First observe that solutions of the quadratic equation $At^2 + 2Bt + C = 0$ are $t = \dfrac{-2B \pm \sqrt{4B^2 - 4AC}}{2A}$.

Further observe that the nature of these solutions is determined by $B^2 - AC$. If the quantity is positive, the solutions are real and distinct; if negative, they are complex conjugate; and if zero, the two solutions are coincident.

The expression $B^2 - AC$ also has the property of invariance with respect to plane rotations

$$x = \bar{x}\cos\theta - \bar{y}\sin\theta$$
$$y = \bar{x}\cos\theta - \bar{y}\sin\theta$$

It has been discovered that with the identifications $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$, we have the partial derivative form $f^2_{xy} - f_{xx}f_{yy}$ that characterizes relative extrema.

The demonstration of invariance of this form can be found in analytic geometric books. However, if you would like to put the problem in the context of the second partial derivative, observe that

$$f_{\bar{x}} = f_x \frac{\partial x}{\partial \bar{x}} + f_x \frac{\partial y}{\partial \bar{x}} = f_x \cos\theta + f_y \sin\theta$$

$$f_{\bar{y}} = f_x \frac{\partial x}{\partial \bar{x}} + f_y \frac{\partial y}{\partial \bar{y}} = -f_x \sin\theta + f_y \cos\theta$$

Then, using the chain rule to compute the second partial derivatives and proceeding by straightforward but tedious calculation, we show that.

$$f^2_{xy} = f_{xx}f_{yy} = f^2_{\bar{x}\bar{y}} - f_{\bar{x}\bar{x}}f_{\bar{y}\bar{y}}$$

The following equivalences are a consequence of this invariant form (independently of direction in the tangent plane at P_0):

$$f^2_{xy} = f_{xx}f_{yy} < 0 \quad \text{and} \quad f_{xx}f_{yy} > 0 \tag{1}$$

$$f^2_{xy} = f_{xx}f_{yy} < 0 \quad \text{and} \quad f_{xx}f_{yy} < 0 \tag{2}$$

The key relation is (1) because in order that this equivalence hold, both terms $f_x f_y$ must have the same sign. We can look to the one-variable case (make the same argument for each coordinate direction) and conclude that there is a relative minimum at P_0 if both partial derivatives are positive and a relative maximum if both are negative. We can make this argument for any pair of coordinate directions because of the invariance under rotation that was established.

If relation (2) holds, then the point is called a *saddle point*. If the quadratic form is zero, no information results.

Observe that this situation is analogous to the one-variable extreme value theory in which the nature of f at x, and with $f'(x) = 0$, is undecided if $f''(x) = 0$.

8.22. Find the relative maxima and minima of $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

$f_x = 3x^2 - 3 = 0$ when $x = \pm 1$, $f_y = 3y^2 - 12 = 0$ when $y = \pm 2$. Then critical points are $P(1, 2)$, $Q(-1, 2)$, $R(1, -2)$, $S(-1, -2)$.

$f_{xx} = 6x$, $f_{yy} = 0$. Then $\Delta = f_{xx}f_{yy} - f^2_{xy} = 36xy$.

At $P(1, 2)$, $\Delta > 0$ and f_{xx} (or f_{yy}) > 0; hence P is a relative minimum point.

At $Q(-1, 2)$, $\Delta < 0$ and Q is neither a relative maximum or minimum point.

At $R(1, -2)$, $\Delta < 0$ and R is neither a relative maximum or minimum point.

At $S(-1, -2)$, $\Delta > 0$ and f_{xx} (or f_{yy}) < 0 so S is a relative maximum point.

Thus, the relative minimum value of $f(x, y)$ occurring at P is 2, while the relative maximum value occurring at S is 38. Points Q and R are *saddle points*.

8.23. A rectangular box, open at the top, is to have a volume of 32 cubic feet. What must be the dimensions so that the total surface is a minimum?

If x, y, and z are the edges (see Fig. 8.7), then

$$\text{Volume of box} = V = xyz = 32 \qquad (1)$$

$$\text{Surface area of box} = S = xy + 2yz + 2xz \qquad (2)$$

or, since $z = 32/xy$ from Equation (1), REMARK, IN THIS CASE $x \neq 0$ AND $y \neq 0$

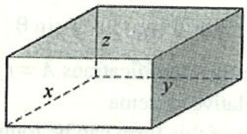

Fig. 8.7

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0 \text{ when } x^2 y = 64 \qquad (3)$$

$$\frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0 \text{ when } xy^2 = 64 \qquad (4)$$

Dividing Equations (3) and (4), we find $y = x$ so that $x^3 = 64$ or $x = y = 4$ and $z = 2$.

For $x = y = 4$, $\Delta = S_{xx}S_{yy} - S_{xy}^2 = \left(\frac{128}{x^3}\right)\left(\frac{128}{y^3}\right) - 1 > 0$ and $s_{xx} = \frac{128}{x^3} > 0$. Hence, it follows that the dimensions 4 feet × 4 feet × 2 feet give the minimum surface.

