CHAPTER 12

Improper Integrals

Definition of an Improper Integral

The functions that generate the Riemann integrals of Chapter 5 are continuous on closed intervals. Thus, the functions are bounded and the intervals are finite. Integrals of functions with these characteristics are called *proper integrals.* When one or more of these restrictions are relaxed, the integrals are said to be *improper.* Categories of improper integrals are established as follows.

The integral $\int_a^b f(x)\,dx$ is called an *improper integral* if

1. $a = -\infty$ or $b = \infty$ or both; i.e., one or both integration limits is infinite.
2. $f(x)$ is unbounded at one or more points of $a \leqq x \leqq b$. Such points are called *singularities* of $f(x)$.

Integrals corresponding to (1) and (2) are called *improper integrals of the first and second kinds*, respectively. Integrals with both conditions (1) and (2) are called *improper integrals of the third kind*

EXAMPLE 1. $\int_0^\infty \sin x^2 dx$ is an improper integral of the first kind.

EXAMPLE 2. $\int_0^4 \frac{dx}{x-3}$ is an improper integral of the second kind.

EXAMPLE 3. $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$ is an improper integral of the third kind.

EXAMPLE 4. $\int_0^1 \frac{\sin x}{x}dx$ is a *proper integral*, since $\lim_{x\to 0+} \frac{\sin x}{x} = 1$.

Improper Integrals of the First Kind (Unbounded Intervals)

If f is integrable on the appropriate domains, then the indefinite integrals $\int_a^x f(t)\,dt$ and $\int_x^a f(t)\,dt$ (with variable upper and lower limits, respectively) are functions. Through them we define three forms of the improper integral of the first kind.

Definition

(a) If f is integrable on $a \leqq x < \infty$, then $\int_a^\infty f(x)dx = \lim_{x\to\infty}\int_a^x f(t)dt$.

(b) If f is integrable on $-\infty < x \leqq a$, then $\int_{-\infty}^a f(x)dx = \lim_{x\to-\infty}\int_x^a f(t)\,dt$.

(c) If f is integrable on $-\infty < x < \infty$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^\alpha f(x)\,dx + \int_\alpha^\infty f(x)\,dx$$
$$= \lim_{x\to-\infty}\int_x^\alpha f(t)\,dt + \lim_{x\to\infty}\int_\alpha^x f(t)\,dt$$

In (c) it is important to observe that

$$\lim_{x\to-\infty}\int_x^{\alpha} f(t)dt + \lim_{x\to\infty}\int_{\alpha}^{x} f(t)dt$$

and

$$\lim_{x\to\infty}\left[\int_{-x}^{\alpha} f(t)dt + \int_{\alpha}^{x} f(t)dt\right]$$

are not necessarily equal.

This can be illustrated with $f(x) = xe^{x^2}$. The first expression is not defined, since neither of the improper integrals (i.e., limits) is defined, while the second form yields the value 0.

EXAMPLE. The function $F(x) = \dfrac{1}{\sqrt{2\pi}} e^{-(x^2/2)}$ is called the *normal density function* and has numerous applications in probability and statistics. In particular (see the bell-shaped curve in Figure 12.1),

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e - \frac{x^2}{2} : dx = 1$$

(See Problem 12.31 for the trick of making this evaluation.)

Perhaps at some point in your academic career you were "graded on the curve." The infinite region under the curve with the limiting area of 1 corresponds to the assurance of getting a grade. C's are assigned to those whose grades fall in a designated central section, and so on. (Of course, this grading procedure is not valid for a small number of students, but as the number increases it takes on statistical meaning.)

In this chapter we formulate tests for convergence or divergence of improper integrals. It will be found that such tests and proofs of theorems bear close analogy to convergence and divergence tests and corresponding theorems for infinite series (see Chapter 11).

