

CHAPTER 2

Sequences

Definition of a Sequence

A sequence is a set of numbers u_1, u_2, u_3, \dots in a definite order of arrangement (i.e., a *correspondence* with the natural numbers or a subset thereof) and formed according to a definite rule. Each number in the sequence is called a *term*; u_n is called the *n*th *term*. The sequence is called *finite* or *infinite* according as there are or are not a finite number of terms. The sequence u_1, u_2, u_3, \dots is also designated briefly by $\{u_n\}$.

- EXAMPLES.**
1. The set of numbers 2, 7, 12, 17, \dots , 32 is a finite sequence; the *n*th term is given by $u_n = 2 + 5(n - 1) = 5n - 3, n = 1, 2, \dots, 7$.
 2. The set of numbers 1, $1/3, 1/5, 1/7, \dots$ is an infinite sequence with *n*th term $u_n = 1/(2n - 1), n = 1, 2, 3, \dots$.

Unless otherwise specified, we shall consider infinite sequences only.

Limit of a Sequence

A number l is called the *limit* of an infinite sequence u_1, u_2, u_3, \dots if for any positive number ϵ we can find a positive number N depending on ϵ such that $|u_n - l| < \epsilon$ for all integers $n > N$. In such case we write $\lim_{n \rightarrow \infty} u_n = l$.

EXAMPLE. If $u_n = 3 + 1/n = (3n + 1)/n$, the sequence is 4, $7/2, 10/3, \dots$ and we can show that $\lim_{n \rightarrow \infty} u_n = 3$.

If the limit of a sequence exists, the sequence is called *convergent*; otherwise, it is called *divergent*. A sequence can converge to only one limit; i.e., if a limit exists, it is unique. See Problem 2.8.

A more intuitive but unrigorous way of expressing this concept of limit is to say that a sequence u_1, u_2, u_3, \dots has a limit l if the successive terms get "closer and closer" to l . This is often used to provide a "guess" as to the value of the limit, after which the definition is applied to see if the guess is really correct.

Theorems on Limits of Sequences

If $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$, then

1. $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = A + B$
2. $\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n = A - B$
3. $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = (\lim_{n \rightarrow \infty} a_n)(\lim_{n \rightarrow \infty} b_n) = AB$