of integrand that follows is suggested by the differential relation $dx = g'(t)\, dt$.] The fundamental theorem enabling us to do this is summarized in the statement

$$\int f(x)\, dx = \int f\{g(t)\}g'(t)\, dt \tag{6}$$

where, after obtaining the indefinite integral on the right, we replace t by its value in terms of x; i.e., $t = g^{-1}(x)$. This result is analogous to the chain rule for differentiation (see Page 76).

The corresponding theorem for definite integrals is

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f\{g(t)\}g'(t)\, dt \tag{7}$$

where $g(\alpha) = a$ and $g(\beta) = b$; i.e., $\alpha = g^{-1}(a)$, $\beta = g^{-1}(b)$. This result is certainly valid if $f(x)$ is continuous in $[a, b]$ and if $g(t)$ is continuous and has a continuous derivative in $\alpha \leq t \leq \beta$.

Integrals of Elementary Functions

The following results can be demonstrated by differentiating both sides to produce an identity. In each case, an arbitrary constant c (which has been omitted here) should be added.

1. $\int u^n\, du = \frac{u^{n+1}}{n+1} \quad n \neq -1$
2. $\int \frac{du}{u} = \ln |u|$
3. $\int \sin u\, du = -\cos u$
4. $\int \cos u\, du = \sin u$
5. $\int \tan u\, du = \ln |\sec u| = -\ln |\cos u|$
6. $\int \cot u\, du = \ln |\sin u|$
7. $\int \sec u\, du = \ln |\sec u + \tan u| = \ln |\tan (u/2 + \pi/4)|$
8. $\int \csc u\, du = \ln |\csc u - \cot u| = \ln |\tan u/2|$
9. $\int \sec^2 u\, du = \tan u$
10. $\int \csc^2 u\, du = -\cot u$
11. $\int \sec u \tan u\, du = \sec u$
12. $\int \csc u \cot u\, du = -\csc u$
13. $\int a^u\, du = \frac{a^u}{\ln a} \quad a > 0, a \neq 1$
18. $\int \coth u\, du = \ln |\sinh u|$
19. $\int \operatorname{sech} u\, du = \tan^{-1}(\sinh u)$
20. $\int \operatorname{csch} u\, du = -\coth^{-1} (\cosh u)$
21. $\int \operatorname{sech}^2 u\, du = \tanh u$
22. $\int \operatorname{csc}h^2 u\, du = -\coth u$
23. $\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u$
24. $\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u$
25. $\int \frac{du}{\sqrt{s^2 - u^2}} = \sin^{-1} \frac{u}{a}$ or $-\cos^{-1} \frac{u}{a}$
26. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln |u + \sqrt{u^2 \pm a^2}|$
27. $\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1} \frac{u}{a}$ or $-\frac{1}{a}\cot^{-1} \frac{u}{a}$
28. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left|\frac{u-a}{u+a}\right|$
29. $\int \frac{du}{u\sqrt{a^2 \pm u^2}} = \frac{1}{a} \ln \left|\frac{u}{a + \sqrt{a^2 \pm u^2}}\right|$
30 $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \cos^{-1} \frac{a}{u}$ or $\frac{1}{a}\sec^{-1} \frac{u}{a}$

14. $\int e^u \, du = e^u$

15. $\int \sinh u \, du = \cosh u$

16. $\int \cosh u \, du = \sinh u$

17. $\int \tanh u \, du = \ln \cosh u$

31. $\int \sqrt{u^2 \pm a^2}\, du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \ln | u + \sqrt{u^2 \pm a^2} |$

32. $\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}$

33. $\int e^{au} \sin bu \, du = \frac{e^{au}(a \sin bu - b \cos bu)}{a^2 + b^2}$

34. $\int e^{au} \cos bu \, du = \frac{e^{au}(a \cos bu + b \sin bu)}{a^2 + b^2}$

Special Methods of Integration

1. Integration by Parts Let u and υ be differentiable functions. According to the product rule for differentials,

$$d(u\upsilon) = u\, d\upsilon + \upsilon\, du$$

Upon taking the antiderivative of both sides of the equation, we obtain

$$u\upsilon = \int u\, d\upsilon + \int \upsilon\, du$$

This is the formula for integration by parts when written in the form

$$\int \upsilon\, dv = uv - \int \upsilon\, du \quad \text{or} \quad \int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)dx$$

where $u = f(x)$ and $\upsilon = g(x)$. The corresponding result for definite integrals over the interval $[a, b]$ is certainly valid if $f(x)$ and $g(x)$ are continuous and have continuous derivatives in $[a, b]$. See Problems 5.17 to 5.19.

