CHAPTER 7

Vectors

Vectors

The foundational ideas for vector analysis were formed independently in the nineteenth century by William Rowan Hamilton and Herman Grassmann. We are indebted to the physicist John Willard Gibbs, who formulated the classical presentation of the Hamilton viewpoint in his Yale lectures, and his student E. B. Wilson, who considered the mathematical material presented in class worthy of organizing as a book (published in 1901). Hamilton was searching for a mathematical language appropriate to a comprehensive exposition of the physical knowledge of the day. His geometric presentation emphasizing magnitude and direction and compact notation for the entities of the calculus was refined in the following years to the benefit of expressing Newtonian mechanics, electromagnetic theory, and so on. Grassmann developed an algebraic and more philosophic mathematical structure which was not appreciated until it was needed for Riemanian (non-Euclidean) geometry and the special and general theories of relativity.

Our introduction to vectors is geometric. We conceive of a vector as a directed line segment $\overrightarrow{PQ}$ from one point *P*, called the *initial point,* to another point *Q*, called the *terminal point*. We denote vectors by boldfaced letters or letters with an arrow over them. Thus, $\overrightarrow{PQ}$ is denoted by **A** or $\vec{A}$, as in Figure 7.1. The *magnitude* or *length* of the vector is then denoted by $|\overrightarrow{PQ}|$, $\overline{PQ}$, $|\mathbf{A}|$ or $|\vec{A}|$.

Vectors are defined to satisfy the geometric properties discussed in the next section.

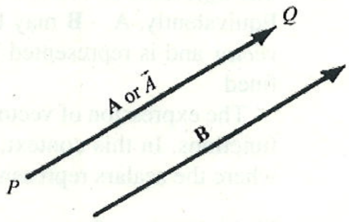

Figure 7.1

Geometric Properties of Vectors

1. Two vectors **A** and **B** are *equal* if they have the same magnitude and direction regardless of their initial points. Thus, **A** = **B** in Figure 7.1.

   In other words, a vector is geometrically represented by any one of a class of commonly directed line segments of equal magnitude. Since any one of the class of line segments may be chosen to represent it, the vector is said to be *free*. In certain circumstances (tangent vectors, forces bound to a point), the initial point is fixed; then the vector is *bound*. Unless specifically stated, the vectors in this discussion are free vectors.

2. A vector having direction opposite to that of vector **A** but with the same magnitude is denoted by –**A** (see Figure 7.2).

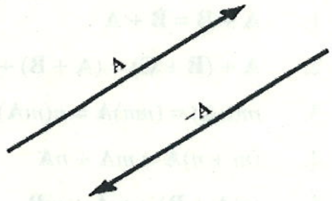

Figure 7.2

3. The *sum* or *resultant* of vectors **A** and **B** of Figure 7.3(*a*) is a vector

C formed by placing the initial point of **B** on the terminal point of **A** and joining the initial point of **A** to the terminal point of **B** [see Figure 7.3(*b*)]. The sum **C** is written **C** = **A** + **B**. The definition here is equivalent to the *parallelogram law* for vector addition, as indicated in Figure 7.3(*c*).

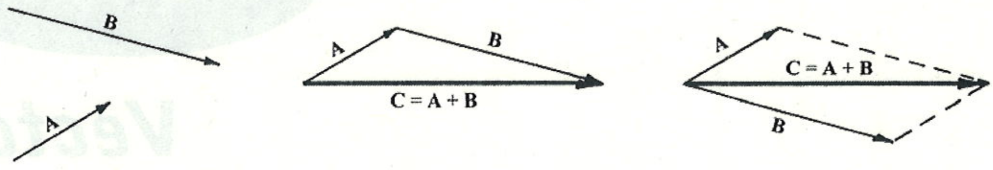

Figure 7.3

Extensions to sums of more than two vectors are immediate. For example, Figure 7.4 shows how to obtain the sum or resultant **E** of the vectors **A**, **B**, **C**, and **D**.

