SOLVED PROBLEMS

Tangent Plane And Normal Line To A Surface

8.1. Find equations for the (a) tangent plane and (b) normal line to the surface $x^2yz + 3y^2 = 2xz^2 - 8z$ at the point $(1, 2, -1)$.

(a) The equation of the surface is $F = x^2yz + 3y^2 - 2xz^2 + 8z = 0$. A normal line to the surface at $(1, 2, -1)$ is

$$\mathbf{N}_0 = \nabla F\big|_{(1,2,-1)} = (2xyz - 2z^2)\mathbf{i} + (x^2z + 6y)\mathbf{j} + (xy - 4xz + 8)\mathbf{k}\big|_{(1,2,-1)}$$

$$= -6\mathbf{i} + 11\mathbf{j} + 14\mathbf{k}$$

Referring to Figure 8.1:
The vector from O to any point (x, y, z) on the tangent plane is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
The vector from O to the point $(1, 2, -1)$ on the tangent plane is $\mathbf{r}_0 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.
The vector $\mathbf{r} - \mathbf{r}_0 = (x-1)\mathbf{i} + (y-2)\mathbf{j} + (z+1)\mathbf{k}$ lies in the tangent plane and is thus perpendicular to $\mathbf{N}_0$.

Then the required equation is

$$(\mathbf{r} - \mathbf{r}_0)\cdot\mathbf{N}_0 = 0 \quad \text{i.e.,} \quad \{(x-1)\mathbf{i} + (y-2)\mathbf{j} + (z+1)\mathbf{k}\}\cdot\{-6\mathbf{i} + 11\mathbf{j} + 14\mathbf{k}\} = 0$$

$$-6(x-1) + 11(y-2) + 14(z+1) = 0 \text{ or } 6x - 11y - 14z + 2 = 0$$

(b) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the vector from O to any point (x, y, z) of the normal $\mathbf{N}_0$. The vector from O to the point $(1, 2, -1)$ on the normal is $\mathbf{r}_0 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The vector $\mathbf{r} - \mathbf{r}_0 = (x-1)\mathbf{i} + (y-2)\mathbf{j} + (z+1)\mathbf{k}$ is collinear with $\mathbf{N}_0$. Then

$$(r - r_0)\times N_0 = 0 \qquad \text{i.e.,} \qquad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x-1 & y-2 & x+1 \\ -6 & 11 & 14 \end{vmatrix} = 0$$

which is equivalent to the equations

$$11(x-1) = -6(y-2),\ 14(y-2) = 11(z+1),\ 14(x-1) = -6(z+1)$$

These can be written as

$$\frac{x-1}{-6} = \frac{y-2}{11} = \frac{z+1}{14}$$

often called the *standard form* for the equations of a line. By setting each of these ratios equal to the parameter t, we have

$$x = 1 - 6t, \quad y = 2 + 11t, \quad z = 14t - 1$$

called the *parametric equations* for the line.

8.2. In what point does the normal line of Problem 8.1(b) meet the plane $x + 3y - 2z = 10$?

Substituting the parametric equations of Problem 8.1(b), we have

$$1 - 6t + 3(2 + 11t) - 2(14t - 1) = 10 \text{ or } t = -1$$

Then $x = 1 - 6t = 7$, $y = 2 + 11t = -9$, $z = 14t - 1 = -15$ and the required point is $(7, -9, -15)$.

8.3. Show that the surface $x^2 - 2yz + y^3 = 4$ is perpendicular to any member of the family of surfaces $x^2 + 1 = (2 - 4a)y^2 + az^2$ at the point of intersection $(1, -1, 2)$.

Let the equations of the two surfaces be written in the form

$$F = x^2 - 2yz + y^3 - 4 = 0 \text{ and } G = x^2 + 1 - (2 - 4a)y^2 - az^2 = 0$$

Then

$$\nabla F = 2x\mathbf{i} + (3y^2 - 2z)\mathbf{j} - 2y\mathbf{k}, \qquad \nabla G = 2x\mathbf{i} - 2(2 - 4a)y\mathbf{j} - 2az\mathbf{k}$$

Thus, the normals to the two surfaces at (1, –1, 2) are given by

$$\mathbf{N}_1 = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k},\ \mathbf{N}_2 = 2\mathbf{i} + 2(2 - 4a)\mathbf{j} - 4a\mathbf{k}$$

Since $\mathbf{N}_1 \cdot \mathbf{N}_2 = (2)(2) - 2(2 - 4a) - (2)(4a) \equiv 0$, it it follows that $\mathbf{N}_1$ and $\mathbf{N}_2$ are perpendicular for all a, and so the required result follows.

