$$

Also,

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix} = (z^2 + x)\mathbf{i} - (z+3)\mathbf{k}$$

and

$$\mathbf{n} = \frac{\nabla(x^2 + y^2 - 2z)}{|\nabla(x^2 + y^2 - 2z)|} = \frac{x\mathbf{i} + y\mathbf{j} - \mathbf{k}}{\sqrt{x^2 + y^2 + 1}}.$$

Then

$$\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = \iint_{\Re} (\nabla \cdot A) \cdot \mathbf{n} \frac{dx\, dy}{|\mathbf{n} \cdot \mathbf{k}|} = \iint_{\Re} (xz^2 + x^2 + z + 3)\, dx\, d$$

$$= \iint_{\Re} \left\{ x\left(\frac{x^2 + y^2}{2}\right)^2 + x^2 + \frac{x^2 + y^2}{2} + 3 \right\} dx\, dy$$

In polar coordinates this becomes

$$\int_{\phi=0}^{2\pi} \int_{\rho=0}^{2} \{(\rho \cos\phi)(\rho^4/2) + \rho^2 \cos^2\phi + \rho^2/2 + 3\}\rho\, d\rho\, d\phi = 20\pi$$

10.28. Prove that a necessary and sufficient condition that $\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$ for every closed curve C is that $\nabla \times \mathbf{A} = 0$ identically.

Sufficiency. Suppose $\nabla \times \mathbf{A} = \mathbf{0}$. Then, by Stokes's theorem,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = 0$$

Necessity. Suppose $\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$ around every closed path C, and assume $\nabla \times \mathbf{A} \neq \mathbf{0}$ at some point P. Then, assuming $\nabla \times \mathbf{A}$ is continuous, there will be a region with P as an interior point, where $\nabla \times \mathbf{A} \neq \mathbf{0}$. Let S be a surface contained in this region whose normal $\mathbf{n}$ at each point has the same direction as $\nabla \times \mathbf{A}$; i.e., $\nabla \times \mathbf{A} = \alpha\mathbf{n}$ where α is a positive constant. Let C be the boundary of S. Then, by Stokes's theorem,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = \alpha \iint_S \mathbf{n} \cdot \mathbf{n}\, dS > 0$$

which contradicts the hypothesis that $\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$ and shows that $\nabla \times \mathbf{A} = 0$.

It follows that $\nabla \times \mathbf{A} = \mathbf{0}$ is also a necessary and sufficient condition for a line integral $\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{r}$ to be independent of the path joining points P_1 and P_2.

10.29. Prove that a necessary and sufficient condition that $\nabla \times \mathbf{A} = \mathbf{0}$ is that $\mathbf{A} = \nabla\phi$.

Sufficiency. If $\mathbf{A} = \nabla\phi$, then $\nabla \times \mathbf{A} = \nabla \times \nabla\phi = \mathbf{0}$ by Problem 7.80.

Necessity. If $\nabla \times \mathbf{A} = \mathbf{0}$, then by Problem 10.28, $\oint_C \mathbf{A} \cdot d\mathbf{r} = \mathbf{0}$ around every closed path and $\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{r}$ is independent of the path joining two points, which we take as (a, b, c) and (x, y, z). Let us define

$$\phi(x, y, z) = \int_{(a,b,c)}^{(x,y,z)} \mathbf{A} \cdot d\mathbf{r} = \int_{(a,b,c)}^{(x,y,z)} A_1 dx + A_2 dy + A_3 dz$$

Then

$$\phi(x + \Delta x, y, z) - \phi(x, y, z) = \int_{(x,y,z)}^{(x+\Delta x,y,z)} A_1 dx + A_2 dy + A_3 dz$$

Since the last integral is independent of the path joining (x, y, z) and $(x + \Delta x, y, z)$, we can choose the path to be a straight line joining these points so that dy and dz are zero. Then

$$\frac{\phi(x + \Delta x, y, z) - \phi(x, y, z)}{\Delta x} = \frac{1}{\Delta x} \int_{(x,y,z)}^{(x+\Delta x,y,z)} A_1 dx = A_1(x + \theta\Delta x, y, z) \quad 0 < \theta < 1$$

where we have applied the law of the mean for integrals.

Taking the limit of both sides as $\Delta x \to 0$ gives $\partial\phi/\partial x = A_1$.

