CHAPTER 6

Partial Derivatives

Functions of Two or More Variables

The definition of a function was given in Chapter 3 (page 43). For us, the distinction for functions of two or more variables is that the domain is a set of n-tuples of numbers. The range remains one-dimensional and is referred to an interval of numbers. If $n = 2$, the domain is pictured as a two-dimensional region. The region is referred to a rectangular Cartesian coordinate system described through number pairs (x, y), and the range variable is usually denoted by z. The domain variables are independent, while the range variable is dependent.

We use the notation $f(x, y)$, $F(x, y)$, etc., to denote the value of the function at (x, y) and write $z = f(x, y)$, $z = F(x, y)$, etc. We also sometimes use the notation $z = z(x, y)$, although it should be understood that in this case z is used in two senses, namely, as a function and as a variable.

EXAMPLE. If $f(x, y) = x^2 + 2y^3$, then $f(3, -1) = (3)^2 + 2(-1)^3 = 7$.

The concept is easily extended. Thus, $w = F(x, y, z)$ denotes the value of a function at (x, y, z) (a point in three-dimensional space), etc.

EXAMPLE. If $z = \sqrt{1 - (x^2 + y^2)}$, the domain for which z is real consists of the set of points (x, y) such that $x^2 + y^2 \leqq 1$, i.e., the set of points inside and on a circle in the xy plane having center at $(0, 0)$ and radius 1.

A three-dimensional rectangular Cartesian coordinate system is obtained by constructing three mutually perpendicular axes (the x, y, and z axes) intersecting in a point (designated by 0 and called the *origin*). This is a natural extension of the rectangular system x, y in the plane. A point in the three-dimensional Cartesian system is represented by the triple of coordinates (x, y, z). The collection of points $P(x, y, z)$, represented by the implicit equation $F(x, y, z) = 0$, is a surface. The term *surface* is used in a very broad sense and requires refinement according to the context in which it is to be used. For example, $x^2 + y^2 + z^2 = r^2$ is the algebraic representation of a surface in the large. This form might be employed in topology to indicate the property of being closed rather than open. In analysis, which is the subject of this outline of advanced calculus, the concern is with portions of a surface—that is, points and their neighborhoods. These may be obtained from implicit representations by imposing restrictions. For example,

$$z = \sqrt{r^2 - (x^2 + y^2)} \text{ with } 1x^2 + y^2 1 < r$$

signifies an open upper hemisphere. Problems in surface theory employ partial derivatives and relate to a point of a surface, the collection of points about it, the tangent plane at the point, and the properties of continuity and differentiability binding this structure. These concepts will be discussed in the following pages.

For functions of more than two variables such geometric interpretation fails, although the terminology is still employed. For example, (x, y, z, w) is a point in four-dimensional space, and $w = f(x, y, z)$ [or $F(x, y, z, w) = 0$] represents a *hypersurface* in four dimensions; thus, $x^2 + y^2 + z^2 + w^2 = a^2$ represents a *hypersphere* in four dimensions with radius $a > 0$ and center at $(0, 0, 0, 0)$. $w = \sqrt{a^2 - (x^2 + y^2 + z^2)}$, $x^2 + y^2 + z^2 \leqq a^2$ describes a function generated from the hypersphere.

Neighborhoods

The set of all points (x, y) such that $|x - x_0| < \delta$, $|y - y_0| < \delta$ where $\delta > 0$ is called a *rectangular* δ *neighborhood* of (x_0, y_0); the set $0 < |x - x_0| < \delta$, $0 < |y - y_0| < \delta$, which excludes (x_0, y_0), is called a *rectangular deleted* δ *neighborhood* of (x_0, y_0). Similar remarks can be made for made for other neighborhoods; e.g., $(x - x_0)^2 + (y - y_0)^2 < \delta^2$ is a *circular* δ *neighborhood* of (x_0, y_0). The term *open ball* is used to designate this circular neighborhood. This terminology is appropriate for generalization to more dimensions. Whether neighborhoods are viewed as circular or square is immaterial, since the descriptions are interchangeable. Simply notice that given an open ball (circular neighborhood) of radius δ there is a centered square whose side is of length less than $\sqrt{2}\,\delta$ that is interior to the open ball, and, conversely, for a square of side δ there is an interior centered circle of radius less than $\delta/2$. (See Figure 6.1.)