8.4. The equation of a surface is given in spherical coordinates by $F(r, \theta, \phi) = 0$, where we suppose that F is continuously differentiable. (a) Find an equation for the tangent plane to the surface at the point (r_0, θ_0, ϕ_0). (b) Find an equation for the tangent plane to the surface $r = 4\cos\theta$ at the point $(2\sqrt{2}, \pi/4, 3\pi/4)$. (c) Find a set of equations for the normal line to the surface in ( ) at the indicated point.

(a) The gradient of Φ in orthogonal curvilinear coordinates is

$$\nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{e}_3$$

where $$\mathbf{e}_1 = \frac{1}{h_1}\frac{\partial\mathbf{r}}{\partial u_1},\quad \mathbf{e}_2 = \frac{1}{h_2}\frac{\partial\mathbf{r}}{\partial u_2},\quad \mathbf{e}_3 = \frac{1}{h_3}\frac{\partial\mathbf{r}}{\partial u_3}$$

(see Pages 170 and 172).

In spherical coordinates $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$, $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = r\sin\theta\cos\phi\mathbf{i} + r\sin\theta\sin\phi\mathbf{j} + r\cos\theta\mathbf{k}$.

Then

$$\begin{cases} \mathbf{e}_1 = \sin\theta\cos\theta\,\mathbf{i} + \sin\theta\sin\theta\,\mathbf{j} + \cos\theta\,\mathbf{k} \\ \mathbf{e}_2 = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k} \\ \mathbf{e}_3 = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \end{cases} \tag{1}$$

and

$$\nabla F = \frac{\partial F}{\partial r}\mathbf{e}_1 + \frac{1}{r}\frac{\partial F}{\partial\theta}\mathbf{e}_2 + \frac{1}{r\sin\theta}\frac{\partial F}{\partial\phi}\mathbf{e}_3 \tag{2}$$

As on Page 196, the required equation is $(\mathbf{r} - \mathbf{r}_0)\cdot\nabla F|_P = 0$.

Now, substituting Equations (1) and (2), we have

$$\begin{aligned} \nabla F|_P = &\left\{\frac{\partial F}{\partial r}\bigg|_P \sin\theta_0\cos\phi_0 + \frac{1}{r_0}\frac{\partial F}{\partial\theta}\bigg|_P \cos\theta_0\cos\phi_0 - \frac{\sin\phi_0}{r_0\sin\theta_0}\frac{\partial F}{\partial\phi}\bigg|_P\right\}\mathbf{i} \\ &+\left\{\frac{\partial F}{\partial r}\bigg|_P \sin\theta_0\sin\phi_0 + \frac{1}{r_0}\frac{\partial F}{\partial\theta}\bigg|_P \cos\theta_0\sin\phi_0 + \frac{\cos\phi_0}{r_0\sin\theta_0}\frac{\partial F}{\partial\phi}\bigg|_P\right\}\mathbf{j} \\ &+\left\{\frac{\partial F}{\partial r}\bigg|_P \cos\theta_0 - \frac{1}{r_0}\frac{\partial F}{\partial\theta}\bigg|_P \sin\theta_0\right\}\mathbf{k} \end{aligned}$$

Denoting the expressions in braces by A, B, C, respectively, so that $\nabla F|_p = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, we see that the required equation is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. This can be written in spherical coordinates by using the transformation equations for x, y, and z in these coordinates.

(b) We have $F = r - 4\cos\theta = 0$. Then $\partial F/\partial r = 1$, $\partial F/\partial\theta = 4\sin\theta$, $\partial F/\partial\phi = 0$.

