Curvilinear Coordinates

Rectangular Cartesian coordinates in the Euclidean plane or in three-dimensional space were mentioned at the beginning of this chapter. Other coordinate systems, the coordinate curves of which are generated from families that are not necessarily linear, are useful. These are called *curvilinear coordinates.*

EXAMPLE 1. Polar coordinates ρ, Φ are related to rectangular Cartesian coordinates through the transformation $x = \rho \cos \Phi$ and $y = \rho \sin \Phi$. The curves $\Phi = \Phi_0$ are radical lines, while $\rho = r_0$ are concentric circles. Among the convenient representations yielded by these coordinates are circles with the center at the origin. A weakness is that representations may not be defined at the origin.

EXAMPLE 2. Spherical coordinates r, Θ, Φ_0 for Euclidean three-dimensional space are generated from rectangular Cartesian coordinates by the transformation equations $x = r \sin \Theta \cos \Phi$, $y = r \sin \Theta \sin \Phi$, and $z = r \cos \Phi$. Again, certain problems lend themselves to spherical coordinates, and also uniqueness of representation can be a problem at the origin. The coordinate surfaces $r = r_0$, $\Theta = \Theta_0$, and $\Phi = \Phi_0$ are spheres, planes, and cones, respectively. The coordinate curves are the intersections of these surfaces, i.e., circles, circles, and lines.

For curvilinear coordinates in higher dimensional spaces, see Chapter 7.

Mean Value Theorem

If $f(x, y)$ is continuous in a closed region and if the first partial derivatives exist in the open region (i.e., excluding boundary points), then

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = hf_x(x_0 + \theta h, y_0 + \theta k) + kf_y(x_0 + \theta h, y_0 + \theta k) \qquad 0 < \theta < 1 \qquad (16)$$

This is sometimes written in a form in which $h = \Delta x = x - x_0$ and $k = \Delta y = y - y_0$.

SOLVED PROBLEMS

Functions and graphs

6.1. If $f(x, y) = x^3 - 2xy + 3y^2$, find: (a) $f\left(\frac{1}{x}, \frac{2}{y}\right)$; (c) $\frac{f(x, y+k) - f(x, y)}{k}$, $k \neq 0$.

(a) $f(-2,3) = (-2)^3 - 2(-2)(3) + 3(3)^2 = -8 + 12 + 27 = 31$

(b) $$f\left(\frac{1}{x}, \frac{2}{y}\right) = \left(\frac{1}{x}\right)^3 - 2\left(\frac{1}{x}\right)\left(\frac{2}{y}\right) + 3\left(\frac{2}{y}\right)^2 = \frac{1}{x^3} - \frac{4}{xy} + \frac{12}{y^2}$$

(c) $$\begin{aligned} \frac{f(x, y+k) - f(x, y)}{k} &= \frac{1}{k}\{[x^3 - 2x(y+k) + 3(y+k)^2] - [x^3 - 2xy + 3y^2]\} \\ &= \frac{1}{k}(x^3 - 2xy - 2kx + 3y^2 + 6ky + 3k^2 - x^2 + 2xy - 3y^2) \\ &= \frac{1}{k}(-2kx = 6ky + 3k^2) = -2x + 6y + 3k. \end{aligned}$$

6.2. Give the domain of definition for which each of the following functions is defined and real, and indicate this domain graphically.

(a) $f(x, y) = \ln\{(16 - x^2 - y^2)(x^2 + y^2 - 4)\}$

The function is defined and real for all points (x, y) such that

$$(16 - x^2 - y^2)(x^2 + y^2 - 4) > 0, \qquad \text{i.e., } 4 < x^2 + y^2 < 16$$

which is the required domain of definition. This point set consists of all points *interior* to the circle of radius 4 with center at the origin and *exterior* to the circle of radius 2 with center at the origin, as in Figure 6.5. The corresponding region, shown shaded in Figure 6.5, is an *open region*.

(b) $f(x, y) = \sqrt{6 - (2x + 3y)}$

The function is defined and real for all points (x, y) such that $2x + 3y \leq 6$, which is the required domain of definition.

The corresponding (unbounded) region of the xy plane is shown shaded in Figure 6.6.

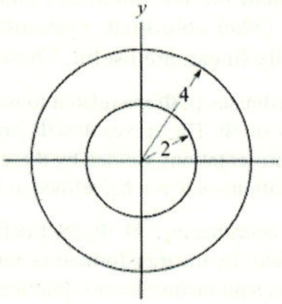

Figure 6.5

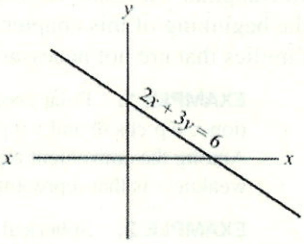

Figure 6.6

6.3. Sketch and name the surface in three-dimensional space represented by each of the following. What are the traces on the coordinate planes?

(a) $2x + 4y + 3z = 12$.