2. Partial Fractions Any rational function $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, with the degree of $P(x)$ less than that of $Q(x)$, can be written as the sum of rational functions having the form $\frac{A}{(ax+b)^r}, \frac{Ax+B}{(ax^2+bx+c)^r}$ where $r = 1, 2, 3, \ldots$, which can always be integrated in terms of elementary functions.

EXAMPLE 1.

$$\frac{3x-2}{(4x-3)(2x-5)^3} = \frac{A}{4x-3} + \frac{B}{(2x+5)^3} + \frac{C}{(2x+5)^2} + \frac{D}{(2x+5)}$$

EXAMPLE 2.

$$\frac{5x^2 - x + 2}{(x^2+2x+4)^2(x-1)} = \frac{Ax+B}{(x^2+2x+4)^2} + \frac{Cx+D}{x^2+2x+4} + \frac{E}{x-1}$$

The constants, A, B, C, etc., can be found by clearing of fractions and equating coefficients of like powers of x on both sides of the equation or by using special methods (see Problem 5.20).

3. Rational Functions of sin *x* and cos *x* These can always be integrated in terms of elementary functions by the substitution $\tan x/2 = u$ (see Problem 5.21).

4. Special Devices Depending on the particular form of the integrand, special devices are often employed (see Problems 5.22 and 5.23).

Improper Integrals

If the range of integration $[a, b]$ is not finite or if $f(x)$ is not defined or not bounded at one or more points of $[a, b]$, then the integral of $f(x)$ over this range is called an *improper integral*. By use of appropriate limiting operations, we may define the integrals in such cases.

EXAMPLE 1

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{M\to\infty} \int_0^M \frac{dx}{1+x^2} = \lim_{M\to\infty} \tan^{-1} x \Big|_0^M = \lim_{M\to\infty} \tan^{-1} M = \pi/2$$

EXAMPLE 2

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0+} \int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0+} 2\sqrt{x}\Big|_\epsilon^1 = \lim_{\epsilon\to 0+} (2 - 2\sqrt{\epsilon}) = 2$$

EXAMPLE 3

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0+} \int_\epsilon^1 \frac{dx}{x} = \lim_{\epsilon\to 0+} \ln x\Big|_\epsilon^1 = \lim_{\epsilon\to 0+} (-\ln \epsilon)$$

Since this limit does not exist, we say that the integral diverges (i.e., does not converge).

For further examples, see Problems 5.29 and 5.74 through 5.76. For further discussion of improper integrals, see Chapter 12.

Numerical Methods for Evaluating Definite Integrals

Numerical methods for evaluating definite integrals are available in case the integrals cannot be evaluated exactly. The following special numerical methods are based on subdividing the interval $[a, b]$ into n equal parts of length $\Delta x = (b - a)/n$. For simplicity we denote $f(a + k\Delta x) = f(x_k)$ by y_k, where $k = 0, 1, 2, \ldots, n$. The symbol $\approx$ means "approximately equal." In general, the approximation improves as n increases.

1. Rectangular Rule

$$\int_a^b f(x)dx \approx \Delta x\{y_0 + y_1 + y_2 + \cdots + y_{n-1}\} \text{ or } \Delta x\{y_1 + y_2 + y_3 + \cdots + y_n\} \tag{8}$$

The geometric interpretation is evident from Figure 5.1. When left endpoint function values $y_0, y_1, \ldots, y_{n-1}$ are used, the rule is called the *left-hand rule*. Similarly, when right endpoint evaluations are employed, it is called the *right-hand rule*.

2. Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\{y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\} \tag{9}$$

This is obtained by taking the mean of the approximations in Equation (*8*). Geometrically, this replaces the curve $y = f(x)$ by a set of approximating line segments.

3. Simpson's Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\{y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\} \tag{10}$$

This formula is obtained by approximating the graph of $y = g(x)$ by a set of parabolic arcs of the form $y = ax^2 + bx + c$. The correlation of two observations lead to Equation (*10*). First,

$$\int_{-h}^{h} [ax^2 + bx + c]dx = \frac{h}{3}[2ah^2 + 6c]$$

The second observation is related to the fact that the vertical parabolas employed here are determined by three nonlinear points. In particular, consider $(-h, y_0)$, $(0, y_1)$, (h, y_2), then $y_0 = a(-h)^2 + b(-h) + c$, $y_1 = c$, $y_2 = ah^2 + bh + c$. Consequently, $y_0 + 4y_1 + y_2 = 2ah^2 + 6c$. Thus, this combination of ordinate values (corresponding to equally spaced domain values) yields the area bounded by the parabola, vertical segments, and the x axis. Now these ordinates may be interpreted as those of the function f whose integral is to be approximated. Then, as illustrated in Figure 5.3:

$$\sum_{k=1}^{n} \frac{h}{3}[y_{k-1} + 4y_k + y_{k+1}] = \frac{\Delta x}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n]$$

The Simpson rule is likely to give a better approximation than the others for smooth curves.