Similarly, we can show that $\partial\phi/\partial y = A_2$, $\partial\phi/\partial z = A_3$. Thus,

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \nabla\phi.$$

10.30. (a) Prove that a necessary and sufficient condition that $A_1\,dx + A_2\,dy + A_3\,dz = d\phi$, an exact differential, is that $\nabla \times \mathbf{A} = \mathbf{0}$ where $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$. (b) Show that in such case,

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} A_1dx + A_2dy + A_3dz = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} d\phi = \phi(x_2,y_2,z_2) - \phi(x_1,y_1,z_1)$$

(a) **Necessity.** If $A_1dx + A_2dy + A_3dz = d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$, then

$$\frac{\partial\phi}{\partial x} = A_1 \tag{1}$$

$$\frac{\partial\phi}{\partial y} = A_2 \tag{2}$$

$$\frac{\partial\phi}{\partial z} = A_3 \tag{3}$$

Then, by differentiating, and assuming continuity of the partial derivatives, we have

$$\frac{\partial A_1}{\partial y} = \frac{\partial A_2}{\partial x}, \quad \frac{\partial A_2}{\partial z} = \frac{\partial A_3}{\partial y}, \quad \frac{\partial A_1}{\partial z} = \frac{\partial A_3}{\partial x}$$

which is precisely the condition $\nabla \times \mathbf{A} = \mathbf{0}$.

Another method: If $A_1\,dx + A_2\,dy + A_3\,dz = d\phi$, then

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \nabla\phi.$$

from which $\nabla \times \mathbf{A} = \nabla \times \nabla\phi = \mathbf{0}$.

Sufficiency. If $\nabla \times \mathbf{A} = \mathbf{0}$, then by Problem 10.29, $\mathbf{A} = \nabla\phi$ and

$$A_1dx + A_2dy + A_3dz = \mathbf{A}\cdot d\mathbf{r} = \nabla\phi \cdot d\mathbf{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

(b) From (a),

$$\phi(x,y,z) = \int_{(a,b,c)}^{(x,y,z)} A_1dx + A_2dy + A_3dz$$

Then, omitting the integrand $A_1\,dx + A_2\,dy + A_3\,dz$, we have

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} = \int_{(a,b,c)}^{(x_2,y_2,z_2)} - \int_{(a,b,c)}^{(x_1,y_1,z_1)} = \phi(x_2,y_2,z_2) - \phi(x_1,y_1,z_1)$$

10.31. (a) Prove that $\mathbf{F} = (2xz^3 + 6y)\mathbf{i} + (6x - 2yz)\mathbf{j} + (3x^2z^2 - y^2)\,\mathbf{k}$ is a conservative force field. (b) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is any path from $(1, -1, 1)$ to $(2, 1, -1)$. (c) Give a physical interpretation of the results.

(a) A force field $\mathbf{F}$ is conservative if the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining any two points. A necessary and sufficient condition that F be conservative is that $\nabla \times \mathbf{F} = \mathbf{0}$.

Since here $\nabla \times \mathrm{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^3 + 6y & 6x - 2yz & 3x^2z^2 - y^2 \end{vmatrix} = \mathbf{0}$, **F** is conservative

(b) **Method 1:** By Problem 10.30, $\mathbf{F}\cdot d\mathbf{r} = (2xz^3 + 6y)dx + (6x - 2yz)dy + (3x^2z^2 - y^2)dz$ is an exact differential $d\phi$, where ϕ is such that

$$\frac{\partial \phi}{\partial x} = 2xz^3 + 6y \tag{1}$$

$$\frac{\partial \phi}{\partial y} = 6x - 2yz \tag{2}$$

$$\frac{\partial \phi}{\partial z} = 3x^2z^2 - y^2 \tag{3}$$

From these we obtain, respectively,

$$\phi = x^2z^3 + 6xy + f_1(y, z)$$

$$\phi = 6xy - y^2z + f_2(x, z)$$

$$\phi = x^2y^2 - y^2z + f_3(x, y)$$

These are consistent if $f_1(y, z) = -y^2z + c$, $f_2(x, z) = x^2z^3 + c$, and $f_3(x, y) = 6xy + c$, in which case $\phi = x^2z^3 + 6xy - y^2z + c$. Thus, by Problem 10.30,

$$\int_{(1,-1,1)}^{(2,1,-1)} \mathbf{F}\cdot d\mathbf{r} = x^2z^3 + 6xy - y^2z + c \,\Big|_{(1,-1,1)}^{(2,1,-1)} = 15$$