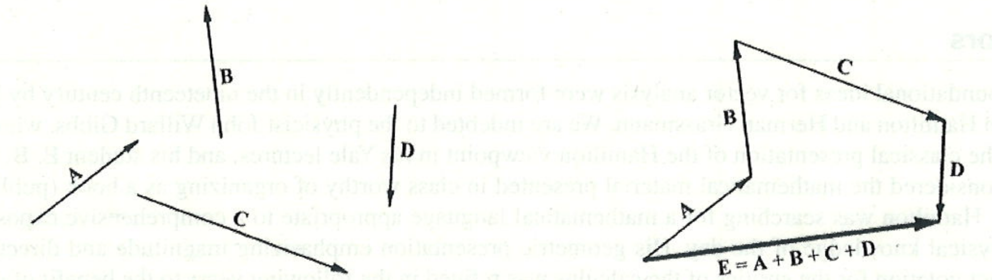

Figure 7.4

4. The *difference* of vectors **A** and **B**, represented by **A** – **B**, is that vector **C** which added to **B** gives **A**. Equivalently, **A** – **B** may be defined as **A** + (–**B**). If **A** = **B**, then **A** – **B** is defined as the *null* or *zero vector* and is represented by the symbol **0**. This has a magnitude of zero, but its direction is not defined.

   The expression of vector equations and related concepts is facilitated by the use of real numbers and functions. In this context, these are called *scalars*. This special designation arises from application where the scalars represent objects that do not have direction, such as mass, length, and temperature.

5. Multiplication of a vector **A** by a scalar m produces a vector m**A** with magnitude $|m|$ times the magnitude of **A** and direction the same as or opposite to that of **A** according as m is positive or negative. If $m = 0$, m**A** = 0, the null vector.

Algebraic Properties of Vectors

The following algebraic properties are consequences of the geometric definition of a vector. (See Problems 7.1 and 7.2.)

If **A**, **B**, and **C** are vectors, and m and n are scalars, then

1.	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	Commutative law for addition
2.	$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$	Associative law for addition
3.	$m(n\mathbf{A}) = (mn)\mathbf{A} = n(m\mathbf{A})$	Associative law for multiplication
4.	$(m + n)\mathbf{A} = m\mathbf{A} + n\mathbf{A}$	Distributive law
5.	$m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B}$	Distributive law

Note that in these laws only multiplication of a vector by one or more scalars is defined. On Pages 164 and 165 we define products of vectors.

Linear Independence and Linear Dependence of a Set of Vectors

That a set of vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_p$ is linearly independent means that $a_1\mathbf{A}_1 + a_2\mathbf{A}_2 + \cdots a_p\mathbf{A}_p = \mathbf{0}$ if and only if $a_1 = a_2 = \ldots = a_p = 0$ (i.e., the algebraic sum is zero if and only if all the coefficients are zero). The set of vectors is linearly dependent when it is not linearly independent.

Unit Vectors

Unit vectors are vectors having unit length. If $\mathbf{A}$ is any vector with magnitude $A = |\mathbf{A}| > 0$, then $\mathbf{A}/|\mathbf{A}|$ is a unit vector. If $\mathbf{a}$ is a unit vector with the same direction and sense as $\mathbf{A}$, then $\mathbf{a} = \mathbf{A}/|\mathbf{A}|$.

Rectangular (Orthogonal) Unit Vectors

The rectangular unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors having the direction of the positive x, y, and z axes of a rectangular coordinate system (see Figure 7.5). The triple $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is said to be a *basis* of the collection of vectors. We use right-handed rectangular coordinate systems unless otherwise specified. Such systems derive their name from the fact that a right-threaded screw rotated through 90° from Ox to Oy will advance in the positive z direction. In general, three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ which have coincident initial points and are not coplanar are said to form a *right-handed system* or *dextral system* if a right-threaded screw rotated through an angle less than 180° from $\mathbf{A}$ to $\mathbf{B}$ will advance in the direction $\mathbf{C}$ (see Figure 7.6).