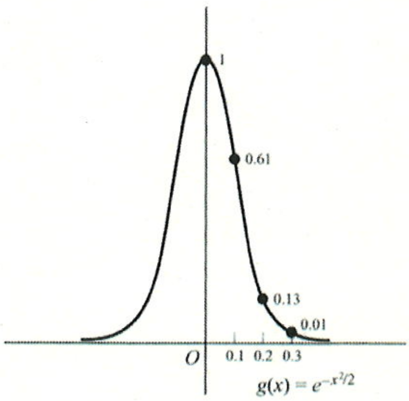

Figure 12.1

Convergence or Divergence of Improper Integrals of the First Kind

Let $f(x)$ be bounded and integrable in every finite interval $a \leq x \leq b$. Then we define

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \tag{1}$$

where b is a variable on the positive real numbers.

The integral on the left is called *convergent* or *divergent* according as the limit on the right does or does not exist. Note that $\int_a^{\infty} f(x)\,dx$ bears close analogy to the infinite series $\sum_{n=1}^{\infty} u_n$, where $u_n = f(n)$, while $\int_a^b f(x)\,dx$ corresponds to the partial sums of such infinite series. We often write M in place of b in Equation (1).

Similarly, we define

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)\,dx \tag{2}$$

where a is a variable on the negative real numbers. And we call the integral on the left convergent or divergent according as the limit on the right does or does not exist.

EXAMPLE 1. $\int_1^{\infty}\frac{dx}{x^2} = \lim_{b\to\infty}\int_1^{b}\frac{dx}{x^2} = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1$ so that $\int_t^{\infty}\frac{dx}{x^2}$ converges to 1.

EXAMPLE 2. $\int_{-\infty}^{a}\cos x\,dx = \lim_{a\to\infty}\int_a^{u}\cos x\,dx = \lim_{a\to\infty}(\sin u - \sin a)$. Since this limit does not exit, $\int_{-\infty}^{a}\cos x\,dx$ is divergent.

In like manner, we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{x_0} f(x)\,dx + \int_{x_0}^{\infty} f(x)\,dx \tag{3}$$

where x_0 is a real number, and we call the integral convergent or divergent according as the integrals on the right converge or not, as in definitions (1) and (2). [See the previous remarks in part (c) of the definition of improper integrals of the first kind.]

Special Improper Integrals of the First Kind

1. **Geometric or exponential integral** $\int_a^{\infty} e^{t-1x}\,dx$, where t is a constant, converges if $t > 0$ and diverges if $t \leqq 0$. Note the analogy with the geometric series if $r = e^{-t}$ so that $e^{-tx} = r^x$.
2. **The p integral of the first kind** $\int_a^{\infty}\frac{dx}{x^p}$, where p is a constant and $a > 0$, converges if $p > 1$ and diverges if $p \leqq 1$. Compare with the p series.

Convergence Tests for Improper Integrals of the First Kind

The following tests are given for cases where an integration limit is ∞. Similar tests exist where an integration limit is $-\infty$ (a change of variable $x = -y$ then makes the integration limit ∞). Unless otherwise specified, we assume that $f(x)$ is continuous and thus integrable in every finite interval $a \leqq x \leqq b$.

1. **Comparison test** for integrals with nonnegative integrands.

   (a) *Convergence.* Let $g(x) \geqq 0$ for all $x \geqq a$, and suppose that $\int_a^{\infty} g(x)\,dx$ converges. Then if $0 \leqq f(x) \leqq g(x)$ for all $x \geqq a$, $\int_a^{\infty} f(x)\,dx$ also converges.

   EXAMPLE. Since $\frac{1}{e^x+1} \leqq \frac{1}{e^x} = e^{-x}$ and $\int_a^{\infty} e^{-x}\,dx$ converges, $\int_0^{\infty}\frac{dx}{e^x+1}$ also converges.

