which is called an *infinite series*. If $\lim_{n\to\infty} S_n = S$ exists, the series is called *convergent* and S is its *sum*; otherwise, the series is called *divergent*.

Further discussion of infinite series and other topics related to sequences is given in Chapter 11.

SOLVED PROBLEMS

Sequences

2.1. Write the first five terms of each of the following sequences.

(a) $\left\{\dfrac{2n-1}{3n+2}\right\}$

(b) $\left\{\dfrac{1-(-1)^n}{n^3}\right\}$

(c) $\left\{\dfrac{(-1)^{n-1}}{2\cdot 4\cdot 6\cdots 2n}\right\}$

(d) $\left\{\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\cdots+\dfrac{1}{2^n}\right\}$

(e) $\left\{\dfrac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}\right\}$

(a) $\dfrac{1}{5}, \dfrac{3}{8}, \dfrac{5}{11}, \dfrac{7}{14}, \dfrac{9}{17}$

(b) $\dfrac{2}{1^3}, 0, \dfrac{2}{3^3}, 0, \dfrac{2}{5^3}$

(c) $1\ \dfrac{1}{2}, \dfrac{-1}{2\cdot 4}, \dfrac{1}{2\cdot 4\cdot 6}, \dfrac{-1}{2\cdot 4\cdot 6\cdot 8}, \dfrac{1}{2\cdot 4\cdot 6\cdot 8\cdot 10}$

(d) $\dfrac{1}{2}, \dfrac{1}{2}+\dfrac{1}{4}, \dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}, \dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}, \dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}+\dfrac{1}{32}$

(e) $\dfrac{x}{1!}, \dfrac{-x^3}{3!}, \dfrac{x^5}{5!}, \dfrac{-x^7}{7!}, \dfrac{x^9}{9!}$

Note that $n! = 1\cdot 2\cdot 3\cdot 4\ldots n$. Thus, $1! = 1$, $3! = 1\cdot 2\cdot 3 = 6$, $5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$, etc. We define $0! = 1$.

2.2. Two students were asked to write an nth term for the sequence 1, 16, 81, 256, . . . and to write the 5th term of the sequence. One student gave the nth term as $u_n = n^4$. The other student, who did not recognize this simple law of formation, wrote $u_n = 10n^3 - 35n^2 + 50n - 24$. Which student gave the correct 5th term?

If $u_n = n^4$, then $u_1 = 1^4 = 1$, $u_2 = 2^4 = 16$, $u_3 = 3^4 = 81$, and $u_4 = 4^4 = 256$, which agrees with the first four terms of the sequence. Hence, the first student gave the 5th term as $u_5 = 5^4 = 625$.

If $u_n = 10n^3 - 35n^2 + 50n - 24$, then $u_1 = 1$, $u_2 = 16$, $u_3 = 81$, and $u_4 = 256$, which also agrees with the first four terms given. Hence, the second student gave the 5th term as $u_5 = 601$.

2.14. Evaluate each of the following, using theorems on limits.

(a) $\lim_{n\to\infty}\frac{3n^2-5n}{5n^2+2n-6}=\lim_{n\to\infty}\frac{3-5/n}{5+2/n-6/n^2}=\frac{3+0}{5+0+0}=\frac{3}{5}$

(b) $\lim_{n\to\infty}\left\{\frac{n(n+2)}{n+1}-\frac{n^3}{n^2+1}\right\}=\lim_{n\to\infty}\left\{\frac{n^3+n^2+2n}{(n+1)(n^2+1)}\right\}=\lim_{n\to\infty}\left\{\frac{1+1/n+2/n^2}{(1+1/n)(1+1/n^2)}\right\}$

$$=\frac{1+0+0}{(1+0)\cdot(1+0)}=1$$

(c) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0$

(d) $\lim_{n\to\infty}\frac{3n^2+4n}{2n-1}=\lim_{n\to\infty}\frac{3=4/n}{2/n-1/n^2}$

Since the limits of the numerator and the denominator are 3 and 0, respectively, the limit does not exist.

Since $\frac{3n^2+4n}{2n-1}>\frac{3n^2}{2n}=\frac{3n}{2}$ can be made larger than any positive number M by choosing $n>N$, we can write, if desired, $\lim_{n\to\infty}\frac{3n^2+4n}{2n-1}=\infty$.

(e) $\lim_{n\to\infty}\left(\frac{2n-3}{2n+7}\right)^4=\left(\lim_{n\to\infty}\frac{2-3/n}{3+7/n}\right)^4=\left(\frac{2}{3}\right)^4=\frac{16}{18}$

(f) $\lim_{n\to\infty}\frac{2n^5-4n^2}{3n^7+n^3-10}=\lim_{n\to\infty}\frac{2/n^2-4/n^5}{3+1/n^4-10/n^7}=\frac{0}{3}=0$

(g) $\lim_{n\to\infty}\frac{1+2\cdot 10^n}{5+3\cdot 10^n}=\lim_{n\to\infty}\frac{10^{-n}+2}{5\cdot 10^{-n}+3}=\frac{2}{3}$ (Compare with Problem 2.5.)

Bounded monotonic sequences

2.15. Prove that the sequence with nth $u_n=\frac{2n-7}{3n+2}$ (a) is monotonic increasing, (b) is bounded above, (c) is bounded below, (d) is bounded, (e) has a limit.

(a) $\{u_n\}$ is monotonic increasing if $u_{n+1}\geqq u_n$, $n=1,2,3,\ldots$ Now

$$\frac{2(n+1)-7}{3(n+1)+2}\geqq\frac{2n-7}{3n+2}\quad\text{if and only if}\quad\frac{2n-5}{2n+5}\geqq\frac{2n-7}{3n+2}$$

or $(2n-5)(3n+2)\geqq(2n-7)(3n+5)$, $6n^2-11n-10\geqq 6n^2-11n-35$, i.e., $-10\geqq -35$, which is true. Thus, by reversal of steps in the inequalities, we see that $\{u_n\}$ is monotonic increasing. Actually, since $-10>-35$, the sequence is strictly increasing.

(b) By writing some terms of the sequence, we may *guess* that an upper bound is 2 (for example). To *prove* this we must show that $u_n\leqq 2$. If $(2n-7)/(3n+2)\leqq 2$, then $2n-7\leqq 6n+4$ or $-4n<11$, which *is* true. Reversal of steps proves that 2 is an upper bound.

(c) Since this particular sequence is monotonic increasing, the first term -1 is a lower bound; i.e., $u_n\geqq -1$, $n=1,2,3,\ldots$ Any number less than -1 is also a lower bound.

(d) Since the sequence has an upper and a lower bound, it is bounded. Thus, for example, we can write $|u_n|\leqq 2$ for all n.

(e) Since every bounded monotonic (increasing or decreasing) sequence has a limit, the given sequence has a limit. In fact, $\lim_{n\to\infty}\frac{2n-7}{3n+2}=\lim_{n\to\infty}\frac{2-7/n}{3+2/n}=\frac{2}{3}$.