Alternatively, we may notice by inspection that

$$\mathbf{F}\cdot d\mathbf{r} = (2xz^3\,dx + 3x^2z^2\,dz) + (6y\,dx + 6x\,dy) - (2yz\,dy + y^2\,dz)$$

$$= d(x^2z^3) + d(6xy) - d(y^2z) = d(x^2z^3 + 6xy - y^2z + c) \text{ from which } \phi \text{ is determined.}$$

Method 2: Since the integral is independent of the path, we can choose any path to evaluate it; in particular, we can choose the path consisting of straight lines from (1, –1, 1) to (2, –1, 1), then to (2, 1, 1) and then to (2, 1, –1). The result is

$$\int_{x=1}^{2} (2x - 6)\,dx + \int_{y=-1}^{1} (12 - 2y)dy + \int_{z=1}^{-1} (12z^2 - 1)dz = 15$$

where the first integral is obtained from the line integral by placing $y = -1$, $z = 1$, $dy = 0$, $dz = 0$; the second integral, by placing $x = 2$, $z = 1$, $dx = 0$, $dz = 0$; and the third integral, by placing $x = 2$, $y = 1$, $dx = 0$, $dy = 0$.

(c) Physically, $\int_C \mathbf{F}\cdot d\mathbf{r}$ represents the work done in moving an object from (1, –1, 1) to (2, 1, –1) along C. In a conservative force field, the work done is independent of the path C joining these points.

Miscellaneous problems

10.32. (a) If $x = f(u, \upsilon)$, $y = g(u, \upsilon)$ defines a transformation which maps a region $\Re$ of the xy plane into a region $\Re'$ of the $u\upsilon$ plane, prove that

$$\iint_{\Re} dx\,dy = \iint_{\Re'} \left|\frac{\partial(x, y)}{\partial(u, \upsilon)}\right| du\,d\upsilon$$

(b) Interpret geometrically the result in (a).

(a) If C (assumed to be a simple closed curve) is the boundary of $\Re$, then by Problem 10.8,

$$\iint_{\Re} dx\,dy = \frac{1}{2}\oint_C x\,dy - y\,dx \tag{1}$$

Under the given transformation, the integral on the right of Equation (1) becomes

$$\frac{1}{2}\oint_{C'} x\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial \upsilon}d\upsilon\right) - y\left(\frac{\partial y}{\partial u}du + \frac{\partial x}{\partial \upsilon}d\upsilon\right) = \frac{1}{2}\int_{C'}\left(x\frac{\partial y}{\partial u} - y\frac{\partial x}{\partial u}\right)du + \left(x\frac{\partial y}{\partial \upsilon} - y\frac{\partial x}{\partial \upsilon}\right)d\upsilon \tag{2}$$

where C' is the mapping of C in the $u\upsilon$ plane (we suppose the mapping to be such that C' is a simple closed curve also).

By Green's theorem, if $\mathfrak{R}'$ is the region in the $u\upsilon$ plane bounded by C', the right side of Equation (2) equals

$$\frac{1}{2}\iint_{\mathfrak{R}'}\left|\frac{\partial}{\partial u}\left(x\frac{\partial y}{\partial \upsilon}-y\frac{\partial x}{\partial \upsilon}\right)-\frac{\partial}{\partial \upsilon}\left(x\frac{\partial y}{\partial u}-y\frac{\partial x}{\partial u}\right)\right| du\,d\upsilon = \iint_{\mathfrak{R}'}\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial \upsilon}-\frac{\partial x}{\partial \upsilon}\frac{\partial y}{\partial u}\right| du\,d\upsilon$$
$$= \iint_{\mathfrak{R}'}\left|\frac{\partial(x,y)}{\partial(u,\upsilon)}\right| du\,d\upsilon$$

where we have inserted absolute value signs so as to ensure that the result is nonnegative, as is $\int\int_{\mathfrak{R}} dx\,dy$.

