Since $\lim_{n\to\infty} v_n = 0$, we can choose P so that $|v_n| < \epsilon/2$ for $n > P$. Then

$$\frac{|v_{P+1}| + |v_{P+2}| + \cdots + |v_n|}{n} < \frac{\varepsilon/2 + \varepsilon/2 + \cdots + \varepsilon/2}{n} = \frac{(n-P)\varepsilon/2}{n} < \frac{\varepsilon}{2} \tag{2}$$

After choosing P, we can choose N so that for $n > N > P$,

$$\frac{|v_1 + v_2 + \cdots + v_P|}{n} < \frac{\varepsilon}{2} \tag{3}$$

Then, using Equations (2) and (3), (*1*) becomes

$$\left|\frac{v_1 + v_2 + \cdots + v_n}{n}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for } n > N$$

thus proving the required result.

2.29. Prove that $\lim_{n\to\infty}(1 + n + n^2)^{1/n} = 1$.

Let $(1 + n + n^2)^{1/n} = 1 + u_n$, where $u_n \geqq 0$. Now, by the binomial theorem,

$$1 + n + n^2 = (1 + u_n)^n = 1 + nu_n + \frac{n(n-1)}{2!}u_n^2 + \frac{n(n-1)(n-2)}{3!}u_n^3 + \cdots + u_n^n$$

Then $1 + n + n^2 > 1 + \frac{n(n-1)(n-2)}{3!}u_n^3$ or $0 < u_n^3 < \frac{6(n^2+n)}{n(n-1)(n-2)}$. Hence, $\lim_{n\to\infty} u_n^3 = 0$ and $\lim_{n\to\infty} u_n = 0$. Thus, $\lim_{n\to\infty}(1 + n + n^2)^{1/n} = \lim_{n\to\infty}(1 + u_n) = 1$.

2.30. Prove that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all constants a.

The result follows if we can prove that $\lim_{n\to\infty} \frac{|a|^n}{n!} = 0$ (see Problem 2.38). We can assume $a \neq 0$.

Let $u_n = \frac{|a|^n}{n!}$. Then $\frac{u_n}{u_{n-1}} = \frac{|a|}{n}$. If n is large enough—say, $n > 2|a|$—and if we call $N = [2|a| + 1]$, i.e., the greatest integer $\leqq 2|a| + 1$, then

$$\frac{u_{N+1}}{u_N} < \frac{1}{2}, \frac{u_{N+2}}{u_{N+1}} < \frac{1}{2}, \ldots, \frac{u_n}{u_{n-1}} < \frac{1}{2}$$

Multiplying these inequalities yields $\frac{u_n}{u_N} < \left(\frac{1}{2}\right)^{n-N}$ or $u_n < \left(\frac{1}{2}\right)^{n-N} u_N$. Since $\lim_{n\to\infty}\left(\frac{1}{2}\right)^{n-N} = 0$ (using Problem 2.7), it follows that $\lim_{n\to\infty} u_n = 0$.

SUPPLEMENTARY PROBLEMS

Sequences

2.31. Write the first four terms of each of the following sequences:

(a) $\left\{\frac{\sqrt{n}}{n+1}\right\}$

(b) $\left\{\frac{(-1)^{n+1}}{n!}\right\}$

(c) $\left\{\frac{(2x)^{n-1}}{(2n-1)^5}\right\}$

(d) $\left\{\frac{(-1)^n x^{2n-1}}{1\cdot3\cdot5\cdots(2n-1)}\right\}$

(e) $\left\{\frac{\cos nx}{x^2+n^2}\right\}$