   (b) *Divergence.* Let $g(x) \geqq 0$ for all $x \geqq a$, and suppose that $\int_a^{\infty} g(x)\,dx$ diverges. Then if $f(x) \leqq g(x)$ for all $x \geqq a$, $\int_a^{\infty} f(x)\,dx$ also diverges.

   EXAMPLE. Since $\frac{1}{\ln x} > \frac{1}{x}$ for $x \geqq 2$ and $\int_2^{\infty}\frac{dx}{x}$ diverges (p integral with $p = 1$), $\int_2^{\infty}\frac{dx}{\ln x}$ also diverges.

2. Quotient test for integrals with nonnegative integrands.

   (a) If $f(x) \geqq$ and $g(x) \geqq 0$, and if $\lim_{x\to\infty}\frac{f(x)}{g(x)} = A \neq 0$ or ∞, then $\int_a^{\infty} f(x)\,dx$ and $\int_a^{\infty} g(x)\,dx$ either both converge or both diverge.

   (b) If $A = 0$ *in* (a) and $\int_a^{\infty} g(x)\,dx$ converges, then $\int_a^{\infty} f(x)\,dx$ converges.

   (c) If $A = \infty$ in (a) and $\int_a^{\infty} g(x)\,dx$ diverges, then $\int_a^{\infty} f(x)\,dx$ diverges.

This test is related to the comparison test and is often a very useful alternative to it. In particular, taking $g(x) = 1/x^p$, we have, from known facts about the p integral, the following theorem.

Theorem 1 Let $\lim_{x\to\infty} x^p f(x) = A$. Then

(i) $\int_a^\infty f(x)\,dx$ converges if $p > 1$ and A is finite.

(ii) $\int_a^\infty f(x)\,dx$ diverges if $p \leqq 1$ and $A \neq 0$ (A may be infinite).

EXAMPLE 1. $\int_0^\infty \frac{x^2\,dx}{4x^4+25}$ converges since $\lim_{x\to\infty} x^2 \cdot \frac{x^2}{4x^4+25} = \frac{1}{4}$.

EXAMPLE 2. $\int_0^\infty \frac{x\,dx}{\sqrt{x^4+x^2+1}}$ diverges since $\lim_{x\to\infty} x \cdot \frac{x}{\sqrt{x^4+x^2+1}} = 1$.

A similar test can be devised using $g(x) = e^{-tx}$.

3. **Series test** for integrals with nonnegative integrands. $\int_a^\infty f(x)dx$ converges or diverges according as Σu_n where $u_n = f(n)$, converges or diverges.

4. **Absolute and conditional convergence.** $\int_a^\infty f(x)dx$ is called *absolutely convergent* if $\int_a^\infty |\,f(x)\,|\,dx$ converges. If $\int_a^\infty f(x)\,dx$ converges but $\int_a^\infty |\,f(x)\,|\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ is called *conditionally convergent*.

Theorem 2 If $\int_a^\infty |\,f(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges. In words, an absolutely convergent integral converges.

EXAMPLE 1. $\int_a^\infty \frac{\cos x}{x^2+1}dx$ is absolutely convergent and thus convergent, since

$$\int_0^\infty \left|\frac{\cos x}{x^2+1}\right| dx \leqq \int_0^\infty \frac{dx}{x^2+1} \text{ and } \int_0^\infty \frac{dx}{x^2+1} \text{ converges.}$$

EXAMPLE 2. $\int_0^\infty \frac{\sin x}{x}dx$ converges (see Problem 12.11), but $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ does not converge (see Problem 12.12). Thus, $\int_0^\infty \frac{\sin x}{x}dx$ dx is conditionally convergent.

Any of the tests used for integrals with nonnegative integrands can be used to test for absolute convergence.

Improper Integrals of the Second Kind

If $f(x)$ becomes unbounded only at the endpoint $x = a$ of the interval $a \leqq x \leqq b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0+} \int_{a+\epsilon}^b f(x)\,dx \tag{4}$$

and define it to be an improper integral of the second kind. If the limit on the right of Equation (4) exists, we call the integral on the left *convergent*; otherwise, it is *divergent*.

