

$$4. \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} = \frac{A}{B} \quad \text{if } \lim_{n \rightarrow \infty} b_n = B \neq 0$$

If $B = 0$ and $A \neq 0$, $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ does not exist.

If $B = 0$ and $A = 0$, $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ may or may not exist.

$$5. \lim_{n \rightarrow \infty} a_n^p = (\lim_{n \rightarrow \infty} a_n)^p = A^p, \quad \text{for } p = \text{any real number if } A^p \text{ exists.}$$

$$6. \lim_{n \rightarrow \infty} p^{a_n} = p^{\lim_{n \rightarrow \infty} a_n} = p^A, \quad \text{for } p = \text{any real number if } p^A \text{ exists.}$$

Infinity

We write $\lim_{n \rightarrow \infty} a_n = \infty$ if for each positive number M we can find a positive number N (depending on M) such that $a_n > M$ for all $n > N$. Similarly, we write $\lim_{n \rightarrow \infty} a_n = -\infty$ if for each positive number M we can find a positive number N such that $a_n < -M$ for all $n > N$. It should be emphasized that ∞ and $-\infty$ are not numbers and the sequences are not convergent. The terminology employed merely indicates that the sequences diverge in a certain manner. That is, no matter how large a number in absolute value that one chooses, there is an n such that the absolute value of a_n is greater than that quantity.

Bounded, Monotonic Sequences

If $u_n \leq M$ for $n = 1, 2, 3, \dots$, where M is a constant (independent of n), we say that the sequence $\{u_n\}$ is *bounded above* and M is called an *upper bound*. If $u_n \geq m$, the sequence is *bounded below* and m is called a *lower bound*.

If $m \leq u_n \leq M$ the sequence is called *bounded*. Often this is indicated by $|u_n| \leq P$. Every convergent sequence is bounded, but the converse is not necessarily true.

If $u_{n+1} \geq u_n$ the sequence is called *monotonic increasing*; if $u_{n+1} > u_n$ it is called *strictly increasing*. Similarly, if $u_{n+1} \leq u_n$ the sequence is called *monotonic decreasing*, while if $u_{n+1} < u_n$ it is *strictly decreasing*.

- EXAMPLES.**
1. The sequence 1, 1.1, 1.11, 1.111, ... is bounded and monotonic increasing. It is also strictly increasing.
 2. The sequence 1, -1, 1, -1, 1, ... is bounded but not monotonic increasing or decreasing.
 3. The sequence -1, -1.5, -2, -2.5, -3, ... is monotonic decreasing and not bounded. However, it is bounded above.

The following theorem is fundamental and is related to the Bolzano-Weierstrass theorem (Chapter 1, Page 7) which is proved in Problem 2.23.

Theorem Every bounded monotonic (increasing or decreasing) sequence has a limit.

Least Upper Bound and Greatest Lower Bound of a Sequence

A number \underline{M} is called the *least upper bound* (l.u.b.) of the sequence $\{u_n\}$ if $u_n \leq \underline{M}$, $n = 1, 2, 3, \dots$ while at least one term is greater than $\underline{M} - \epsilon$ for any $\epsilon > 0$.