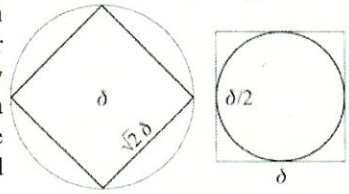

Figure 6.1

A point (x_0, y_0) is called a *limit point, accumulation point,* or *cluster point* of a point set S if every deleted δ neighborhood of (x_0, y_0) contains points of S. As in the case of one-dimensional point sets, every bounded infinite set has at least one limit point (the Bolzano-Weierstrass theorem; see Chapter 1). A set containing all its limit points is called a *closed set.*

Regions

A point P belonging to a point set S is called an *interior point* of S if there exists a deleted δ neighborhood of P all of whose points belong to S. A point P not belonging to S is called an *exterior point* of S if there exists a deleted δ neighborhood of P all of whose points do not belong to S. A point P is called a *boundary point* of S if every deleted δ neighborhood of P contains points belonging to S and also points not belonging to S.

If any two points of a set S can be joined by a path consisting of a finite number of broken line segments all of whose points belong to S, then S is called a *connected set.* A *region* is a connected set which consists of interior points or interior and boundary points. A *closed region* is a region containing all its boundary points. An *open region* consists only of interior points. The complement of a set S in the xy plane is the set of all points in the plane not belonging to S. (See Figure 6.2.)

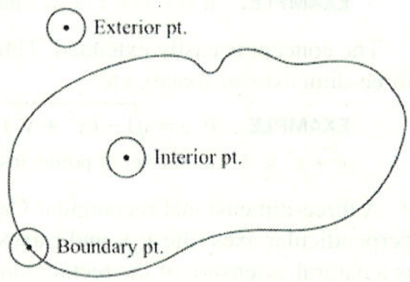

Figure 6.2

Examples of some regions are shown graphically in Figure 6.3(a), (b), and (c) . The rectangular region of Figure 6.1(a), including the boundary, represents the sets of points $a \leq x \leq b$, $c \leq y \leq d$ which is a natural extension of the closed interval $a \leq x \leq b$ for one dimension. The set $a < x < b$, $c < y < d$ corresponds to the boundary being excluded.

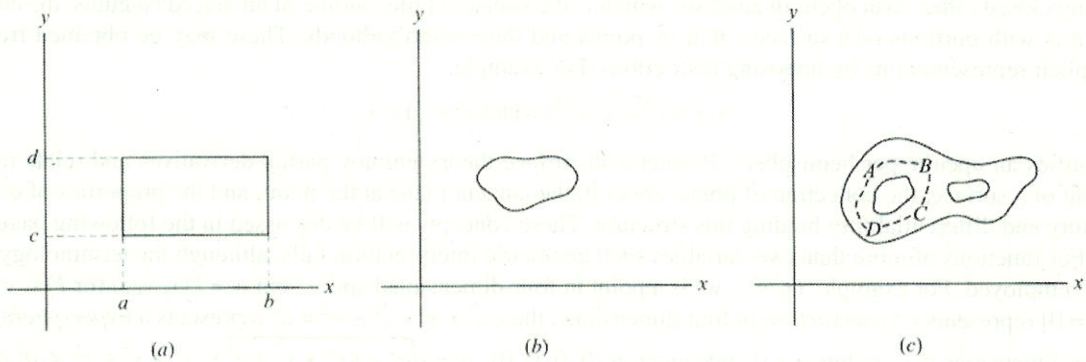

Figure 6.3

In the regions of Figure 6.3(a) and (b), any *simple closed curve* (one which does not intersect itself anywhere) lying inside the region can be shrunk to a point which also lies in the region. Such regions are called *simply connected regions*. In Figure 6.3(c), however, a simple closed curve $ABCD$ surrounding one of the "holes" in the region cannot be shrunk to a point without leaving the region. Such regions are called *multiply connected regions*.

Limits

Let $f(x, y)$ be defined in a deleted δ neighborhood of (x_0, y_0) [i.e., $f(x, y)$ may be undefined at (x_0, y_0)]. We say that l is the *limit* of $f(x,y)$ as x approaches x_0 and y approaches y_0 [or (x, y) approaches (x_0, y_0)] and write $\lim_{\substack{x\to x_0\\ y\to y_0}} f(x, y) = l$ [or $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l$] if for any positive number ϵ we can find some positive number δ [depending on δ and (x_0, y_0), in general] such that $|f(x, y) - l| < \epsilon$ whenever $0 < |x - x_0| < \delta$ and $0 < |y - y_0| < \delta$.

