2.44. Evaluate each of the following, using theorems on limits:

(a) $\lim_{n\to\infty} \frac{4-2n-3n^2}{2n^2+n}$ (d) $\lim_{n\to\infty} \frac{4\cdot 10^n - 3\cdot 10^{2n}}{3\cdot 10^{n-1} + 2\cdot 10^{2n-1}}$

(b) $\lim_{n\to\infty} \sqrt[3]{\frac{(3-\sqrt{n})(\sqrt{n}+2)}{8n-4}}$ (e) $\lim_{n\to\infty}(\sqrt{n^2+n}-n)$

(c) $\lim_{n\to\infty} \frac{\sqrt{3n^2-5n+4}}{2n-7}$ (f) $\lim_{n\to\infty}(2^n+3^n)^{1/n}$

Ans. (a) –3/2 (b) –1/2 (c) $\sqrt{3}/2$ (d) –15 (e) 1/2 (f) 3

Bounded monotonic sequences

2.45. Prove that the sequence with nth term $u_n = u_n = \sqrt{n}/(n+1)$. (a) is monotonic decreasing, (b) is bounded below, (c) is bounded above, and (d) has a limit.

2.46. If $u_n = \frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \cdots + \frac{1}{n+n}$, prove that $\lim_{n\to\infty} u_n$ exists and lies between 0 and 1.

2.47. If $u_n = \sqrt{u_n+1}$, $u_1 = 1$, prove that $\lim_{n\to\infty} u = \frac{1}{2}(1+\sqrt{5})$.

2.48. If $u_{n+1} = \frac{1}{2}(u_n + p/u_n)$ where $p > 0$ and $u_1 > 0$, prove that $\lim_{n\to\infty} u_n = \sqrt{p}$. Show how this can be used to determine $\sqrt{2}$.

2.49. If u_n is monotonic increasing (or monotonic decreasing), prove that S_n/n, where $S_n = u_1 + u_2 + \cdots + u_n$ is also monotonic increasing (or monotonic decreasing).

Least upper bound, greatest lower bound, limit superior, limit inferior

2.50. Find the l.u.b., g.l.b., lim sup ($\overline{\lim}$), and lim inf ($\underline{\lim}$) for each sequence:

(a) $-1, \frac{1}{3}, -\frac{1}{5}, \frac{1}{7}, \ldots, (-1)^n/(2n-1), \ldots$ (c) $1, -3, 5, -7, \ldots, (-1)^{n-1}(2n-1), \ldots$

(b) $\frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots, (-1)^{n+1}(n+1)/(n+2), \ldots$ (d) $1, 4, 1, 16, 1, 36, \ldots, n^{1+(-1)^n}, \ldots$

Ans. (a) $\frac{1}{3}$, –1, 0,0 (b) 1, –1, 1, –1 (c) none, none, +∞, – ∞ (d) none, 1, +∞, 1

2.51. Prove that a bounded sequence $\{u_n\}$ is convergent if and only if $\overline{\lim}\, u_n = \underline{\lim}\, u_n$.

Infinite series

2.52. Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$.

Ans. 2