Applications

The use of the integral as a limit of a sum enables us to solve many physical and geometrical problems such as determination of areas, volumes, arc lengths, moments of intertia, and centroids.

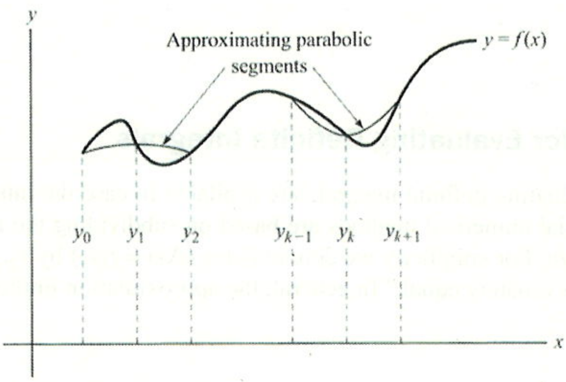

Figure 5.3

Arc Length

As you walk a twisting mountain trail, it is possible to determine the distance covered by using a pedometer. To create a geometric model of this event, it is necessary to describe the trail and a method of measuring distance along it. The trail might be referred to as a *path*, but in more exacting geometric terminology the word *curve* is appropriate. That segment to be measured is an arc of the curve. The arc is subject to the following restrictions:

1. It does not intersect itself (i.e., it is a simple arc).

2. There is a tangent line at each point.
3. The tangent line varies continuously over the arc.

These conditions are satisfied with a parametric representation $x = f(t)$, $y = g(t)$, $z = h(t)$, $a \leqq t \leqq b$, where the functions f, g, and h have continuous derivatives that do not simultaneously vanish at any point. This arc is in Euclidean three-dimensional space and is discussed in Chapter 10. In this introduction to curves and their arc length, we let $z = 0$, thereby restricting the discussion to the plane.

A careful examination of your walk would reveal movement on a sequence of straight segments, each changed in direction from the previous one. This suggests that the length of the arc of a curve is obtained as the limit of a sequence of lengths of polygonal approximations. (The polygonal approximations are characterized by the number of divisions $n \to \infty$ and no subdivision is bound from zero. (See Figure 5.4.)

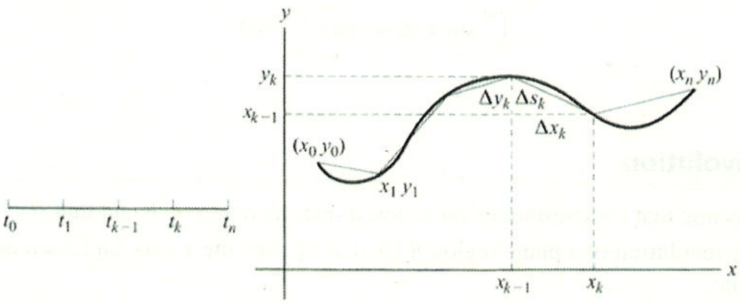

Figure 5.4

Geometrically, the measurement of the kth segment of the arc $0 \leqq t \leqq s$ is accomplished by employing the Pythagorean theorem; thus, the measure is defined by

$$\lim_{n\to\infty} \sum_{k=1}^{n} \{(\Delta x_k)^2 + (\Delta y_k)^2\}^{1/2}$$

or, equivalently,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left\{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right\}^{1/2} (\Delta x_k)$$

where $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$.

Thus, the length of the arc of a curve in rectangular Cartesian coordinates is

$$L = \int_a^b \{[f'(t)^2] + [g'(t)]^2\}^{1/2}\, dt = \int \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{1/2} dt$$

(This form may be generalized to any number of dimensions.)

Upon changing the variable of integration from t to x we obtain the planar form

$$L = \int_{f(a)}^{f(b)} \left\{1 + \left[\frac{dy}{dx}\right]^2\right\}^{1/2}$$

(This form is appropriate only in the plane.)

The generic differential formula $ds^2 = dx^2 + dy^2$ is useful, in that various representations algebraically arise from it. For example,

$$\frac{ds}{dt}$$

expresses instantaneous speed.

Area

Area was a motivating concept in introducing the integral. Since many applications of the integral are geometrically interpretable in the context of area, an extended formula is listed and illustrated here.

Let f and g be continuous functions whose graphs intersect at the graphical points corresponding to $x = a$ and $x = b$, $a < b$. If $g(x) \varepsilon f(x) \varepsilon f(x)$ on $[a, b]$, then the area bounded by $f(x)$ and $g(x)$ is

$$A = \int_a^b \{g(x) - f(x)\}dx$$

If the functions intersect in (a, b), then the integral yields an algebraic sum. For example, if $g(x) = \sin x$ and $f(x) = 0$ then

$$\int_0^{2\pi} \sin x \, dx = \cos x \Big|_0^{2\pi} = 0$$

Volumes of Revolution

Disk Method Assume that f is continuous on a closed interval $a \leq x \leq b$ and that $f(x) \varepsilon 0$. Then the solid realized through the revolution of a plane region R [bound by $f(x)$, the x axis, and $x = a$ and $x = b$] about the x axis has the volume

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

This method of generating a volume is called the *disk method* because the cross sections of revolution are circular disks. See Figure 5.5(a).