Since $r_0 = 2\sqrt{2}$, $\theta_0 = \pi/4$, $\phi_0 = 3\pi/4$, we have from (a), $\nabla F|_p = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} = -\mathbf{i} + \mathbf{j}$.

From the transformation equations, the given point has rectangular coordinates $(-\sqrt{2}, \sqrt{2}, 2)$, and so $r - r_0 = (x + \sqrt{2})\mathbf{i} + (y - \sqrt{2})\mathbf{j} + (z - 2)\mathbf{k}$.

The required equation of the plane is thus $-(x + \sqrt{2}) + (y - \sqrt{2}) = 0$ or $y - x = 2\sqrt{2}$. In spherical coordinates this becomes $r \sin\theta \sin\phi - r \sin\theta \cos\phi = 2\sqrt{2}$.

In rectangular coordinates the equation $r = 4\cos\theta$ becomes $x^2 + y^2 + (z-2)^2) = 4$ and the tangent plane can be determined from this as in Problem 8.1. In other cases, however, it may not be so easy to obtain the equation in rectangular form, and in such cases the method of part (*a*) is simpler to use.

(c) The equations of the normal line can be represented by

$$\frac{x+\sqrt{2}}{-1} = \frac{y-\sqrt{2}}{1} = z = 2$$

Tangent Line and Normal Plane to a Curve

8.5. Find equations for (a) the tangent line and (b) the normal plane to the curve $x = t - \cos t$, $y = 3 + \sin 2t$, $z = 1 + \cos 3t$ at the point where $t = 1/2\pi$.

(a) The vector from origin O (see Figure 8.2) to any point of curve C is $\mathbf{R} = (t - \cos t)\mathbf{i} + (3 + \sin 2t)\mathbf{j} + (1 + \cos 3t)\mathbf{k}$. Then a vector tangent to C at the point where $t = \frac{1}{2}\pi$ is

$$\mathbf{T}_0 = \left.\frac{d\mathbf{R}}{dt}\right|_{t=1/2\pi} = (1+\sin t)\,i + 2\cos 2t\,\mathbf{j} - 3\sin 3t\,\mathbf{k}\,|_{t=1/2\pi} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

The vector from O to the point where $t = 1/2\pi$ is $\mathbf{r}_0 = 1/2\pi\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

The vector from O to any point (x, y, z) on the tangent line is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Then $\mathbf{r} - \mathbf{r}_0 = (x - \frac{1}{2}\pi)\mathbf{i} + y - 3)\mathbf{j} + (z - 1)\,\mathbf{k}$ is collinear with $\mathbf{T}_0$, so that the required equation is

$$(\mathbf{r} - \mathbf{r}_0) \times \mathbf{T}_0 = \mathbf{0}, \quad \text{i.e.,} \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x - \frac{1}{2}\pi & y-3 & z-1 \\ 2 & -2 & 3 \end{vmatrix} = 0$$

and the required equations are $\dfrac{x - \frac{1}{2}\pi}{2} = \dfrac{y-3}{-2} = \dfrac{z-1}{3}$ or, in parametric form, $x = 2t + \frac{1}{2}\pi$, $y = 3 - 2t$, $z = 3t + 1$.

(b) Let $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the vector from O to any point (x, y, z) of the normal plane. The vector from O to the point where $t = \frac{1}{2}\pi$ is $\mathbf{r}_0 = \frac{1}{2}\pi\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. The vector $\mathbf{r} - \mathbf{r}_0 = (x - \frac{1}{2}\pi)\mathbf{i} + (y-3)\mathbf{j} + (z-1)\mathbf{k}$ lies in the normal plane and, hence, is perpendicular to $\mathbf{T}_0$. Then the required equation is $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{T}_0 = 0$ or $2(x - \frac{1}{2}\pi) - 2(y-3) + 3(z-1) = 0$.

8.6. Find equations for (a) the tangent line and (b) the normal plane to the curve $3x^2y + y^2z = -2$, $2xz - x^2y = 3$ at the point $(1, -1, 1)$.