If desired, we can use the deleted circular neighborhood open ball $0 < (x - x_0)^2 + (y - y_0)^2 < \delta^2$ instead of the deleted rectangular neighborhood.

EXAMPLE. Let $f(x, y) = \begin{cases} 3xy & \text{if } (x, y) \neq (1, 2) \\ 0 & \text{if } (x, y) = (1, 2) \end{cases}$. As $x \to 1$ and $y \to 2$ [or $(x, y) \to (1, 2)$], $f(x, y)$ gets closer to $3(1)(2) = 6$ and we *suspect* that $\lim_{\substack{x\to 1\\ x\to 2}} f(x, y) = 6$. To *prove* this, we must show that the preceding definition of limit, with $l = 6$, is satisfied. Such a proof can be supplied by a method similar to that of Problem 6.4.

Note that $\lim_{\substack{x\to 1\\ x\to 2}} f(x, y) \neq f(1, 2)$ since $f(1, 2) = 0$. The limit would, in fact, be 6 even if $f(x, y)$ were not defined at (1, 2). Thus, the existence of the limit of $f(x, y)$ as $(x, y) \to (x_0, y_0)$ is in no way dependent on the existence of a value of $f(x, y)$ at (x_0, y_0).

Note that in order for $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ to exist, it must have the same value regardless of the approach of (x, y) to (x_0, y_0). It follows that if two different approaches give different values, the limit cannot exist (see Problem 6.7). This implies, as in the case of functions of one variable, that if a limit exists it is unique.

The concept of one-sided limits for functions of one variable is easily extended to functions of more than one variable.

EXAMPLE 1. $\lim_{\substack{x\to 0+\\ y\to 1}} \tan^{-1}(y/x) = \pi/2$, $\lim_{\substack{x\to 0-\\ y\to 1}} \tan^{-1}(y/x) = -\pi/2$.

EXAMPLE 2. $\lim_{\substack{x\to 0\\ y\to 1}} \tan^{-1}(y/x)$ does not exist, as is clear from the fact that the two different approaches of Example 1 give different results.

In general, the theorems on limits, concepts of infinity, etc., for functions of one variable (see Page 25) apply as well, with appropriate modifications, to functions of two or more variables.

Iterated Limits

The *iterated limits* $\lim_{x\to x_0}\left\{\lim_{y\to y_0} f(x, y)\right\}$ and $\lim_{y\to y_0}\left\{\lim_{x\to x_0} f(x, y)\right\}$ [also denoted by $\lim_{x\to x_0}\lim_{y\to y_0} f(x, y)$ and $\lim_{y\to y_0}\lim_{x\to x_0} f(x, y)$, respectively] are not necessarily equal. Although they must be equal if $\lim_{\substack{x\to x_0\\ y\to y_0}} f(x, y)$ is to exist, their equality does not guarantee the existence of this last limit.

EXAMPLE. If $(x, y) = \dfrac{x-y}{x+y}$, then $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}\dfrac{x-y}{x+y}\right) = \lim\limits_{x\to 0}(1) = 1$ and $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}\dfrac{x-y}{x+y}\right) = \lim\limits_{y\to 0}(-1) = -1$.

Thus, the iterated limits are not equal and so $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y)$ cannot exist.

Continuity

Let $f(x, y)$ be defined in a δ neighborhood of (x_0, y_0) [i.e., $f(x, y)$ must be defined *at* (x_0, y_0) as well as near it]. We say that $f(x, y)$ is *continuous* at (x_0, y_0) if for any positive number δ we can find some positive number δ [depending on δ and (x_0, y_0) in general] such that $|f(x, y) - f(x_0, y_0)| < \delta$ whenever $|x - x_0| < \delta$ and $|y - y_0| < \delta$, or, alternatively, $(x - x_0)^2 + (y - y_0)^2 < \delta^2$.

Note that three conditions must be satisfied in order for $f(x, y)$ to be continuous at (x_0, y_0):

1. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = l$; i.e., the limit exists as $(x, y) \to (x_0, y_0)$.
2. $f(x_0, y_0)$ must exist; i.e., $f(x, y)$ is defined at (x_0, y_0).
3. $l = f(x_0, y_0)$.

If desired, we can write this in the suggestive form $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x, y) = f(\lim\limits_{x\to x_0} x, \lim\limits_{y\to y_0} y)$.