Lagrange multipliers for maxima and minima

8.24. Consider $F(x, y, z)$ subject to the constraint condition $G(x, y, z) = 0$. Prove that a necessary condition that $F(x, y, z)$ have an extreme value is that $F_xG_y - F_yG_x = 0$.

Since $G(x, y, z) = 0$, we can consider z as a function of x and y—say, $z = f(x, y)$. A necessary condition that $F[x, y, f(x, y)]$ have an extreme value is that the partial derivatives with respect to x and y be zero. This gives

$$F_x + F_z z_x = 0 \qquad (1)$$

$$F_y + F_z Z_y = 0 \qquad (2)$$

Since $G(x, y, z) = 0$, we also have

$$G_x + G_z z_x = 0 \qquad (3)$$

$$G_y + G_z z_y = 0 \qquad (4)$$

From Equations (1) and (3) we have

$$F_x G_x - F_x G_x = 0 \qquad (5)$$

and from Equations (2) and (4) we have

$$F_y G_z - F_z G_y = 0 \qquad (6)$$

Then from Equations (5) and (6) we find $F_xG_y - F_yG_x = 0$.

These results hold only if $F_z \neq 0$, $G_z \neq 0$.

8.25. Referring to Problem 8.24, show that the stated condition is equivalent to the conditions $\phi_x = 0$, $\phi_y = 0$ where $\phi = F + \lambda G$ and λ is a constant.

If $\phi_x = 0$, $F_x + \lambda G_x = 0$. If $\phi_y = 0$, $F_y + \lambda G_y = 0$. Elimination of λ between these equations yields $F_xG_y - F_y G_x = 0$.

The multiplier λ is the *Lagrange multiplier*. If desired, we can consider equivalently $\phi = \lambda F + G$ where $\phi_x = 0$, $\phi_y = 0$.

8.26. Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$, $z = 0$.

We must find the minimum value of $x^2 + y^2$ (the square of the distance from the origin to any point in the xy plane) subject to the constraint $x^2 + 8xy + 7y^2 = 225$.

According to the method of Lagrange multipliers, we consider $\phi = x^2 + 8xy + 7y^2 - 225 + \lambda(x^2 + y^2)$.

Then

$$\phi_x = 2x + 8y + 2\lambda x = 0 \quad \text{or} \quad (\lambda + 1)x + 4y = 0 \qquad (1)$$

$$\phi_y = 8x + 14y + 2\lambda y = 0 \quad \text{or} \quad 4x + (\lambda + 7)y = 0 \qquad (2)$$

From Equations (1) and (2), since $(x, y) \neq (0, 0)$, we must have

$$\begin{vmatrix} \lambda+1 & 4 \\ 4 & \lambda+7 \end{vmatrix} = 0, \text{ i.e., } \lambda^2 + 8\lambda - 9 = \text{ or } \lambda = 1, -9$$

Case 1: $\lambda = 1$. From Equation (*1*) or (2), $x = -2y$, and substitution in $x^2 + 8xy + 7y^2 = 225$ yields $-5y^2 = 225$, for which no real solution exists.

Case 2: $\lambda = -9$. From Equation (*1*) or (2), $y = 2x$, and substitution in $x^2 + 8xy + 7y^2 = 225$ yields $45x^2 = 225$. Then $x^2 = 5$, $y^2 = 4x^2 = 20$ and so $x^2 + y^2 = 25$. Thus, the required shortest distance is $\sqrt{25} = 5$.

8.27. (a) Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the constraint conditions $x^2/4 + y^2/5 + z^2/25 = 1$ and $z = x + y$. (b) Give a geometric interpretation of the result in (*a*).

(a) We must find the extrema of $F = x^2 + y^2 + z^2$ subject to the constraint conditions $\phi_1 = \frac{x^2}{4} + \frac{x^2}{5} + \frac{z^2}{25} - 1 = 0$ and $\phi_2 = x + y - z = 0$. In this case we use two Lagrange multipliers λ_1, λ_2 and consider the function

$$G = F + \lambda_1\phi_1 + \lambda_2\phi_2 = x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right) + \lambda_2(x + y - z)$$

Taking the partial derivatives of G with respect to x, y, z and setting them equal to zero, we find

$$G_x = 2x + \frac{\lambda_1 x}{2} + \lambda_2 = 0, \quad G_y = 2y + \frac{2\lambda_1 y}{5} + \lambda_2 = 0, \quad G_x = 2z + \frac{2\lambda_1 z}{25} - \lambda_2 = 0 \qquad (1)$$

Solving these equations for x, y, z, we find

$$x = \frac{-2\lambda_2}{\lambda_1 + 4}, \quad y = \frac{-5\lambda_2}{2\lambda_1 + 10}, \quad z = \frac{25\lambda_2}{2\lambda_1 + 50} \qquad (2)$$

From the second constraint condition, $x + y - z = 0$, we obtain, on division by λ_2, assumed different from zero (this is justified, since otherwise we would have $x = 0$, $y = 0$, $z = 0$, which would not satisfy the first constraint condition), the result

$$\frac{2}{\lambda_1 + 4} + \frac{5}{2\lambda_1 + 10} + \frac{25}{2\lambda_1 + 50} = 0$$

Multiplying both sides by $2(\lambda_1 + 4)(\lambda_1 + 5)(\lambda_1 + 5)(\lambda_1 + 25)$ and simplifying yields

$$17\lambda^2_1 + 245\lambda_1 + 750 = 0 \text{ or } (\lambda_1 + 10)(17\lambda_1 + 75) = 0$$

from which $\lambda_1 = -10$ or $-75/17$.