In general, we can show (see Problem 10.83) that

$$\iint_{\mathfrak{R}} F(x,y)\,dx\,dy = \iint_{\mathfrak{R}'} F\{f(u,\upsilon),g(u,\upsilon)\}\left|\frac{\partial(x,y)}{\partial(u,\upsilon)}\right| du\,d\upsilon \tag{3}$$

(b) $\iint_{\mathfrak{R}} dx\,dy$ and $\iint_{\mathfrak{R}'}\left|\frac{\partial(x,y)}{\partial(u,\upsilon)}\right| du\,d\upsilon$ represent the area of region R, the first expressed in rectangular coordinates, the second in curvilinear coordinates. See Page 225, and the introduction of the differential element of surface area for an alternative to (a).

10.33. Let $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2}$. (a) Calculate $\nabla \times \mathbf{F}$. (b) Evaluate $\oint \mathbf{F}\cdot d\mathbf{r}$ around any closed path and explain the results.

(a) $$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} = \mathbf{0} \text{ in any region excluding } (0,0).$$

(b) $\oint \mathbf{F}\cdot d\mathbf{r} = \oint_C \dfrac{-y\,dx = x\,dy}{x^2+y^2}$. Let $x = \rho\cos\phi$, $y = \rho\sin\phi$, where (ρ, ϕ) are polar coordinates. Then

$$dx = -\rho\sin\phi\,d\phi + d\rho\cos\phi, \quad dy = \rho\cos\phi\,d\phi + d\rho\sin\phi$$

and so

$$\frac{-y\,dx = x\,dy}{x^2+y^2} = d\phi = d\left(\text{are}\tan\frac{y}{x}\right)$$

For a closed curve *ABCDA* [see Figure 10.22 (*a*)] surrounding the origin, $\phi = 0$ at A and $\phi = 2\pi$ after a complete circuit back to A. In this case the line integral equals $\int_0^{2\pi} d\phi = 2\pi$.

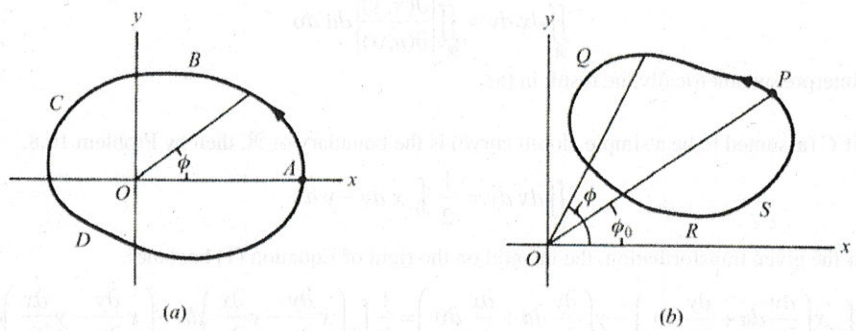

Figure 10.22

For a closed curve *PQRSP* [see Figure 10.22(*b*)] not surrounding the origin, $\phi = \phi_0$ at P and $\phi = \phi_0$ after a complete circuit back to P. In this case the line integral equals $\int_{\phi_0}^{\phi_0} d\phi = 0$.

Since $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, $\nabla \times \mathbf{F} = \mathbf{0}$ is equivalent to $\partial P/\partial y = \partial Q/\partial x$, the results would seem to contradict those of Problem 10.11. However, no contradiction exists, since $P = \dfrac{-y}{x^2 + y^2}$ and $Q = \dfrac{x}{x^2 + y^2}$ do not have continuous derivatives throughout any region including (0, 0), and this was assumed in Problem 10.11.

10.34. If div **A** denotes the divergence of a vector field **A** at a point P, show that

$$\operatorname{div} \mathbf{A} = \lim_{\Delta V \to 0} \frac{\iint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, dS}{\Delta V}$$

where ΔV is the volume enclosed by the surface ΔS and the limit is obtained by shrinking ΔV to the point P.

By the divergence theorem,

$$\iiint_{\Delta V} \operatorname{div} \mathbf{A}\, dV = \iint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, dS$$

By the mean value theorem for integrals, the left side can be written

$$\overline{\operatorname{div} \mathbf{A}} \iiint_{\Delta V} dV = \overline{\operatorname{div} \mathbf{A}}\, \Delta V$$

where $\overline{\operatorname{div} \mathbf{A}}$ is some value intermediate between the maximum and minimum of div **A** throughout ΔV. Then

$$\overline{\operatorname{div} \mathbf{A}} = \frac{\iint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, dS}{\Delta V}$$

Taking the limit as $\Delta V \to 0$ such that P is always interior to ΔV, $\overline{\operatorname{div} \mathbf{A}}$ approaches the value div **A** at point P: hence,

$$\operatorname{div} \mathbf{A} = \lim_{\Delta V \to 0} \frac{\iint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, dS}{\Delta V}$$

This result can be taken as a starting point for defining the divergence of **A**, and from it all the properties may be derived, including proof of the divergence theorem. We can also use this to extend the concept of divergence to coordinate systems other than rectangular (see Page 170).