Trace on xy plane ($z = 0$) is the straight line $x + 2y = 6$, $z = 0$.
Trace on yz plane ($x = 0$) is the straight line $4y + 3z = 12$, $x = 0$.
Trace on xz plane ($y = 0$) is the straight line $2x + 3z = 12$, $y = 0$.
These are represented by AB, BC, and AC in Figure 6.7.

The surface is a plane intersecting the x, y, and z axes in the points $A(6, 0, 0)$, $B(0, 3, 0)$, and $C(0, 0, 4)$. The lengths $\overline{OA} = 6$, $\overline{OB} = 3$, and $\overline{OC} = 4$ are called the *x, y, and z intercepts*, respectively.

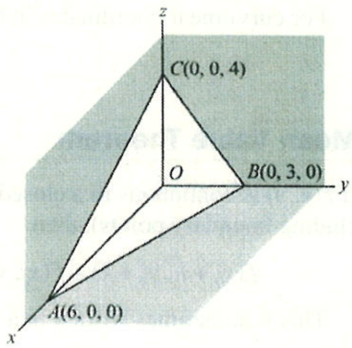

Figure 6.7

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Trace on xy plane ($z = 0$) is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, z = 0.$

Trace on yz plane ($x = 0$) is the hyperbola $\dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1, x = 0.$

Trace on xz plane ($y = 0$) is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1, y = 0.$

Trace on any plane $z = p$ parallel to the xy plane is the ellipse $\dfrac{x^2}{a^2(1 + p^2/c^2)} + \dfrac{y^2}{b^2(1 + p^2/c^2)} = 1$.

As $|p|$ increases from zero, the elliptic cross section increases in size.
The surface is a *hyperboloid of one sheet* (see Figure 6.8).

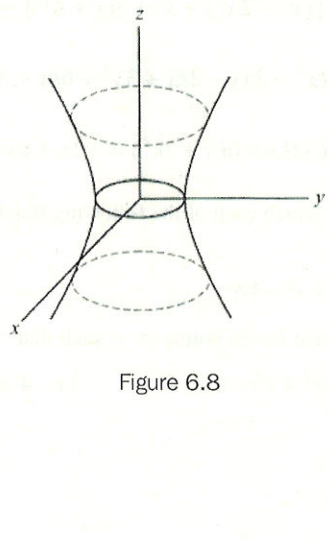

Figure 6.8

Partial derivatives

6.8. If $f(x, y) = 2x^2 - xy + y^2$, find (a) $\partial f/\partial x$ and (b) $\partial f/\partial y$ at (x_0, y_0) directly from the definition.

(a) $$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0) = \lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$= \lim_{h\to 0}\frac{[2(x_0+h)^2 - (x_0+h)y_0 + y_0^2] = [2x_0^2 - x_0y_0 + y_0^2]}{h}$$

$$= \lim_{h\to 0}\frac{4hx_0 + 2h^2 - hy_0}{h} = \lim_{h\to 0}(4x_0 + 2h - y_0) = 4x_0 - y_0$$

(b) $$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = f_y(x_0, y_0) = \lim_{k\to 0}\frac{f(x_0, y_0+k) - f(x_0, y_0)}{k}$$

$$= \lim_{k\to 0}\frac{[2x_0^2 - x_0(y_0+k) + (y_0+k)^2] - [2x_0^2 - x_0y_0 + y_0^2]}{k}$$

$$= \lim_{k\to 0}\frac{-kx_0 + 2ky_0 + k^2}{k} = \lim_{k\to 0}(-x_0 + 2y_0 + k) = -x_0 + 2y_0$$

Since the limits exist for all points (x_0, y_0), we can write $f_x(x, y) = f_x = 4x - y$, $f_y(x, y) = f_y = -x + 2y$, which are themselves functions of x and y.

Note that *formally* $f_x(x_0, y_0)$ is obtained from $f(x, y)$ by differentiating with respect to x, keeping y constant and then putting $x = x_0$, $y = y_0$. Similarly, $f_y(x_0, y_0)$ is obtained by differentiating f with respect to y, keeping x constant. This procedure, while often lucrative in practice, need not always yield correct results (see Problem 6.9). It will work if the partial derivatives are continuous.

6.9. Let $f(x, y) = \begin{cases} xy/(x^2+y^2) & (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$. Prove that (a) both $f_x(0, 0)$ and $f_y(0, 0)$ exist but that (b) $f(x, y)$ is discontinuous at (0, 0).

(a) $$f_x(0, 0) = \lim_{h\to 0}\frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

$$f_x(0, 0) = \lim_{k\to 0}\frac{f(0, 0) - f(0, 0)}{k} = \lim_{k\to 0}\frac{0}{k} = 0$$

(b) Let $(x, y) \to (0, 0)$ along the line $y = mx$ in the xy plane. Then $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y) = \lim\limits_{x\to 0}\dfrac{mx^2}{x^2 + m^2x^2} = \dfrac{m}{1+m^2}$

so that the limit depends on m and, hence, on the approach; therefore, it does not exist. Hence, $f(x, y)$ is not continuous at (0, 0).