EXAMPLE. If $f(x, y) = \begin{cases} 3xy & (x, y) \neq (1, 2) \\ 0 & (x, y) = (1, 2) \end{cases}$, then $\lim\limits_{(x,y)\to(1,2)} f(x, y) = 6 \neq f(1, 2)$. Hence, $f(x, y)$ is not continuous at $(1, 2)$. If we redefine the function so that $f(x, y) = 6$ for $(x, y) = (1, 2)$, then the function is continuous at $(1, 2)$.

If a function is not continuous at a point (x_0, y_0), it is said to be *discontinuous* at (x_0, y_0), which is then called a *point of discontinuity*. If, as in the preceding example, it is possible to redefine the value of a function at a point of discontinuity so that the new function is continuous, we say that the point is a *removable discontinuity* of the old function. A function is said to be *continuous in a region* $\mathfrak{R}$ of the xy plane if it is continuous at every point of $\mathfrak{R}$.

Many of the theorems on continuity for functions of a single variable can, with suitable modification, be extended to functions of two more variables.

Uniform Continuity

In the definition of continuity of $f(x, y)$ at (x_0, y_0), δ depends on δ and also (x_0, y_0) in general. If in a region $\mathfrak{R}$ we can find a δ which depends only on δ but not on any particular point (x_0, y_0) in $\mathfrak{R}$ (i.e., the same δ will work for *all* points in $\mathfrak{R}$), then $f(x, y)$ is said to be *uniformly continuous* in $\mathfrak{R}$. As in the case of functions of one variable, it can be proved that a function which is continuous in a closed and bounded region is uniformly continuous in the region.

Partial Derivatives

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant, is called the *partial derivative* of the function with respect to the variable. Partial derivatives of $f(x, y)$ with respect to x and y are denoted by $\dfrac{\partial}{\partial x}\left[\text{or } f_x, f_x(x, y), \left.\dfrac{\partial f}{\partial x}\right|_y\right]$ and $\dfrac{\partial f}{\partial y}\left[\text{or } f_y, f_y(x, y), \left.\dfrac{\partial f}{\partial y}\right|_x\right]$, respectively, the latter notations being used when needed to emphasize which variables are held constant.

By definition,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}, \quad \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} \tag{1}$$

when these limits exist. The derivatives evaluated at the particular point (x_0, y_0) are often indicated by $\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0)$ and $\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = f_x(x_0, y_0)$, respectively.

EXAMPLE. If $f(x, y) = 2x^3 + 3xy^2$, then $f_x = \partial f/\partial x = 6x^2 + 3y^2$ and $f_y = \partial f/\partial y = 6xy$. Also, $f(1, 2) = 6(1)^2 + 3(2)^2 = 18$, $f_y(1, 2) = 6(1)(2) = 12$.

If a function f has continuous partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ in a region, then f must be continuous in the region. However, the existence of these partial derivatives alone is not enough to guarantee the continuity of f (see Problem 6.9).

Higher-Order Partial Derivatives

If $f(x, y)$ has partial derivatives at each point (x, y) in a region, then $\partial f/\partial x$ and $\partial f/\partial y$ are themselves functions of x and y, which may also have partial derivatives. These second derivatives are denoted by are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}, \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = f_{yx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = f_{xy} \tag{2}$$

If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$ and the order of differentiation is immaterial; otherwise they may not be equal (see Problems 6.13 and 6.41).

EXAMPLE. If $f(x, y) = 2x^3 + 3xy^2$ (see preceding example), then $f_{xx} = 12x$, $f_{yy} = 6x$, and $f_{xy} = 6y = f_{yx}$. In such case $f_{xx}(1, 2) = 12$, $f_{yy}(1, 2) = 6$, and $f_{xy}(1, 2) = f_{yx}(1, 2) = 12$.

In a similar manner, higher order derivatives are defined. For example, $\frac{\partial^3 f}{\partial x^2 \partial y} = f_{yxx}$ is the derivative of f taken once with respect to y and twice with respect to x.

Differentials

(The section on differentials in Chapter 4 should be read before beginning this one.)