Similarly, if $f(x)$ becomes unbounded only at the endpoint $x = b$ of the interval $a \leqq x \leqq b$, then we extend the category of improper integrals of the second kind.

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0+} \int_a^{b-e} f(x)\,dx \tag{5}$$

Note: Be alert to the word *unbounded*. This is distinct from *undefined*. For example, $\int_a^1 \frac{\sin x}{x}dx = \lim_{\epsilon\to 0} \int_\epsilon^1 \frac{\sin x}{x}dx$ is a proper integral, since $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and, hence, is bounded as $x \to 0$ even though the function is undefined at $x = 0$. In such case the integral on the left of Equation (5) is called convergent or divergent according as the limit on the right exists or does not exist.

Finally, the category of improper integrals of the second kind also includes the case where $f(x)$ becomes unbounded only at an interior point $x = x_0$ of the interval $a \leqq x \leqq b$; then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon_1 \to 0+} \int_a^{x_0-\epsilon_1} f(x)dx + \lim_{\epsilon_2 \to 0+} \int_{x_0+\epsilon_2}^b f(x)dx \tag{6}$$

The integral on the left of Equation (6) converges or diverges according as the limits on the right exist or do not exist.

Extensions of these definitions can be made in case $f(x)$ becomes unbounded at two or more points of the interval $a \leqq x \leqq b$.

Cauchy Principal Value

It may happen that the limits on the right of Equation (6) do not exist when ϵ_1 and ϵ_2 aproach zero independently. In such case it is possible that by choosing $\epsilon_1 = \epsilon_2 = \epsilon$ in (6), i.e., writing

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0+} \left\{ \int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx \right\} \tag{7}$$

the limit does exist. If the limit on the right of Equation (7) does exist, we call this limiting value the *Cauchy principal value* of the integral on the left. See Problem 12.14.

EXAMPLE. The natural logarithm (i.e., base e) may be defined as follows:

$$\ln x = \int_t^x \frac{dt}{t}, \qquad 0 < x < \infty$$

Since $f(x) = \dfrac{1}{x}$ is unbounded as $x \to 0$, this is an improper integral of the second kind (see Figure 12.2).

Also, $\int_0^\infty \dfrac{dt}{t}$ is an integral of the third kind, since the interval to the right is unbounded.

Now $\lim\limits_{\epsilon \to 0} \int_\epsilon^1 \dfrac{dt}{t} = \lim\limits_{\epsilon \to 0} [\ln 1 - \ln \epsilon] \to -\infty$ as $\epsilon \to 0$; therefore, this improper integral of the second kind is divergent. Also, $\int_t^\infty \dfrac{dt}{t} = \lim\limits_{x \to \infty} \int_1^x \dfrac{dt}{t} = \lim\limits_{x \to \infty} [\ln x - \ln 1] \to \infty$; this integral (which is of the first kind) also diverges.

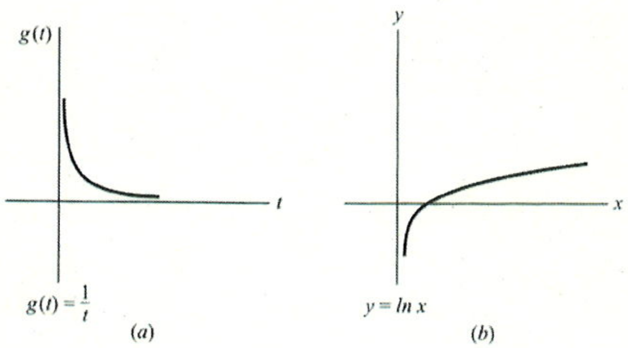

Figure 12.2

Special Improper Integrals of the Second Kind

1. $\int_a^b \dfrac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges if $p \geqq 1$.

