SUPPLEMENTARY PROBLEMS

Derivatives

4.37. Use the definition to compute the derivatives of each of the following functions at the indicated point: (a) $(3x - 4)/(2x + 3)$, $x = 1$, (b) $x^3 - 3x^2 + 2x - 5$, $x = 2$, (c) $\sqrt{x}$, $x = 4$, and (d) $\sqrt[3]{6x-4}$, $x = 2$.

Ans. (a) 17/25, (b) 2, (c) $\frac{1}{4}$, (d) $\frac{1}{2}$

4.38. Show from definition that (a) $\frac{d}{dx}x^4 = 4x^3$ and (b) $\frac{d}{dx}\frac{3+x}{3-x} = \frac{6}{(3-x)^2}$, $x \neq 3$.

4.39. Let $f(x) = \begin{cases} x^3 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ Prove that (a) $f(x)$ is continuous at $x = 0$. (b) $f(x)$ has a derivative at $x = 0$, and (c) $f'(x)$ is continuous at $x = 0$.

4.40. Let $f(x) = \begin{cases} xe^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Determine whether $f(x)$ (a) is is continuous at $x = 0$, and (b) has a derivative at $x = 0$.

Ans. (a) Yes (b) Yes, 0

4.41. Give an alternative proof of the theorem in Problem 4.3, using "ϵ, δ" definitions.

4.42. If $f(x) = e^x$, show that $f'(x_0) = e^{x_0}$ depends on the result $\lim_{h\to 0}(e^h - 1)/h = 1$.

4.43. Use the results $\lim_{h\to 0}(\sin h)/h = 1$. $\lim_{h\to 0}(1 - \cos h)/h = 0$ to prove that if $f(x) = \sin x$, $f'(x_0) = \cos x_0$.

Right-and left-hand derivatives

4.44. Let $f(x) = x\,|x|$. (a) Calculate the right-hand derivative of $f(x)$ at $x = 0$. (b) Calculate the left-hand derivative of $f(x)$ at $x = 0$. (c) Does $f(x)$ have a derivative at $x = 0$? (d) Illustrate the conclusions in (a), (b), and (c) from a graph.

Ans. (a) 0 (b) 0 (c) Yes, 0

4.45. Discuss the (*a*) continuity and (*b*) differentiability of $f(x) = x^p \sin 1/x$, $f(0) = 0$, where p is any positive number. What happens in case p is any real number?

4.46. Let $f(x) = \begin{cases} 2x - 3, & 0 \leqq x \leqq 2 \\ x^2 - 3, & 2 < x \leqq 4 \end{cases}$. Discuss the (*a*) continuity and (*b*) differentiability of $f(x)$ in $0 \leqq x \leqq 4$.

4.47. Prove that the derivative of $f(x)$ at $x = x_0$ exists if and only if $f'_+(x_0) = f'_-(x_0)$.

4.48. (a) Prove that $f(x) = x^3 - x^2 + + 5x - 6$ is differentiable in $a \leqq x \leqq b$, where a and b are any constants.
(b) Find equations for the tangent lines to the curve $y = x^3 - x^2 + 5x - 6$ at $x = 0$ and $x = 1$. Illustrate by means of a graph. (c) Determine the point of intersection of the tangent lines in (b). (d) Find $f'(x), f''(x), f'''(x), f^{(IV)}(x), \ldots$

Ans. (b) $y = 5x - 6$, $y = 6x - 7$ (c) $(1, -1)$ (d) $3x^2 - 2x + 5$, $6x - 2$, 6, 0, 0, 0, . . .

4.49. If $f(x) = x^2 |x|$, discuss the existence of successive derivatives of $f(x)$ at $x = 0$.

Differentials

4.50. If $y = f(x) = x + 1/x$, find (a) Δy, (b) dy, (c) $\Delta y - dy$, (d) $(\Delta y - dy)/\Delta x$, and (e) dy/dx.

Ans. (a) $\Delta x - \dfrac{\Delta x}{x(x+\Delta x)}$ (b) $\left(1 - \dfrac{1}{x^2}\right)\Delta x$ (c) $\dfrac{(\Delta x)^2}{x^2(x+\Delta x)}$ (d) $\dfrac{\Delta x}{x^2(x+\Delta x)}$ (e) $1 - \dfrac{1}{x^2}$

Note: $\Delta x = dx$.

4.51. If $f(x) = x^2 + 3x$, find (a) Δy, (b) dy, (c) $\Delta y/\Delta x$, (d) dy/dx, and (e) $(\Delta y - dy)/\Delta x$, if $x = 1$ and $\Delta x = .01$.

Ans. (a) .0501, (b) .05, (c) 5.01, (d) 5, (e) .01

4.52. Using differentials, compute approximate values for each of the following: (a) sin 31°, (b) ln(1.12), (c) $\sqrt[5]{36}$.

Ans. (a) 0.515, (b) 0.12, (c) 2.0125

4.53. If $y = \sin x$, evaluate (a) Δy and (b) dy. (c) Prove that $(\Delta y - dy)/\Delta x \to 0$ as $\Delta x \to 0$.

Differentiation rules and elementary functions

4.54. Prove the following:

(a) $\dfrac{d}{dx}\{f(x)+g(x)\} = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$

(b) $\dfrac{d}{dx}\{f(x)-g(x)\} = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$

(c) $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

4.55. Evaluate (a) $\dfrac{d}{dx}\{x^3 \ln(x^2 - 2x + 5)\}$ at $x = 1$ and (b) $\dfrac{d}{dx}\{\sin^2(3x + \pi/6\}$ at $x = 0$.

Ans. (a) $3\ln 4$ (b) $\dfrac{3}{2}\sqrt{3}$

4.56. Derive these formulas: (a) $\dfrac{d}{dx}a^u = a^u \ln a \dfrac{du}{dx}$, $a > 0$, $a \neq 1$; $\dfrac{d}{dx}\csc u = -\csc u \cot u \dfrac{du}{dx}$; and (c) $\dfrac{d}{dx}\tanh u = \operatorname{sech}^2 u \dfrac{du}{dx}$ where u is a differentiable function of x.

4.57. Compute (a) $\dfrac{d}{dx}\tan^{-1} x$, (b) $\dfrac{d}{dx}\csc^{-1} x$, (c) $\dfrac{d}{dx}\sinh^{-1} x$, and (d) $\dfrac{d}{dx}\coth^{-1} x$, paying attention to the use of principal values.

4.58. If $y = x^x$, compute dy/dx. (Hint: Take logarithms before differentiating.)

Ans. $x^x(1 + \ln x)$