Let $\Delta x = dx$ and $\Delta y = dy$ be increments given to x and y, respectively. Then

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y) = \Delta f \tag{3}$$

is called the *increment* in $z = f(x, y)$. If $f(x, y)$ has continuous first partial derivatives in a region, then

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + \epsilon_1\, dx + \epsilon_2 dy = \Delta f \tag{4}$$

where ϵ_1 and ϵ_2 approach zero as Δx and Δy approach zero (see Problem 6.14). The expression

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad \text{or} \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{5}$$

is called the *total differential* or simply the *differential* of z or f, or the *principal part* of Δz or Δf. Note that $\Delta z \neq dz$ in general. However, if $\Delta x = dx$ and $\Delta y = dy$ are "small," then dz is a close approximation of Δz (see Problem 6.15). The quantities dx and dy—called *differentials* of x and y, respectively—need not be small.

The form $dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$ signifies a linear function with the independent variables dx and dy and the dependent range variable dz. In the one-variable case, the corresponding linear function represents the tangent line to the underlying curve. In this case, the underlying entity is a surface and the linear function generates the tangent plane at P_0. In a small enough neighborhood, this tangent plane is an approximation of the surface (i.e., the linear representation of the surface at P_0). If y is held constant, then we obtain the curve of intersection of the surface and the coordinate plane $y = y_0$. The differential form reduces to $dz = f_x(x_0, y_0)$ dx (i.e., the one-variable case). A similar statement follows when x is held constant. (See Figure 6.4.)

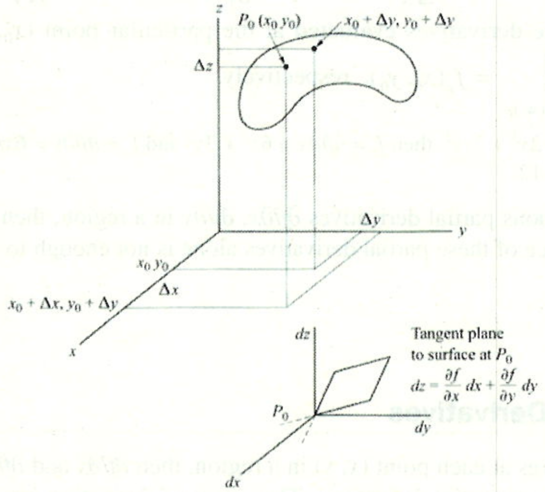

Figure 6.4

If f is such that Δf (or Δz) can be expressed in the form of Equation (4) where ϵ_1 and ϵ_2 approach zero as Δx and Δy approach zero, we call f *differentiable* at (x, y). The mere existence of f_x and f_y does not in itself guarantee differentiability; however, continuity of f_x and f_y does (although this condition happens to be slightly stronger than necessary). In case f_x and f_y are continuous in a region $\Re$, we say that f is *continuously differentiable* in $\Re$.

Theorems on Differentials

In the following, we assume that all functions have continuous first partial derivatives in a region $\Re$; i.e., the functions are continuously differentiable in $\Re$.

1. If $z = f(x_1, x_2, \ldots, x_n)$, then

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n \tag{6}$$

regardless of whether the variables $x_1, x_2, \ldots, x_n$ are independent or dependent on other variables (see Problem 6.20). This is a generalization of result in Equation (5). In Euqtion (6) we often use z in place of f.

2. If $f(x_1, x_2, \ldots, x_n) = c$, a constant, then $df = 0$. Note that in this case $x_1, x_2, \ldots, x_n$ cannot all be independent variables.

3. The expression $P(x, y)dx + Q(x, y)dy$, or, briefly, $P\,dx + Q\,dy$, is the differential of $f(x, y)$ if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. In such case, $P\,dx + Q\,dy$ is called an *exact differential*.

   Note: Observe that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ implies that $\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y}$.

4. The expression $P(x, y, z)\,dx + Q(x, y, z)dy + R(x, y, z)dz$, or, briefly, $P\,dx + Q\,dy + R\,dz$, is the differential of $f(x, y, z)$ if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$. In such case, $P\,dx + Q\,dy + R\,dz$ is called an *exact differential*.

Proofs of Theorems 3 and 4 are best supplied by methods of later chapters (see Problems 10.13 and 10.30).

Differentiation of Composite Functions

Let $z = f(x, y)$ where $x = g(r, s)$, $y = h(r, s)$ so that z is a function of r and s. Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \tag{7}$$

In general, if $u = F(x_1, \ldots, x_n)$ where $x_1 = f_1(r_1, \ldots, r_p), \ldots, x_n = f_n(r_1, \ldots, r_p)$, then

$$\frac{\partial u}{\partial r_k} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial r_k} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial r_k} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial u}{\partial r_k} \qquad k = 1, 2, \ldots, p \tag{8}$$

If, in particular, $x_1, x_2, \ldots, x_n$ depend on only one variable s, then

$$\frac{du}{ds} = \frac{\partial u}{\partial x_1}\frac{dx_1}{ds} + \frac{\partial u}{\partial x_2}\frac{dx_2}{ds} + \cdots + \frac{\partial u}{\partial x_n}\frac{dx_n}{ds} \tag{9}$$

These results, often called *chain rules*, are useful in transforming derivatives from one set of variables to another.