(a) The equations of the surfaces intersecting in the curve are

$$F = 3x^2y + y^2z + 2 = 0, \quad G = 2xz - x^2y - 3 = 0$$

The normals to each surface at the point $P(1, -1, 1)$ are, respectively,

$$\mathbf{N}_1 = \nabla F|_P = 6xy\,\mathbf{i} + (3x^2 + 2yz)\mathbf{j} + y^2\mathbf{k} = -6 + \mathbf{j} + \mathbf{k}$$

$$\mathbf{N}_2 = \nabla G|_P = (2z - 2xy)\mathbf{i} - x^2\mathbf{j} + 2x\mathbf{k} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

Then a tangent vector to the curve at P is

$$\mathbf{T}_0 = \mathbf{N}_1 \times \mathbf{N}_2 = (-6\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (4 - \mathbf{j} + 2\mathbf{k}) = 3\mathbf{i} + 16\mathbf{j} + 2\mathbf{k}$$

Thus, as in Problem 8.5(a), the tangent line is given by

$$(\mathbf{r}-\mathbf{r}_0)\times\mathbf{T}_0=0 \text{ or } \{(x-1)\mathbf{i}+(y+1)\mathbf{j}+(z-1)\mathbf{k}\}\times\{3\mathbf{i}+16\mathbf{j}+2\mathbf{k}\}=\mathbf{0}$$

i.e.,

$$\frac{x-1}{3}=\frac{y+1}{16}=\frac{z-1}{2} \quad \text{or} \quad x=1+3t,\quad y=16t-1,\quad z=2t+1$$

(b) As in Problem 8.5(b), the normal plane is given by

$$(r-\mathbf{r}_0)\cdot\mathbf{T}_0=0 \text{ or } \{(x-1)\mathbf{i}+(y+1)\mathbf{j}+(z-1)\mathbf{k}\}\cdot\{3\mathbf{i}+16\mathbf{j}+2\mathbf{k}\}=0$$

i.e.,

$$3(x-1)+16(y+1)+2(z-1)=0 \text{ or } 3x+16y+2z=-11$$

The results in (a) and (b) can also be obtained by using Equations (7) and (10), respectively, from Page 197.

8.7. Establish equation (10), from Page 197.

Suppose the curve is defined by the intersection of two surfaces whose equations are $F(x, y, z)=0$, $G(x, y, z)=0$, where we assume F and G to be continuously differentiable.

The normals to each surface at point P are given, respectively, by $\mathbf{N}_1=\nabla F\,|_P$ and $\mathbf{N}_2=\nabla G\,|_P$. Then a tangent vector to the curve at P is $\mathbf{T}_0=\mathbf{N}_1\times\mathbf{N}_2=\nabla F\,|_P\times\nabla G\,|_P$. Thus, the equation of the normal plane is $(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{T}_0=0$. Now

$$\mathbf{T}_0=\nabla F\,|_P\times\nabla G\,|_P=\{(F_x i+F_y j+F_z k)\times(G_x i+G_y j+G_z k)\}\,|_P$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ F_x & F_y & F_z\\ G_x & G_y & G_z\end{vmatrix}_P=\begin{vmatrix}F_y & F_z\\ G_y & G_z\end{vmatrix}_P\mathbf{i}+\begin{vmatrix}F_z & F_x\\ G_x & G_x\end{vmatrix}_P\mathbf{j}+\begin{vmatrix}F_x & F_y\\ G_x & G_y\end{vmatrix}_P\mathbf{k}$$

and so the required equation is

$$(r-r_0)\,.\,\nabla F\,|_P=0 \quad or \quad \begin{vmatrix}F_y & F_z\\ G_y & G_z\end{vmatrix}_P(x-x_0)+\begin{vmatrix}F_z & F_x\\ G_z & G_x\end{vmatrix}_P(y-y_0)+\begin{vmatrix}F_x & F_y\\ G_x & G_y\end{vmatrix}_P(z-z_0)=0$$

Envelopes

8.8. Prove that the envelope of the family $\phi(x, y, \alpha)=0$, if it exists, can be obtained by solving simultaneously the equations $\phi=0$ and $\phi_\alpha=0$.

