**2.14.** Evaluate each of the following, using theorems on limits.

(a) 
$$\lim_{n \to \infty} \frac{3n^2 - 5n}{5n^2 + 2n - 6} = \lim_{n \to \infty} \frac{3 - 5/n}{5 + 2/n - 6/n^2} = \frac{3 + 0}{5 + 0 + 0} = \frac{3}{5}$$

(b) 
$$\lim_{n \to \infty} \left\{ \frac{n(n+2)}{n+1} - \frac{n^3}{n^2 + 1} \right\} = \lim_{n \to \infty} \left\{ \frac{n^3 + n^2 + 2n}{(n+1)(n^2 + 1)} \right\} = \lim_{n \to \infty} \left\{ \frac{1 + 1/n + 2/n^2}{(1 + 1/n)(1 + 1/n^2)} \right\}$$
$$= \frac{1 + 0 + 0}{(1 + 0) \cdot (1 + 0)} = 1$$

$$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

(d) 
$$\lim_{n \to \infty} \frac{3n^2 + 4n}{2n - 1} = \lim_{n \to \infty} \frac{3 = 4/n}{2/n - 1/n^2}$$

Since the limits of the numerator and the denominator are 3 and 0, respectively, the limit does not exist.

Since  $\frac{3n^2 + 4n}{2n - 1} > \frac{3n^2}{2n} = \frac{3n}{2}$  can be made larger than any positive number M by choosing n > N, we can

write, if desired,  $\lim_{n\to\infty} \frac{3n^2 + 4n}{2n - 1} = \infty$ 

(e) 
$$\lim_{n \to \infty} \left( \frac{2n-3}{2n+7} \right)^4 = \left( \lim_{n \to \infty} \frac{2-3/n}{3+7/n} \right)^4 = \left( \frac{2}{3} \right)^4 = \frac{16}{18}$$

(f) 
$$\lim_{n \to \infty} \frac{2n^5 - 4n^2}{3n^7 + n^3 - 10} = \lim_{n \to \infty} \frac{2/n^2 - 4/n^5}{3 + 1/n^4 - 10/n^7} = \frac{0}{3} = 0$$

(g) 
$$\lim_{n\to\infty} \frac{1+2\cdot 10^n}{5+3\cdot 10^n} = \lim_{n\to\infty} \frac{10^{-n}+2}{5\cdot 10^{-n}+3} = \frac{2}{3}$$
 (Compare with Problem 2.5.)

#### **Bounded monotonic sequences**

- 2.15. Prove that the sequence with *n*th  $u_n = \frac{2n-7}{3n+2}$  (a) is monotonic increasing, (b) is bounded above, (c) is bounded below, (d) is bounded (e) has a limit.
  - (a)  $\{u_n\}$  is monotonic increasing if  $u_{n+1} \ge u_n$ ,  $n = 1, 2, 3, \dots$  Now

$$\frac{2(n+1)-7}{3(n+1)+2} \ge \frac{2n-7}{3n+2} \text{ if and only if } \frac{2n-5}{2n+5} \ge \frac{2n-7}{3n+2}$$

or  $(2n-5)(3n+2) \ge (2n-7)(3n+5)$ ,  $6n^2-11n-10 \ge 6n^2-11n-35$ , i.e.,  $-10 \ge -35$ , which is true. Thus, by reversal of steps in the inequalities, we see that  $\{u_n\}$  is monotonic increasing. Actually, since -10 > -35, the sequence is strictly increasing.

- (b) By writing some terms of the sequence, we may *guess* that an upper bound is 2 (for example). To *prove* this we must show that  $u_n \le 2$ . If  $(2n-7)/(3n+2) \le 2$ , then  $2n-7 \le 6n+4$  or -4n < 11, which is true. Reversal of steps proves that 2 is an upper bound.
- (c) Since this particular sequence is monotonic increasing, the first term -1 is a lower bound; i.e.,  $u_n \ge -1$ ,  $n = 1, 2, 3, \ldots$  Any number less than -1 is also a lower bound.
- (d) Since the sequence has an upper and a lower bound, it is bounded. Thus, for example, we can write  $|u_n| \le 2$  for all n.
- (e) Since every bounded monotonic (increasing or decreasing) sequence has a limit, the given sequence has a limit. In fact,  $\lim_{n\to\infty} \frac{2n-7}{3n+2} = \lim_{n\to\infty} \frac{2-7/n}{3+2/n} = \frac{2}{3}$ .

×

**2.16.** A sequence  $\{u_n\}$  is defined by the recursion formula  $u_{n+1}$   $\sqrt{3u_n}$ ,  $u_1 = 1$ . (a) Prove that  $\lim_{n \to \infty} u_n$  exists. (b) Find the limit in (a).