Higher derivatives are obtained by repeated application of the chain rules.

Euler's Theorem on Homogeneous Functions

A function represented by $F(x_1, x_2, \ldots, x_2, \ldots, x_n)$ is called *homogeneous of degree p* if, for all values of the parameter λ and some constant p, we have the identity

$$F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^p F(x_1, x_2, \ldots, x_n) \tag{10}$$

EXAMPLE. $F(x, y) = x^4 + 2xy^3 - 5y^4$ is homogeneous of degree 4, since

$$F(\lambda x, \lambda y) = (\lambda x)^4 + 2(\lambda x)(\lambda y)^3 - 5(\lambda y)^4 = \lambda^4(x^4 + 2xy^3 - 5y^4) = \lambda^4 F(x, y)$$

Euler's theorem on homogeneous functions states that if $F(x_1, x_2, \ldots, x_n)$ is homogeneous of degree p, then (see Problem 6.25)

$$x_1\frac{\partial F}{\partial x_1} + x_2\frac{\partial F}{\partial x_2} + \cdots + x_n\frac{\partial F}{\partial x_n} = pF \tag{11}$$

Implicit Functions

In general, an equation such as $F(x, y, z) = 0$ defines one variable—say, z—as a function of the other two variables x and y. Then z is sometimes called an *implicit function* of x and y, as distinguished from an *explicit function f*, where $z = f(x, y)$, which is such that $F[x, y, f(x, y)] \equiv 0$.

Differentiation of implicit functions requires considerable discipline in interpreting the independent and dependent character of the variables and in distinguishing the intent of one's notation. For example, suppose that in the implicit equation $F[x, y, f(x, z)] = 0$, the independent variables are x and y and that $z = f(x, y)$. In order to find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, we initially write the following [observe that $F(x, t, z)$ is zero for all domain pairs (x, y); i.e., it is a constant]:

$$0 = dF = F_x\,dx + F_y\,dy + F_z\,dz$$

and then compute the partial derivatives F_x, F_y, F_z as though y, y, z constituted an independent set of variables. At this stage we invoke the dependence of z on x and y to obtain the differential form $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. Upon substitution and some algebra (see Problem 6.30), the following results are obtained:

$$\frac{\partial f}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial f}{\partial y} = -\frac{F_y}{F_z}$$

EXAMPLE. If $0 = F(x, y, z) = x^2z + yz^2 + 2xy^2 - z^3$ and $z = f(x, y)$, then $F_x = 2xz + 2y^2$, $F_y = z^2 + 4xy$, $F_z = x^2 + 2yz - 3z^2$. Then

$$\frac{\partial f}{\partial x} = -\frac{(2xz + 2y^2)}{x^2 + 2yz - 3z^2}, \qquad \frac{\partial f}{\partial y} = -\frac{(z^2 + 4xy)}{x^2 + 2yz - 3x^2}$$

Observe that f need not be known to obtain these results. If that information is available, then (at least theoretically) the partial derivatives may be expressed through the independent variables x and y.

Jacobians

If $F(u, \upsilon)$ and $G(u, \upsilon)$ are differentiable in a region, the *Jacobian determinant*, or the *Jacobian*, of F and G with respect to u and υ is the second-order functional determinant defined by

$$\frac{\partial(F, G)}{\partial(u, \upsilon)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial \upsilon} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial \upsilon} \end{vmatrix} = \begin{vmatrix} F_u & F_\upsilon \\ G_u & G_\upsilon \end{vmatrix} \tag{12}$$

Similarly, the third-order determinant

$$\frac{\partial(F, G, H)}{\partial(u, \upsilon, w)} = \begin{vmatrix} F_u & F_\upsilon & F_w \\ G_u & G_\upsilon & G_w \\ H_u & H_\upsilon & H_w \end{vmatrix}$$

is called the Jacobian of F, G, and H with respect to u, υ, and w. Extensions easily made.