Assume parametric equations of the envelope to be $x=f(\alpha)$, $y=g(\alpha)$. Then $\phi(f(\alpha), g(\alpha), \alpha)=0$ identically, so, upon differentiating with respect to α (assuming that ϕ, f, and g have continuous derivatives), we have

$$\phi_x f'(\alpha)+\phi_y g'(\alpha)+\phi_\alpha=0 \tag{1}$$

The slope of any member of the family $\phi(x, y, \alpha)=0$ at (x, y) is given by $\phi_x\,dx+\phi_y\,dy=0$ or $\dfrac{dy}{dx}=-\dfrac{\phi_x}{\phi_y}$.

The slope of the envelope at (x, y) is $\dfrac{dy}{dx}=\dfrac{dy/d\alpha}{dx/d\alpha}=\dfrac{g'(\alpha)}{f'(\alpha)}$. Then at any point where the envelope and a member of the family are tangent, we must have

$$-\frac{\phi_x}{\phi_y}=\frac{g'(\alpha)}{f'(\alpha)} \quad \text{or} \quad \phi_x f'(\alpha)+\phi_y g'(\alpha)=0 \tag{2}$$

Comparing Equations (2) with (1), we see that $\phi_\alpha=0$, and the required result follows.

(b) Percent error (or relative error) in $T = \dfrac{dT}{T} = \dfrac{-0.0444}{2.846} = -1.56\%.$

Another method: Since $\ln T = \ln 2\pi + \frac{1}{2}\ln l - \frac{1}{2}\ln g$,

$$\frac{dT}{T} = \frac{1}{2}\frac{dl}{l} - \frac{1}{2}\frac{dg}{g} = \frac{1}{2}\left(\frac{-0.05}{2}\right) - \frac{1}{2}\left(\frac{+0.06}{9.75}\right) = -1.56\% \qquad (2)$$

as before. Note that Equation (2) can be written

$$\text{Percent error in } T = \frac{1}{2}\text{ Percent error in } l - \frac{1}{2}\text{ Percent error in } g$$

Miscellaneous problems

8.29. Evaluate $\displaystyle\int_0^1 \frac{x-1}{\ln x}\,dx.$

In order to evaluate this integral, we resort to the following device. Define

$$\phi(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx \qquad \alpha > 0$$

Then by Leibniz's rule

$$\phi'(\alpha) = \int_0^1 \frac{\partial}{\partial\alpha}\left(\frac{x^\alpha - 1}{\ln x}\right)dx = \int_0^1 \frac{x^\alpha \ln x}{\ln x}\,dx = \int_0^1 dx = \frac{1}{\alpha+1}$$

Integrating with respect to α, $\phi(\alpha) = \ln(\alpha + 1) + c$. But since $\phi(0) = 0$, $c = 0$, and so $\phi(\alpha) = \ln(\alpha + 1)$. Then the value of the required integral is $\phi(1) = \ln 2$.

The applicability of Leibniz's rule can be justified here, since if we define $F(x, \alpha) = (x^\alpha - 1)/\ln x$, $0 < x < 1$, $F(0, \alpha) = 0$, $F(1, \alpha) = \alpha$, then $F(x, \alpha)$ is continuous in both x and α for $0 \leq x \leq 1$ and all finite $\alpha > 0$.

8.30. Find constants a and b for which $F(a, b) = \displaystyle\int_0^\pi \{\sin x - (ax^2 + bx)\}^2\,dx$ is a minimum.

The necessary conditions for a minimum are $\partial F/\partial a = 0$. Performing these differentiations, we obtain

$$\frac{\partial F}{\partial a} = \int_0^\pi \frac{\partial}{\partial a}\{\sin x - (ax^2 + bx)\}^2\,dx = -2\int_0^\pi x^2\{\sin x - (ax^2 + bx)\}\,dx = 0$$

$$\frac{\partial F}{\partial b} = \int_0^\pi \frac{\partial}{\partial b}\{\sin x - (ax^2 + bx)\}^2\,dx = -2\int_0^\pi x\{\sin x - (ax^2 + bx)\}\,dx = 0$$