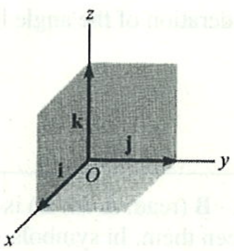

Figure 7.5

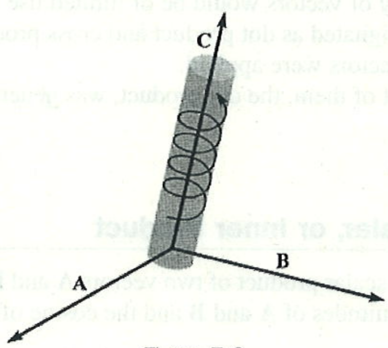

Figure 7.6

Components of a Vector

Any vector $\mathbf{A}$ in three dimensions can be represented with initial point at the origin O of a rectangular coordinate system (see Figure 7.7). Let (A_1, A_2, A_3) be the rectangular coordinates of the terminal point of vector $\mathbf{A}$ with initial point at O. The vectors $A_1\mathbf{i}$, $A_2\mathbf{j}$, and $A_3\mathbf{k}$ are called the *rectangular component vectors*, or simply *component vectors*, of $\mathbf{A}$ in the x, y, and z directions, respectively. A_1, A_2, and A_3 are called the *rectangular components*, or simply *components*, of $\mathbf{A}$ in the x, y, and z directions, respectively.

The vectors of the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are perpendicular to one another, and they are unit vectors. The words *orthogonal* and *normal*, respectively, are used to describe these characteristics; hence, the set is what is called an *orthonormal basis*.

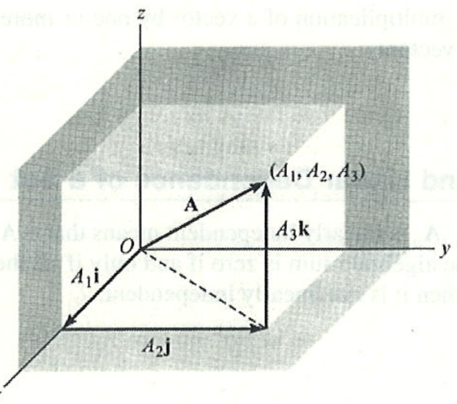

Figure 7.7

It is easily shown to be linearly independent. In an *n*-dimensional space, any set of *n* linearly independent vectors is a basis. The further characteristic of a basis is that any vector of the space can be expressed through it. It is the basis representation that provides the link between the geometric and the algebraic expressions of vectors and vector concepts.

The sum or resultant of $A_1\mathbf{i}$, $A_2\,\mathbf{j}$, and $A_3\mathbf{k}$ is the vector **A**, so that we can write

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \tag{1}$$

The magnitude of **A** is

$$A = |\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2} \tag{2}$$

In particular, the *position vector* or *radius vector* **r** from *O* to the point (*x, y, z* is written

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3}$$

and has magnitude $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

A theory of vectors would be of limited use without a process of multiplication. In fact, two binary processes, designated as dot product and cross product, were created to meet the geometric and physical needs to which vectors were applied.

The first of them, the dot product, was generated from consideration of the angle between two vectors.

Dot, Scalar, or Inner Product

The dot or scalar product of two vectors **A** and **B**, denoted by **A** · **B** (read: **A** dot **B**) is defined as the product of the magnitudes of **A** and **B** and the cosine of the angle between them. In symbols,

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta, \qquad 0 \leqq \theta \leqq \pi \tag{4}$$

Assuming that neither **A** nor **B** is the zero vector, an immediate consequence of the definition is that **A** · **B** = 0 if and only if **A** and **B** are perpendicular. Note that **A** · **B** is a scalar and not a vector.

The following laws are valid:

1. $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ Commutative law for dot products
2. $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ Distributive Law
3. $m(\mathbf{A}\cdot\mathbf{B}) = (m\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(m\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B})m$, where m is a scalar
4. $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$

The equivalence of this component form the dot product with the geometric definition 4 following from the law of cosines. (See Figure 7.8).

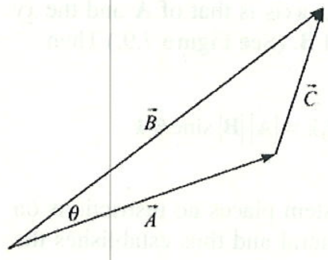

Figure 7.8

In particular,

$$|\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}|\,|\mathbf{B}| \cos\theta$$

Since $\mathbf{C} = \mathbf{B} - \mathbf{A}$, its components are $B_1 - A_1$, $B_2 - A_2$, $B_3 - A_3$ and the square of its magnitude is

$$|\mathbf{B}|^2 + |\mathbf{A}|^2 - 2(A_1B_1 + A_2B_2 + A_3B_3$$

When this representation for $|\mathbf{C}^2|$ is placed in the original equation and cancellations are made, we obtain

$$A_1B_1 + A_2B_2 + A_3B_3 = |\mathbf{A}|\,|\mathbf{B}| \cos\theta.$$

The second form of vector multiplication—that is, the cross product—emerged from Hamilton's theory of quaternions (1844). Algebraically, the cross product is an example of a noncommutative operation. Geometrically, it generates a vector perpendicular to the initial pair of vectors, and its physical value is illustrated in electromagnetic theory, where it aids in the representation of a magnetic field perpendicular to the direction of an electric current.