Partial Derivatives Using Jacobians

Jacobians often prove useful in obtaining partial derivatives of implicit functions. Thus, for example, given the simultaneous equations

$$F(x, y, u, \upsilon) = 0, \qquad G(x, y, u, \upsilon) = 0$$

we may, in general, consider u and υ as functions of x and y. In this case, we have (see Problem 6.31)

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F, G)}{\partial(x, \upsilon)}}{\frac{\partial(F, G)}{\partial(u, \upsilon)}}, \quad \frac{\partial u}{\partial y} = -\frac{\frac{\partial(F, G)}{\partial(y, \upsilon)}}{\frac{\partial(F, G)}{\partial(u, \upsilon)}}, \quad \frac{\partial \upsilon}{\partial x} = -\frac{\frac{\partial(F, G)}{\partial(u, x)}}{\frac{\partial(F, G)}{\partial(u, \upsilon)}}, \quad \frac{\partial \upsilon}{\partial y} = -\frac{\frac{\partial(F, G)}{\partial(u, y)}}{\frac{\partial(F, G)}{\partial(u, \upsilon)}}$$

The ideas are easily extended. Thus, if we consider the simultaneous equations

$$F(u, \upsilon, w, x, y) = 0, \qquad G(u, \upsilon, w, x, y)\ y) = 0, \qquad H(u, \upsilon, w, x, y) = 0$$

we may, for example, consider u, υ, and w as functions of x and y. In this case,

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F, G, H)}{\partial(x, \upsilon, w)}}{\frac{\partial(F, G, H)}{\partial(u, \upsilon, w)}}, \qquad \frac{\partial w}{\partial y} = -\frac{\frac{\partial(F, G, H)}{\partial(u, \upsilon, y)}}{\frac{\partial(F, G, H)}{\partial(u, \upsilon, w)}}$$

with similar results for the remaining partial derivatives (see Problem 6.33).

Theorems on Jacobians

In the following we assume that all functions are continuously differentiable.

1. A necessary and sufficient condition that the equations $F(u, \upsilon, x, y, z) = 0$ and $G(u, \upsilon, x, y, z) = 0$ can be solved for u and υ (for example) is that $\dfrac{\partial(F, G)}{\partial(u, \upsilon)}$ is not identically zero in a region $\Re$.
   Similar results are valid for m equations in n variables, where $m < n$.
2. If x and y are functions of u and υ while u and υ are functions of and s, then (see Problem 6.43)
$$\frac{\partial(x, y)}{\partial(r, s)} = \frac{\partial(x, y)}{\partial(u, \upsilon)} \frac{\partial(u, \upsilon)}{\partial(r, s)} \tag{13}$$
   This is an example of a *chain rule* for Jacobians. These ideas are capable of generalization (see Problems 6.107 and 6.109, for example).
3. If $u = f(x, y)$ and $\upsilon = g(x, y)$, then a necessary and sufficient condition that a functional relation of the form $\phi(u, \upsilon) = 0$ exists between u and υ is that $\dfrac{\partial(u, \upsilon)}{\partial(x, y)}$ be identically zero Similar results. hold for n functions of n variables.

Further discussion of Jacobians appears in Chapter 7, where vector interpretations are employed.

Transformations

The set of equations
$$\begin{cases} x = F(u, \upsilon) \\ y = F(u, \upsilon) \end{cases} \tag{14}$$
defines, in general, a *transformation* or *mapping* which establishes a correspondence between points in the $u\upsilon$ and xy planes. If to each point in the $u\upsilon$ plane there corresponds one and only one the xy plane, and conversely, we speak of a *one-to-one* transformation or mapping. This will be so if F and G are continuously differentiable, with Jacobian not identically zero in a region. In such case (which we assume unless otherwise stated), Equations (14) are said to define a *continuously differentiable transformation* or *mapping*.

The words *transformation* and *mapping* describe the same mathematical concept in different ways. A *transformation* correlates one coordinate representation of a region of space with another. A *mapping* views this correspondence as a correlation of two distinct regions.

Under the transformation (14) a closed region $\Re$ of the xy plane is, in general, mapped into a closed region $\Re'$ of the $u\upsilon$ plane. Then if ΔA_{xy} and $\Delta A_{u\upsilon}$ denote, respectively, the areas of these regions, we can show that
$$\lim \frac{\Delta A_{xy}}{\Delta A_{u\upsilon}} = \left| \frac{\partial(x, y)}{\partial(u, \upsilon)} \right| \tag{15}$$
where lim denotes the limit as ΔA_{xy} (or $\Delta A_{u\upsilon}$) approaches zero. The Jacobian on the right of Equation (15) is often called the *Jacobian of the transformation* (14).