From these we find

$$\begin{cases} \alpha\displaystyle\int_0^\pi x^4\,dx + b\int_0^\pi x^3\,dx = \int_0^\pi x^2 \sin x\,dx \\ \alpha\displaystyle\int_0^\pi x^3\,dx + b\int_0^\pi x^2\,dx = \int_0^\pi x\sin x\,dx \end{cases}$$

or

$$\begin{cases} \dfrac{\pi^5 a}{5} + \dfrac{\pi^4 b}{4} = \pi^2 - 4 \\ \dfrac{\pi^4 a}{4} + \dfrac{\pi^3 b}{3} = \pi \end{cases}$$

Solving for a and b, we find

$$a = \frac{20}{\pi^3} - \frac{320}{\pi^5} \approx -0.40065. \qquad b = \frac{240}{\pi^4} - \frac{12}{\pi^2} \approx 1.24798$$

We can show that for these values, $F(a, b)$ is indeed a minimum using the sufficiency conditions on Page 200.

The polynomial $ax^2 + bx$ is said to be a *least square approximation* of $\sin x$ over the interval $(0, \pi)$. The ideas involved here are of importance in many branches of mathematics and their applications.

SUPPLEMENTARY PROBLEMS

Tangent plane and normal line to a surface

8.31. Find the equations of (a) the tangent plane and (b) the normal line to the surface $x^2 + y^2 = 4z$ at $(2, -4, 5)$.

Ans. (a) $x - 2y - z = 5$ (b) $\dfrac{x-2}{1} = \dfrac{y+4}{-2} = \dfrac{z-5}{-1}$

8.32. If $z = f(x, y)$, prove that the equations for the tangent plane and normal line at point $P(x_0, y_0, z_0)$ are given, respectively, by (a) $z - z_0 = f_x\big|_P (x - x_0) + f_y\big|_P (y - y_0)$ and (b) $\dfrac{x - x_0}{f_x\big|_P} = \dfrac{y - y_0}{f_x\big|_P} = \dfrac{z - z_0}{-1}$.

8.33. Prove that the acute angle γ between the z axis and the normal line to the surface $F(x, y, z) = 0$ at any point is given by $\sec\gamma = \sqrt{F_x^2 + F_y^2\ F_z^2}\,/\,|F_z|$.

8.34. The equation of a surface is given in cylindrical coordinates by $F(\rho, \phi, z) = 0$, where F is continuously differentiable. Prove that the equations of (a) the tangent plane and (b) the normal line at the point $P(\rho_0, \phi_0, z_0)$ are given, respectively, by $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ and $\dfrac{x - x_0}{A} = \dfrac{y - y_0}{B} = \dfrac{z - z_0}{C}$

where $x_0 = \rho_0 \cos \phi_0$, $y_0 = \rho_0 \sin \phi_0$ and $A = F_\rho\big|p \cos\phi_0 - \dfrac{1}{\rho} F_\phi\big|p \sin\phi_0$, $B = F_\rho\big|p \sin\phi_0 + \dfrac{1}{\rho} F_\phi\big|p \cos\phi_0$, and $C = F_z\big|_P$.

8.35. Use Problem 8.34 to find the equation of the tangent plane to the surface $\pi z = \rho\phi$ at the point where $\rho = 2$, $\phi = \pi/2$, $z = 1$. To check your answer, work the problem using rectangular coordinates.

Ans. $2x - \pi y + 2\pi z = 0$

Tangent line and normal plane to a curve

8.36. Find the equations of (a) the tangent line and (b) the normal plane to the space curve $x = 6 \sin t$, $y = 4 \cos 3t$. $z = 2 \sin 5t$ at the point where $t = \pi/4$.

Ans. (a) $\dfrac{x - 3\sqrt{2}}{3} = \dfrac{y + 2\sqrt{2}}{-6} = \dfrac{z + \sqrt{2}}{-5}$ (b) $3x - 6y - 5z = 26\sqrt{2}$

8.37. The surfaces $x + y + z = 3$ and $x^2 - y^2 + 2z^2 = 2$ intersect in a space curve. Find the equations of (a) the tangent line and (b) the normal plane to this space curve at the point $(1, 1, 1)$.

Ans. (a) $\dfrac{x-1}{-3} = \dfrac{y-1}{1} = \dfrac{z-1}{2}$ (b) $3x - y - 2z = 0$