2. $\int_a^b \dfrac{dx}{(b-x)^p}$ converes if $p < 1$ and diverges if $p \geqq 1$.

These can be called *p integrals of the second kind*. Note that when $p \leqq 0$ the integrals are proper.

Convergence Tests for Improper Integrals of the Second Kind

The following tests are given for the case where $f(x)$ is unbounded only at $x = a$ in the interval $a \leqq x \leqq b$. Similar tests are available if $f(x)$ is unbounded at $x = b$ or at $x = x_0$ where $a < x_0 < b$.

1. **Comparison test** for integrals with nonnegative integrands.

   (a) *Convergence.* Let $g(x)$ ε 0 for $a < x \leqq b$, and suppose that $\int_a^b g(x)\,dx$ converges. Then if $0 \leqq f(x) \leqq g(x)$ for $a < x \leqq b$, $\int_a^b f(x)\,dx$ also converges.

EXAMPLE. $\dfrac{1}{\sqrt{x^4-1}} < \dfrac{1}{\sqrt{x-1}}$ for $x > 1$. Then since $\int_1^5 \dfrac{dx}{\sqrt{x-1}}$ converges (p integral with $a = 1$, $p = \dfrac{1}{2}$), $\int_1^5 \dfrac{dx}{\sqrt{x^4-1}}$ also converges.

   (b) *Divergence.* Let $g(x) \geqq 0$ for $a < x \leqq b$, and suppose that $\int_a^b g(x)\,dx$ diverges. Then if $f(x) \geqq g(x)$ for $a < x \geqq b$, $\int_a^b f(x)dx$ also diverges.

EXAMPLE. $\dfrac{\ln x}{(x-3)^4} > \dfrac{1}{(x-3)^4}$ for $x > 3$. Then since $\int_3^b \dfrac{dx}{(x-3)^4}$ diverges (p integral with $a = 3$, $p = 4$), $\int_3^b \dfrac{\ln x}{(x-3)^4}dx$ also diverges.

2. **Quotient test** for integrals with nonnegative integrands.

   (a) If $f(x) \geqq 0$ and $g(x) \geqq 0$ for $a < x \leqq b$, and if $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = A \neq 0$ or ∞, then $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either both converge or both diverge.

   (b) If $A = 0$ in (a), and $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.

   (c) If $A = \infty$ in (a), and $\int_a^b g(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges.

This test is related to the comparison test and is a very useful alternative to it. In particular, taking $g(x) = 1/(x-a)^p$ we have, from known facts about the p integral, the following theorems.

Theorem 3. Let $\lim\limits_{x\to a+} (x-a)^p f(x) = A$. Then

(i) $\int_a^b f(x)dx$ converges if $p < 1$ and A is finite.

(ii) $\int_a^b f(x)dx$ diverges if $p \geqq 1$ and $A \neq 0$ (A may be infinite).

If $f(x)$ becomes unbounded only at the upper limit, these conditions are replaced by those in Theorem 4.

Theorem 4. Let $\lim\limits_{x\to b-} (b-x)^p f(x) = B$. Then

(i) $\int_a^b f(x)dx$ converges of $p < 1$ and B is finite.

(ii) $\int_a^b f(x)dx$ diverges if $p \geqq 1$ and $B \neq 0$ (B may be infinite).

EXAMPLE 1. $\int_1^5 \dfrac{dx}{\sqrt{x^4-1}}$ converges, since $\lim\limits_{x\to 1+} (x-1)^{1/2} \cdot \dfrac{1}{(x^4-1)^{1/2}} = \lim\limits_{x\to 1+} \sqrt{\dfrac{x-1}{x^4-1}} = \dfrac{1}{2}$.

EXAMPLE 2. $\int_0^3 \dfrac{dx}{(3-x)\sqrt{x^2+1}}$ diverges, since $\lim\limits_{x\to 3-} (3-x) \cdot \dfrac{1}{(3-x)\sqrt{x^2+1}} = \dfrac{1}{\sqrt{10}}$.

