3.63. If $[x]$ = largest integer $\leq x$, evaluate (a) $\lim_{x \to 2+} \{x - [x]\}$ and (b) $\lim_{x \to 2-} \{x - [x]\}$.

Ans. (a) 0 and (b) 1

3.64. If $\lim_{x \to x_0} f(x) = A$, prove that (a) $\lim_{x \to x_0} \{f(x)\}^2 = A^2$ and (b) $\lim_{x \to x_0} \sqrt[3]{f(x)} = \sqrt[3]{A}$. What generalizations of these do you suspect are true? Can you prove them?

3.65. If $\lim_{x \to x_0} f(x)$, = A and lim $g(x)$, = B, prove that (a) $\lim_{x \to x_0} \{f(x) - g(x)\} = A - B$ and (b) $\lim_{x \to x_0} \{af(x) + bg(x)\} = aA + bB$, where a, b = any constants.

3.66. If the limits of $f(x)$, $g(x)$, and $h(x)$ are A, B, and C respectively, prove that (a) $\lim_{x \to x_0} \{f(x) + g(x) + h(x)\} = A + B + C$ and (b) $\lim_{x \to x_0} f(x)g(x)h(x) = ABC$. Generalize these results.

3.67. Evaluate each of the following using the theorems on limits.

(a) $\lim_{x \to 1/2} \left\{ \dfrac{2x^2 - 1}{(3x + 2)(5x - 3)} - \dfrac{2 - 3x}{x^2 - 5x + 3} \right\}$

(b) $\lim_{x \to \infty} \dfrac{(3x - 1)(2x + 3)}{(5x - 3)(4x + 5)}$

(c) $\lim_{x \to -\infty} \left(\dfrac{3x}{x - 1} - \dfrac{2x}{x + 1} \right)$

(d) $\lim_{x \to 1} \dfrac{1}{x - 1} \left(\dfrac{1}{x + 3} - \dfrac{2x}{3x + 5} \right)$

Ans. (a) –8/21 (b) 3/10 (c) 1 (d) 1/32

3.68. Evaluate $\lim_{h \to 0} \dfrac{\sqrt[3]{8 + h} - 2}{h}$. (Hint: Let $8 + h = x^3$.)

Ans. 1/12

3.69. If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B \neq 0$, prove directly that $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{A}{B}$.

3.70. Given $\lim_{x \to 0} \dfrac{\sin 3x}{x} = 1$, evaluate:

(a) $\lim_{x \to 0} \dfrac{\sin 3x}{x}$

(b) $\lim_{x \to 0} \dfrac{1 - \cos x}{x}$

(c) $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$

(d) $\lim_{x \to 3} (x - 3) \csc \pi x$

(e) $\lim_{x \to 0} \dfrac{6x - \sin 2x}{2x + 3 \sin 4x}$

(f) $\lim_{x \to 0} \dfrac{\cos ax - \cos bx}{x^2}$

(g) $\lim_{x \to 0} \dfrac{1 - 2\cos x + \cos 2x}{x^2}$

(h) $\lim_{x \to 1} \dfrac{3 \sin \pi x - \sin 3\pi x}{x^3}$

Ans. (a) 3 (b) 0 (c) $\dfrac{1}{2}$ (d) $-1/\pi$ (e) $\dfrac{2}{7}$ (f) $\dfrac{1}{2}(b^2 - a^2)$ (g) –1 (h) $4\pi^3$

3.71. If $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$, prove that (a) $\lim_{x\to 0} \frac{e^{-ax} - e^{-bx}}{x} = b - a$, (b) $\lim_{x\to 0} \frac{a^x - b^x}{x} = \ln \frac{a}{b}$, $a, b > 0$, and (c) $\lim_{x\to 0} \frac{\tanh ax}{x} = a$.

3.72. Prove that $\lim_{x\to x_0} f(x) = l$ if and only if $\lim_{x\to x_0^+} f(x) = l$.

Continuity

In the following problems, assume the largest possible domain unless otherwise stated.

3.73. Prove that $f(x) = x^2 - 3x + 2$ is continuous at $x = 4$.

3.74. Prove that $f(x) = 1/x$ is continuous (a) at $x = 2$ and (b) in $1 \leqq x \leqq 3$.

3.75. Investigate the continuity of each of the following functions at the indicated points:

(a) $f(x) = \frac{\sin x}{x}$; $x \neq 0$, $f(0) = 0$; $x = 0$

(b) $f(x) = x - |x|$; $x = 0$

(c) $f(x) = \frac{x^3 - 8}{x^2 - 4}$; $x \neq 2$, $f(2) = 3$; $x = 2$

(d) $f(x) = \begin{cases} \sin \pi x, & 0 < x < 1 \\ \ln & 1 < x < 2 \end{cases}$; $x = 1$

Ans. (a) discontinuous, (b) continuous, (c) continuous, (d) discontinuous

3.76. If $[x]$ = greatest integer $\leqq x$, investigate the continuity of $f(x) = x - [x]$ in the interval (a) $1 < x < 2$ and (b) $1 \leqq x \leqq 2$.

3.77. Prove that $f(x) = x^3$ is continuous in every finite interval.

3.78. If $f(x)/g(x)$ and $g(x)$ are continuous at $x = x_0$, prove that $f(x)$ must be continuous at $x = x_0$.

3.79. Prove that $f(x) = (\tan^{-1} x)/x$, $f(0) = 1$ is continuous at $x = 0$.

3.80. Prove that a polynomial is continuous in every finite interval.

3.81. If $f(x)$ and $g(x)$ are polynomials, prove that $f(x)/g(x)$ is continuous at each point $x = x_0$ for which $g(x_0) \neq 0$.

3.82. Give the points of discontinuity of each of the following functions.

(a) $f(x) = \frac{x}{(x-2)(x-4)}$

(b) $f(x) = x^2 \sin 1/x$, $x \neq 0$, $f(0) = 0$

(c) $f(x) = \sqrt{(x-3)(6-x)}$, $3 \leqq x \leqq 6$

(d) $f(x) = \frac{1}{1 + 2\sin x}$

Ans. (a) $x = 2, 4$ (b) none (c) none (d) $x = 7\pi/6 \pm 2m\pi$, $11\pi/6 \pm 2m\pi$, $m = 0, 1, 2, \ldots$