Note that unlike the situation for functions of one variable, the existence of the first partial derivatives at a point *does not* imply continuity at the point.

Note also that if $(x, y) \neq (0, 0)$, $f_x = \dfrac{y^2 - x^2y}{(x^2+y^2)^2}$, $f_y = \dfrac{x^3 - xy^2}{(x^2+y^2)^2}$ and $f_x(0, 0)$, $f_y(0, 0)$ cannot be computed from them by merely letting $x = 0$ and $y = 0$. See the remark at the end of Problem 4.5(b).

6.10. If $\phi(x, y) = x^3y + e^{xy^2}$, find (a) ϕ_x, (b) ϕ_y, ϕ_{xx}, (d) ϕ_{yy}, (e) ϕ_{xy}, and (f) ϕ_{yx}.

(a) $$\phi_x = \frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(x^3y + e^{xy^2}) = 3x^2y + e^{xy^2}\cdot y^2 = 3x^2y + y^2e^{xy^2}$$

(b) $$\phi_y = \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(x^3y + e^{xy^2}) = x^3 + e^{xy^2}\cdot 2xy = x^3 + 2xy\,e^{xy^2}$$

(c) $$\phi_{xx} = \frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2y + y^2\,e^{xy^2}) = 6xy + y^2(e^{xy^2}\cdot y^2) = 6xy + y^4e^{xy^2}$$

(d) $\phi_{yy} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial}{\partial y}(x^3 + 2xy\, e^{xy^2}) = 0 + 2xy \cdot \dfrac{\partial}{\partial y}(e^{xy^2}) + e^{xy^2}\dfrac{\partial}{\partial y}(2xy)$

$= 2xy \cdot e^{xy^2} \cdot 2xy + e^{xy^2} \cdot 2x = x^2y^2 e^{xy^2} + 2x\, e^{xy^2}$

(e) $\phi_{xy} = \dfrac{\partial^2 \phi}{\partial y\, \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \phi}{\partial x}\right) = \dfrac{\partial}{\partial y}(3x^2 y + y^2\, e^{xy^2}) = 3x^2 + y^2 \cdot e^{xy^2} \cdot 2xy + e^{xy^2} \cdot 2y$

$= 3x^2 + 2xy^3\, e^{xy^2} + 2y\, e^{xy^2}$

(f) $\phi_{yx} = \dfrac{\partial^2 \phi}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \phi}{\partial y}\right) = \dfrac{\partial}{\partial x}(x^3 + 2xy\, e^{xy^2}) = 3x^2 + y^2 \cdot e^{xy^2} \cdot 2xy + e^{xy^2} \cdot 2y$

$= 3x^2 + 2xy^3\, e^{xy^2} + 2y\, e^{xy^2}$

Note that $\phi_{xy} = \phi_{yx}$ in this case. This is because the second partial derivatives exist and are continuous for all (x, y) in a region $\Re$. When this is not true, we may have $\phi_{xy} \neq \phi_{yx}$ (see Problem 6.41, for example).

6.11. Show that $U(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ satisfies Laplace's partial differential equation $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2} = 0.$

We assume here that $(x, y, z) \neq (0, 0, 0)$. Then

$$\frac{\partial U}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} \cdot 2x = -x(x^2 + y^2 + z^2)^{-3/2}$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}[-x(x^2 + y^2 + z^2)^{-3/2}] = (-x)[-\frac{3}{2}(x^2 + y^2 + z^2)^{-5/2} \cdot 2x] + (x^2 + y^2 + z^2)^{-3/2} \cdot (-1)$$

$$= \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

Similarly,

$$\frac{\partial^2 U}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}, \qquad \frac{\partial^2 U}{\partial x^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$$

Adding,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

6.12. If $z = x^2 \tan^{-1} \dfrac{y}{x}$, find $\dfrac{\partial^2 z}{\partial x\, \partial y}$ at (1, 1).

$$\frac{\partial z}{\partial y} = x^2 \cdot \frac{1}{1 + (y/x)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = x^2 \cdot \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x} = \frac{x^3}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x^3}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(3x^2) - (x^3)(2x)}{(x^2 + y^2)^2} = \frac{2 \cdot 3 - 1 \cdot 2}{2^2} = 1 \text{ at } (1, 1)$$

The result can be written $z_{xy}(1, 1) = 1$.

Note: In this calculation we are using the fact that z_{xy} is continuous at (1, 1) (see the remark at the end of Problem 6.9).

6.13. If $f(x, y)$ is defined in a region $\Re$ and if f_{xy} and f_{yx} exist and are continuous at a point of $\Re$, prove that $f_{xy} = f_{yx}$ at this point.

Let (x_0, y_0) be the point of $\Re$. Consider

$$G = f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) - f(x_0 - h, y_0) + f(x_0, y_0)$$