Physically, $\left(\iiint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, dS \right) / \Delta V$ represents the flux or net outflow per unit volume of the vector **A** from the surface ΔS. If div **A** is positive in the neighborhood of a point P, it means that the outflow from P is positive, and we call P a *source*. Similarly, if div **A** is negative in the neighborhood of P, the outflow is really an inflow, and P is called a *sink*. If in a region there are no sources or sinks, then div $\mathbf{A} = 0$, and we call **A** a *solenoidal* vector field.

SUPPLEMENTARY PROBLEMS

Line Integrals

10.35. Evaluate $\int_{(1,1)}^{(4,2)} (x + y)dx + (y - x)dy$ along (a) the parabola $y^2 = x$, (b) a straight line, (c) straight lines from (1, 1) to (1, 2) and then to (4, 2), and (d) the curve $x = 2t^2 + t + 1$, $y = t^2 + 1$.

Ans. (a) 34/3 (b) 11 (c) 14 (d) 32/3

For a closed curve $ABCDA$ [see Fig. 10-10(b)] surrounding the origin, $\theta = 0$ at A and $\theta = 2\pi$ after a complete circuit back to A. In this case the line integral equals $\int_0^{2\pi} d\theta = 2\pi$.

Since $y = P\,dx + Q\,dy$ is not exact in any region containing the origin, the results would seem to contradict those of Problem 10.11. However, no contradiction exists since $P = \frac{-y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$ do not have continuous derivatives throughout any region including (0, 0), and this was assumed in Problem 10.11.

10.34. If div **A** denotes the divergence of a vector field **A** at a point P, show that

$$\operatorname{div} \mathbf{A} = \lim_{\Delta V \to 0} \frac{\iint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, dS}{\Delta V}$$

where ΔV is the volume enclosed by the surface ΔS and the limit is obtained by shrinking ΔV to the point P.

By the divergence theorem,

$$\iiint_{\Delta V} \operatorname{div} \mathbf{A}\, dV = \iint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, dS$$

By the mean value theorem for integrals, the left side can be written

$$\overline{\operatorname{div} \mathbf{A}} \iiint_{\Delta V} dV = \overline{\operatorname{div} \mathbf{A}}\, \Delta V$$

where $\overline{\operatorname{div} \mathbf{A}}$ is some value intermediate between the maximum and minimum of div **A** throughout ΔV. Then

$$\overline{\operatorname{div} \mathbf{A}} = \frac{\iint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, dS}{\Delta V}$$

Taking the limit as $\Delta V \to 0$ such that P is always interior to ΔV, $\overline{\operatorname{div} \mathbf{A}}$ approaches the value div **A** at point P; hence,

$$\operatorname{div} \mathbf{A} = \lim_{\Delta V \to 0} \frac{\iint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, dS}{\Delta V}$$

This result can be taken as a starting point for defining the divergence of **A**, and from it all the properties may be derived, including proof of the divergence theorem. We can also use this to extend the concept of divergence to coordinate systems other than rectangular (see Page 170).

Physically, $\left(\iint_{\Delta S} \mathbf{A} \cdot \mathbf{n}\, dS\right) / \Delta V$ represents the flux or net outflow per unit volume of the vector **A** from the surface ΔS. If div **A** is positive in the neighborhood of a point P, it means that the outflow from P is positive and we call P a source. Similarly, if div **A** is negative in the neighborhood of P, the outflow is really an inflow and P is called a sink. If in a region there are no sources or sinks, then div **A** = 0 and we call **A** a solenoidal vector field.

Line Integrals

10.36. Evaluate $\int_{(1,1)}^{(4,2)} (x+y)\,dx + (y-x)\,dy$ along (a) the parabola $y^2 = x$, (b) a straight line, (c) straight lines from (1, 1) to (1, 2) and then to (4, 2), and (d) the curve $x = 2t^2 + t + 1$, $y = t^2 + 1$.

Ans. (a) 34/3 (b) 11 (c) 14 (d) 32/3