Cross or Vector Product

The cross or vector product of **A** and **B** is a vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ (read: **A** cross **B**). The magnitude of $\mathbf{A} \times \mathbf{B}$ is defined as the product of the magnitudes of **A** and **B** and the sine of the angle between them. The direction of the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is perpendicular to the plane of **A** and **B**, and such that **A**, **B**, and **C** form a right-handed system. In symbols,

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta \mathbf{u}, \ 0 \le \theta \le \pi \tag{5}$$

where **u** is a unit vector indicating the direction of $\mathbf{A} \times \mathbf{B}$. If $\mathbf{A} = \mathbf{B}$ or if **A** is parallel to **B**, then $\sin\theta = 0$ and $\mathbf{A} \times \mathbf{B} = 0$.

The following laws are valid:

1. $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$ (Commutative law for cross products fails)
2. $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ Distributive Law
3. $m(\mathbf{A} \times \mathbf{B}) = (m\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (m\mathbf{B}) = (\mathbf{A} \times \mathbf{B})m$, where m is a scalar

Also, the following consequences of the definition are important:

4. $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

The equivalence of this component representation (5) and the geometric definition may be seen as follows. Choose a coodinate system such that the direction of the x-axis is that of **A** and the xy plane is the plane of the vectors **A** and **B**. (See Figure 7.9.) Then

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ A_1 & 0 & 0 \\ B_1 & B_2 & 0 \end{vmatrix} = A_1B_2\mathrm{k} = |\mathbf{A}|\,|\mathbf{B}| \text{ sine } \theta\ k$$

Since this choice of coordinate system places no restrictions on the vectors **A** and **B**, the result is general and thus establishes the equivalence.

6. $|\mathbf{A}\times\mathbf{B}|$ = the area of a parallelogram with sides **A** and **B**.
7. If $\mathbf{A}\times\mathbf{B} = 0$ and neither **A** nor **B** is a null vector, then **A** and **B** are parallel.

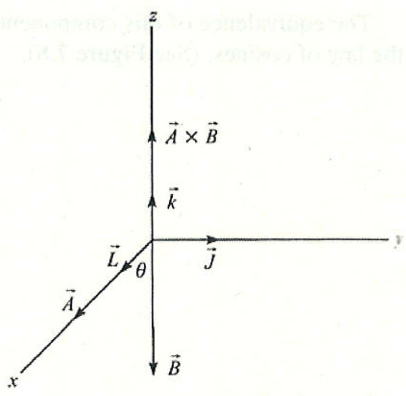

Figure 7.9

Triple Products

Dot and cross multiplication of three vectors, **A**, **B**, and **C** may produce meaningful products of the form $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. The following laws are valid:

1. $(\mathbf{A}\cdot\mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$ in general
2. $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$ = volume of a parallelepiped having **A**, **B**, and **C** as edges, or the negative of this volume according as **A**, **B**, and **C** do or do not form a right-handed handed system. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ and $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$, then

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \tag{6}$$

3. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ (Associative law for cross products fails)
4. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

   $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$

The product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is called the *scalar triple product* or *box product* and may be denoted by [**ABC**]. The product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is called the *vector triple product.*

In $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ parentheses are sometimes omitted and we write $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$. However, parentheses must be used in $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ (see Problem 7.29). Note that $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$. This is often expressed by stating that in a scalar triple product the dot and the cross can be interchanged without affecting the result (see Problem 7.26).

Axiomatic Approach to Vector Analysis

From the preceding remarks it is seen that a vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is determined when its three components (x, y, z) relative to some coordinate system are known. In adopting an axiomatic approach, it is thus quite natural for us to make the following assumptions.

Definition A three-dimensional vector is an *ordered triplet* of real numbers with the following properties. If $\mathbf{A} = (A_1, A_2, A_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$, then