7.33. If $\mathbf{A} = x^2 \sin y\mathbf{i} + z^2 \cos y\mathbf{j} - xy^2\mathbf{k}$, find $d\mathbf{A}$.

Method 1:

$$\frac{\partial \mathbf{A}}{\partial x} = 2x\sin y\mathbf{i} - y^2\mathbf{k}, \quad \frac{\partial \mathbf{A}}{\partial y}\ x^2 \cos y\mathbf{i} - z^2 \sin y\mathbf{j} - 2xy\mathbf{k}, \quad \frac{\partial \mathbf{A}}{\partial z} = 2z\cos y\mathbf{j}$$

$$d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial x}dx + \frac{\partial \mathbf{A}}{\partial y}dy + \frac{\partial \mathbf{A}}{\partial z}dz$$

$$= (2x\sin y\mathbf{i} - y^2\mathbf{k})dx + (x^2 \cos y\mathbf{i} - z^2 \sin y\mathbf{j} - 2xy\mathbf{k})dy + (2z\cos y\mathbf{j})dz$$

$$= (2x\sin y\, dx + x^2 \cos y\, dy)\mathbf{i} + (2z\cos y\, dz - z^2 \sin y\, dy)\mathbf{j} - (y^2 dx + 2xy\, dy)\mathbf{k}$$

Method 2:

$$d\mathbf{A} = d(x^2 \sin y)\mathbf{i} + d(z^2 \cos y)\mathbf{j} - d(xy^2)\mathbf{k}$$

$$= (2x \sin y\, dx + x^2 \cos y\, dy)\mathbf{i} + (2z \cos y\, dz - z^2 \sin y\, dy)\mathbf{j} - (y^2\, dx + 2xy\, dy)\mathbf{k}$$

Gradient, divergence, and curl

7.34. If $\phi = x^2yz^3$ and $\mathbf{A} = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$, find (a) $\nabla\phi$, (b) $\nabla \cdot \mathbf{A}$, (c) $\nabla \times \mathbf{A}$, (d) div $(\phi\mathbf{A})$, (e) curl $(\phi\mathbf{A})$.

(a) $$\nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \frac{\partial}{\partial x}(x^2yz^3)\mathbf{i} + \frac{\partial}{\partial x}(x^2yz^3)\mathbf{j} + \frac{\partial}{\partial z}(x^2yz^3)\mathbf{k}$$

$$= 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}$$

(b) $$\nabla \cdot \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) = z - 2y$$

(c) $$\nabla \times \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -y^2 & 2x^2y \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial z}(-y^2)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(2x^2y)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(xz)\right)\mathbf{k}$$

$$= 2x^2\mathbf{i} + (x - 4xy)\mathbf{j}$$

(d) $$\text{div}\,(\phi\mathbf{A}) = \nabla \cdot (\phi\mathbf{A}) = \nabla \cdot (x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^3yz^4) + \frac{\partial}{\partial y}(-x^2y^3z^3) + \frac{\partial}{\partial z}(2x^4y^2z^3)$$

$$= 3x^2yz^4 - 3x^2y^2z^3 + 6x^4y^2z^2$$

(e) $$\text{curl}\,(\phi\mathbf{A}) = \nabla \times (\phi\mathbf{A}) = \nabla \times (x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3yz^4 & -x^2y^3z^3 & 2x^4y^2z^3 \end{vmatrix}$$

$$= (4x^4yz^3 - 3x^2y^3z^2)\mathbf{i} + (4x^3yz^3 - 8x^3y^2z^3)\mathbf{j} - (2xy^3z^3 + x^3z^4)\mathbf{k}$$