If we solve transformation (14) for u and υ in terms of x and y, we obtain the transformation $u = f(x, y)$, $\upsilon = g(x, y)$, often called the *inverse transformation* corresponding to (14). The Jacobians $\dfrac{\partial(u, \upsilon)}{\partial(x, y)}$ and $\dfrac{\partial(x, y)}{\partial(u, \upsilon)}$ of these transformations are reciprocals of each other (see Problem 6.43). Hence, if one Jacobian is different from zero in a region, the inverse exists and is not zero.

These ideas can be extended to transformations in three or higher dimensions. We deal further with these topics in Chapter 7, where use is made of the simplicity of vector notation and interpretation.

Curvilinear Coordinates

Rectangular Cartesian coordinates in the Euclidean plane or in three-dimensional space were mentioned at the beginning of this chapter. Other coordinate systems, the coordinate curves of which are generated from families that are not necessarily linear, are useful. These are called *curvilinear coordinates.*

EXAMPLE 1. Polar coordinates ρ, Φ are related to rectangular Cartesian coordinates through the transformation $x = \rho \cos \Phi$ and $y = \rho \sin \Phi$. The curves $\Phi = \Phi_0$ are radical lines, while $\rho = r_0$ are concentric circles. Among the convenient representations yielded by these coordinates are circles with the center at the origin. A weakness is that representations may not be defined at the origin.

EXAMPLE 2. Spherical coordinates r, Θ, Φ_0 for Euclidean three-dimensional space are generated from rectangular Cartesian coordinates by the transformation equations $x = r \sin \Theta \cos \Phi$, $y = r \sin \Theta \sin \Phi$, and $z = r \cos \Phi$. Again, certain problems lend themselves to spherical coordinates, and also uniqueness of representation can be a problem at the origin. The coordinate surfaces $r = r_0$, $\Theta = \Theta_0$, and $\Phi = \Phi_0$ are spheres, planes, and cones, respectively. The coordinate curves are the intersections of these surfaces, i.e., circles, circles, and lines.

For curvilinear coordinates in higher dimensional spaces, see Chapter 7.

Mean Value Theorem

If $f(x, y)$ is continuous in a closed region and if the first partial derivatives exist in the open region (i.e., excluding boundary points), then

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = hf_x(x_0 + \theta h, y_0 + \theta k) + kf_y(x_0 + \theta h, y_0 + \theta k) \qquad 0 < \theta < 1 \qquad (16)$$

This is sometimes written in a form in which $h = \Delta x = x - x_0$ and $k = \Delta y = y - y_0$.

SOLVED PROBLEMS

Functions and graphs

6.1. If $f(x, y) = x^3 - 2xy + 3y^2$, find: (a) $f\left(\frac{1}{x}, \frac{2}{y}\right)$; (c) $\frac{f(x, y+k) - f(x, y)}{k}$, $k \neq 0$.

(a) $f(-2,3) = (-2)^3 - 2(-2)(3) + 3(3)^2 = -8 + 12 + 27 = 31$

(b) $$f\left(\frac{1}{x}, \frac{2}{y}\right) = \left(\frac{1}{x}\right)^3 - 2\left(\frac{1}{x}\right)\left(\frac{2}{y}\right) + 3\left(\frac{2}{y}\right)^2 = \frac{1}{x^3} - \frac{4}{xy} + \frac{12}{y^2}$$

(c) $$\frac{f(x, y+k) - f(x, y)}{k} = \frac{1}{k}\{[x^3 - 2x(y+k) + 3(y+k)^2] - [x^3 - 2xy + 3y^2]\}$$
$$= \frac{1}{k}(x^3 - 2xy - 2kx + 3y^2 + 6ky + 3k^2 - x^2 + 2xy - 3y^2)$$
$$= \frac{1}{k}(-2kx = 6ky + 3k^2) = -2x + 6y + 3k.$$

6.2. Give the domain of definition for which each of the following functions is defined and real, and indicate this domain graphically.

(a) $f(x, y) = \ln\{(16 - x^2 - y^2)(x^2 + y^2 - 4)\}$

The function is defined and real for all points (x, y) such that

$$(16 - x^2 - y^2)(x^2 + y^2 - 4) > 0, \qquad \text{i.e., } 4 < x^2 + y^2 < 16$$