Case 1: $\lambda_1 = -10$.

From (2), $x = \frac{1}{3}\lambda_2$, $y = \frac{1}{2}\lambda_2$, $z = 5/6\lambda_2$. Substituting in the first constraint condition, $x^2/4 + y^2/5 + z^2/25 = 1$, yields $\lambda^2_2 = 180/19$ or $\lambda_2 = \pm 6\sqrt{5/19}$. This gives the two critical points

$$\left(2\sqrt{5/19}\,,3\sqrt{5/19}\,,5\sqrt{5/19}\right)\quad \left(-2\sqrt{5/19}\,,-3\sqrt{5/19}\,,-5\sqrt{5/19}\right)$$

The value of $x^2 + y^2 + z^2$ corresponding to these critical points is $(20 + 45 + 125)/19 = 10$.

Case 2: $\lambda_1 = -75/17$.

From (2), $x = 34/7\,\lambda_2$, $y = -17/4\lambda_2$, $z = 17/28\lambda_2$. Substituting in the first constraint condition, $x^2/4 + y^2/5 + z^2/25 = 1$, yields $\lambda_2 = \pm 140/(17\sqrt{646})$ which give the critical points

$$\left(40/\sqrt{646}\,,-35\sqrt{646}\,,5/\sqrt{646}\right)\quad \left(-40/\sqrt{646}\,,-35\sqrt{646}\,,-5/\sqrt{646}\right)$$

The value of $x^2 + y^2 + z^2$ corresponding to these is $(1600 + 1225 + 25)/646 = 75/17$.

Thus, the required maximum value is 10 and the minimum value is 75/17.

(b) Since $x^2 + y^2 + z^2$ represents the square of the distance of (x, y, z) from the origin (0, 0, 0), the problem is equivalent to determining the largest and smallest distances from the origin to the curve of intersection of the ellipsoid $x^2/4 + y^2/5 + z^2/25 = 1$ and the plane $z = x + y$. Since this curve is an ellipse, we have the interpretation that $\sqrt{10}$ and $\sqrt{75/17}$ are the lengths of the semimajor and semiminor axes of this ellipse.

The fact that the maximum and minimum values happen to be given by $-\lambda_1$ in both Case 1 and Case 2 is more than a coincidence. It follows, in fact, on multiplying Equations (*1*) by x, y, and z in succession and adding, for we then obtain

$$2x^2 + \frac{\lambda_1 x^2}{2} + \lambda_2 x + 2y^2 + \frac{2\lambda_1 y^2}{5} + \lambda_2 y + 2z^2 + \frac{2\lambda_1 z^2}{25} - \lambda_2 z = 0$$

i.e.,

$$x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25}\right) + \lambda_2\,(x + y - z) = 0$$

Then, using the constraint conditions, we find $x^2 + y^2 + z^2 = -\lambda_1$.

For a generalization of this problem, see Problem 8.76.

Applications to errors

8.28. The period T of a simple pendulum of length l is given by $T = 2\sqrt{l/g}$. Find (a) the error and (b) the percent error made in computing T by using $l = 2$ m and $g = 9.75$ m/ sec², if the true values are $l = 19.5$m and $g = 9.81$ m/sec².

(a) $T = 2\pi l^{1/2} g^{-1/2}$. Then

$$dT = (2\pi\, g^{-1/2}(\frac{1}{2}\,l^{-1/2}\;dl) + (2\pi\, l^{1/2})(-\frac{1}{2}g^{-3/2}\;dg) = \frac{\pi}{\sqrt{lg}}\,dl - \pi\sqrt{\frac{1}{g^3}}\,dg \qquad (1)$$

Error in $g = \Delta g = dg = +\,0.06$; error in $l = \Delta l = dl = -\,0.5$

The error in T is actually ΔT, which is in this case approximately equal to dT. Thus, we have from Equation (1),

$$\text{Error in } T = dT = \frac{\pi}{\sqrt{(2)(9.75)}}(-0.05) - \pi\sqrt{\frac{2}{(9.75)^3}}(+0.06) = -\,0.0444 \text{ sec (approx.)}$$

The value of T for $l = 2$, $g = 9.75$ is $T = 2\pi\sqrt{\dfrac{2}{9.75}} = 2.846$ sec (approx.)