3. **Absolute and conditional convergence.** $\int_a^b f(x)\,dx$ is called *absolute convergent* if $\int_a^b |\, f(x)\,|\, dx$ converges. If $\int_a^b f(x)\,dx$ converges but $\int_a^b |\, f(x)\,|\, dx$ diverges, then $\int_a^b f(x)\,dx$ is called *conditionally convergent.*

Theorem 5. If $\int_a^b |f(x)|\,dx$ converges, then $\int_a^b f(x)\,dx$ converges. In words, an absolutely convergent integral converges.

EXAMPLE. $\left|\dfrac{\sin x}{\sqrt[3]{x-\pi}}\right| \leqq \dfrac{1}{\sqrt[3]{x-\pi}}$ and $\int_\pi^{4\pi} \dfrac{dx}{\sqrt[3]{x-\pi}}$ converges $\left(p \text{ integral with } a=\pi, p=\dfrac{1}{3}\right)$, it follows that $\int_\pi^{4\pi} \left|\dfrac{\sin x}{\sqrt[3]{x-\pi}}\right| dx$ converges and thus $\int_\pi^{4\pi} \dfrac{\sin x}{\sqrt[3]{x-\pi}} dx$ converges (absolutely).

Any of the tests used for integrals with nonnegative integrands can be used to test for absolute convergence.

Improper Integrals of the Third Kind

Improper integrals of the third kind can be expressed in terms of improper integrals of the first and second kinds, and, hence, the question of their convergence or divergence is answered by using results already established.

Improper Integrals Containing a Parameter, Uniform Convergence

Let

$$\phi(\alpha) = \int_a^\infty f(x,\alpha)\,dx \tag{8}$$

This integral is analogous to an infinite series of functions. In seeking conditions under which we may differentiate or integrate $\phi(\alpha)$ with respect to α, it is convenient to introduce the concept of convergence for integrals by analogy with infinite series.

We shall suppose that the integral (8) converges for $\alpha_1 \leqq \alpha \leqq \alpha_2$, or, briefly, $[\alpha_1, \alpha_2]$.

Definition. The integral (8) is said to be *uniformly convergent* in $[\alpha_1, \alpha_2]$ if for each $\epsilon > 0$ we can find a number N depending on ϵ but not on α, such that

$$\left|\phi(\alpha) - \int_a^u f(x,\alpha)\,dx\right| < \epsilon \qquad \text{for all } u > N \text{ and all } \alpha \text{ in } [\alpha_1, \alpha_2]$$

This can be restated by noting that $\left|\phi(\alpha) - \int_a^u f(x,\alpha)\,dx\right| = \left|\int_u^\infty f(x,\alpha)\,dx\right|$, which is analogous in an infinite series to the absolute value of the remainder after N terms.

This definition and the properties of uniform convergence to be developed are formulated in terms of improper integrals of the first kind. However, analogous results can be given for improper integrals of the second and third kinds.

Special Tests for Uniform Convergence of Integrals

1. **Weierstrass M test.** If we can find a function $M(x) \varepsilon\, 0$ such that
   (a) $|f(x,\alpha)| \leqq M(x)\ \alpha_1 \leqq \alpha \leqq \alpha_2, x > a$
   (b) $\int_a^\infty M(x)\,dx$ converges

   then $\int_a^\infty f(x,\alpha)\,dx$ is uniformly and absolutely convergent in $\alpha_1 \leqq \alpha \leqq \alpha_2$.

   EXAMPLE. Since $\left|\dfrac{\cos\alpha x}{x^2+1}\right| \leqq \dfrac{1}{x^2+1}$ and $\int_0^\infty \dfrac{dx}{x^2+1}$ converges, it follows that $\int_0^\infty \dfrac{\cos\alpha x}{x^2+1} dx$ is uniformly and absolutely convergent for all real values of α.