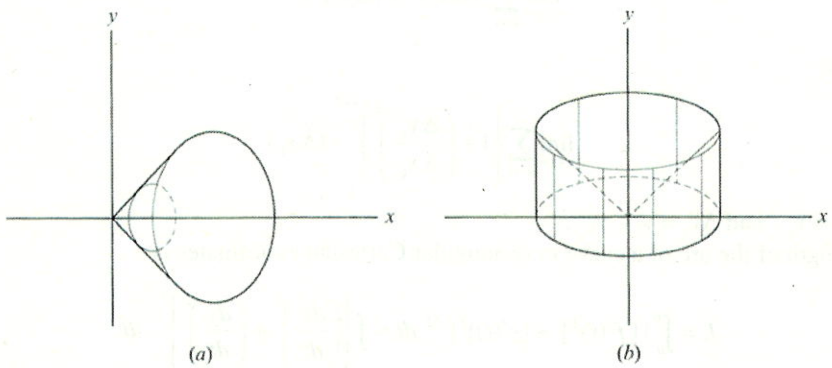

Figure 5.5

EXAMPLE. A solid cone is generated by revolving the graph of $y = kx$, $k > 0$, and $0 \leq x \leq b$ about the x axis. Its volume is

$$V = \pi \int_0^b k^2x^2 \, dx = \pi \frac{k^3x^3}{3}\Big|_0^b = \pi \frac{k^3b^3}{3}$$

Shell Method Suppose f is a continuous function on $[a, b]$, $a \varepsilon 0$, satisfying the condition $f(x) \varepsilon 0$. Let R be a plane region bounded by $f(x)$, $x = a$, $x = b$, and the x axis. The volume obtained by orbiting R about the y axis is

$$V = \int_a^b 2\pi x f(x) dx$$

This method of generating a volume is called the *shell method* because of the cylindrical nature of the vertical lines of revolution. See Figure 5.5(b).

EXAMPLE. If the region bounded by $y = kx$, $0 \leqq x \leqq b$, and $x = b$ (with the same conditions as in the previous example) is orbited about the y axis, the volume obtained is

$$V = 2\pi \int_0^b x(kx)dx = 2\pi k \frac{x^3}{3}\Big|_0^b = 2\pi k \frac{b^3}{3}$$

By comparing this example with that in the section on the disk method, it is clear that for the same plane region the disk method and the shell method produce different solids and, hence, different volumes.

Moment of Inertia Moment of inertia is an important physical concept that can be studied through its idealized geometric form. This form is abstracted in the following way from the physical notions of kinetic energy $K = 1/2\, m\upsilon^2$ and angular velocity $\upsilon = \omega r$ (m represents mass and υ signifies linear velocity). Upon substituting for υ,

$$K = \frac{1}{2} m\omega^2 r^2 = \frac{1}{2}(mr^2)\omega^2$$

When this form is compared to the original representation of kinetic energy, it is reasonable to identify mr^2 as rotational mass. It is this quantity, $I = mr^2$, that we call the *moment of inertia.*

Then in a purely geometric sense, we denote a plane region R described through continuous functions f and g on $[a, b]$, where $a > 0$ and $f(x)$ and $g(x)$ intersect at a and b only. For simplicity, assume $g(x)\ \varepsilon\ f(x) > 0$. Then

$$I = \int_a^b x^2[g(x) - f(x)]\,dx$$

By idealizing the plane region R as a volume with uniform density *one*, the expression $[f(x) - g(x)]\,dx$ stands in for mass and r^2 has the coordinate representation x^2. See Problem 5.25(b) for more details.

SOLVED PROBLEMS

Definition of a definite integral

5.1. If $f(x)$ is continuous in $[a, b]$, prove that $\lim\limits_{n\to\infty} \frac{b-a}{n} \sum\limits_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x)\,dx.$

Since $f(x)$ is continuous, the limit exists independent of the mode of subdivision (see Problem 5.31). Choose the subdivision of $[a, b]$ into n equal parts of equal length $\Delta x = (b - a)/n$ see Figure 5.1. Let $\xi_k = a + k(b - a)/n$, $k = 1, 2, \ldots, n$. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x)\,dx$$

5.2. Express $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f\left(\frac{k}{n}\right)$ as a definite integral.

Let $a = 0$, $b = 1$ in Problem 5.1. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx$$