L'Hospital's Rule

4.75. Evaluate each of the following limits.

(a) $\lim_{x\to 0}\dfrac{x-\sin x}{x^3}$

(b) $\lim_{x\to 0}\dfrac{e^{2x}-2^{ex}+1}{\cos 3x-2\cos 2x+\cos x}$

(c) $\lim_{x\to 1}(x^2-1)\tan \pi x/2$

(d) $\lim_{x\to\infty} x^3e^{-2x}$

(e) $\lim_{x\to 0+} x^3 \ln x$

(f) $\lim_{x\to 0}(3^x-2^x)/x$

(g) $\lim_{x\to\infty}(1-3/x)^{2x}$

(h) $\lim_{x\to\infty}(1+2x)^{1/3x}$

(i) $\lim_{x\to 0}(1/x-\csc x)$

(j) $\lim_{x\to 0} x^{\sin x}$

(k) $\lim_{x\to\infty}(1/x^2-\cot^2 x)$

(l) $\lim_{x\to 0}\dfrac{\tan^{-1}x-\sin^{-1}x}{x(1-\cos x)}$

(m) $\lim_{x\to\infty} x \ln\left(\dfrac{x+3}{x-3}\right)$

(n) $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$

(o) $\lim_{x\to\infty}(x+e^x+e^{2x})^{1/x}$

(p) $\lim_{x\to 0+}(\sin x)^{1/\ln x}$

Ans. (a) $\frac{1}{6}$ (b) -1 (c) $-4/\pi$ (d) 0 (e) 0 (f) $\ln 3/2$ (g) e^{-6} (h) 1 (i) 0 (j) 1 (k) $\frac{2}{3}$ (l) $\frac{1}{3}$ (m) 6 (n) $e^{-1/6}$ (o) e^2 (p) e

Miscellaneous problems

4.76. Prove that $\sqrt{\dfrac{1-x}{1+x}} < \dfrac{\ln(1+x)}{\sin^{-1}x} < 1$ if $0 < x < 1$.

4.77. If $\Delta f(x) = f(x+\Delta x) - f(x)$, (a) prove that $\Delta\{\Delta f(x)\} = \Delta^2 f(x) = f(x+2\Delta x) - 2f(x+\Delta x) + f(x)$; (b) derive an expression for $\Delta^n f(x)$ where n is any positive integer; and (c) show that $\lim_{\Delta x\to 0}\dfrac{\Delta^n f(x)}{(\Delta x)^n} = f^{(n)}(x)$ if this limit exists.

4.78. Complete the analytic proof mentioned at the end of Problem 4.36.

4.79. Find the relative maximum and minima of $f(x) = x^2$, $x > 0$.

Ans. $f(x)$ has a relative minimum when $x = e^{-1}$.

4.80. A train moves according to the rule $x = 5t^3 + 30t$, where t and x are measured in hours and miles, respectively. (a) What is the acceleration after 1 minute? (b) What is the speed after 2 hours?

4.81. A stone thrown vertically upward has the law of motion $x = -16t^2 + 96t$. (Assume that the stone is at ground level at $t = 0$, that t is measured in seconds, and that x is measured in feet.) (*a*) What is the height of the stone at $t = 2$ seconds? (*b*) To what height does the stone rise? (*c*) What is the initial velocity, and what is the maximum speed attained?

4.82. A particle travels with constant velocities v_1 and v_2 in mediums I and II, respectively (see Figure 4.11). Show that in order to go from point P to point Q in the least time, it must follow path PAQ where A is such that

$$(\sin\theta_1)/(\sin\theta_2) = v_1/v_2$$