(a) The terms of the sequence are  $u_1 = 1$ ,  $u_2 = \sqrt{3u_1} = 3^{1/2}$ ,  $u_3 = \sqrt{3u_2} = 3^{1/2+1/4}$ , ...

The *n*th term is given by  $u_n = 3^{1/2+1/4+\cdots+1/2n-1}$ , as can be proved by mathematical induction (Chapter 1). Clearly,  $u_{n+1} \ge u_n$ . Then the sequence is monotonic increasing.

By Problem 1.14,  $u_n \le 3^1 = 3$ , i.e.,  $u_n$  is bounded above. Hence,  $u_n$  is bounded (since a lower bound is zero).

Thus, a limit exists, since the sequence is bounded and monotonic increasing.

(b) Let x = required limit. Since  $\lim_{n \to \infty} u_{n+1} = \lim_{n \to \infty} \sqrt{3u_n}$ , we have  $x = \sqrt{3x}$  and x = 3. (The other possibility, x = 0, is excluded, since  $u_n \ge 1$ .)

Another method:  $\lim_{n \to \infty} 3^{1/2 + 1/4 + \dots + 1/2^{n-1}} = \lim_{n \to \infty} 3^{1 - 1/2^n} = 3 \lim_{n \to \infty} (1 - 1/2^n) = 3^1 = 3$ 

**2.17.** Verify the validity of the entries in the following table.

SEQUENCE	BOUNDED	MONOTONIC INCREASING	MONOTONIC DECREASING	LIMIT EXISTS
$2, 1.9, 1.8, 1.7, \ldots, 2 - (n-1)/10 \ldots$	No	No	Yes	No
$1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots$	Yes	No	No	No
$\frac{1}{2}$ , $-\frac{1}{3}$ , $\frac{1}{4}$ , $-\frac{1}{5}$ ,, $(-1)^{n-1}/(n+1)$ ,	Yes	No	No	Yes (0)
$.6, .66, .666, \ldots, \frac{2}{3} (1 - 1/10^n), \ldots$	Yes	Yes	No	Yes $(\frac{2}{3})$
$-1, +2, -3, +4, -5, \ldots, (-1)^n n, \ldots$	No	No	No	No

× 2.18.

Prove that the sequence with the *n*th term  $u_n = \left(1 + \frac{1}{n}\right)^n$  is monotonic, increasing, and bounded, and thus a limit exists. The limit is denoted by the symbol e.

*Note:*  $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ , where  $e \cong 2.71828 \cdots$  was introduced in the eighteenth century by Leonhart

Euler as the base for a system of logarithms in order to simplify certain differentiation and integration formulas.

By the binomial theorem, if n is a positive integer (see Problem 1.95),

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^2 + \dots + \frac{n(n-1)\cdots(n-n+1)}{n!}x^n$$

Letting x = 1/n,

$$u^{n} = \left(1 + \frac{1}{n}\right)^{n} = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^{2}} + \dots + \frac{n(n-1)\cdots(n-n+1)}{n!}\frac{1}{n^{n}}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)$$

$$+ \dots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{n-1}{n}\right)$$

Since each term beyond the first two terms in the last expression is an increasing function of n, it follows that the sequence  $u_n$  is a monotonic increasing sequence.

It is also clear that

$$\left(1+\frac{1}{n}\right)^n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}< 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}< 3$$

by Problem 1.14.

Thus,  $u_n$  is bounded and monotonic increasing, and so has a limit which we denote by e. The value of e = 2.71828...

2.19. Prove that 
$$\lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e$$
, where  $x \to \infty$  in any manner whatsoever (i.e., not necessarily along the positive integers, as in Problem 2.18).

If 
$$n = \text{largest integer} \le x$$
, then  $n \le x \le n+1$  and  $\left(1+\frac{1}{n+1}\right)^n \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^{n+1}$ . Since 
$$\lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1} / \left(1+\frac{1}{n+1}\right) = e \text{ and } \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) = e$$
, it follows that  $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$ .

### Least upper bound, greatest lower bound, limit superior, limit inferior

- **2.20.** Find the (a) l.u.b., (b) g.l.b., (c)  $\limsup (\overline{\lim})$ , and (d)  $\liminf (\underline{\lim})$  for the sequence 2, -2, 1, -1, 1, -1, 1, -1, 1, -1, ...
  - (a) l.u.b. = 2, since all terms are less than equal to 2, while at least one term (the 1st) is greater than  $2 \epsilon$  for any  $\epsilon > 0$ .
  - (b) g.l.b. = -2, since all terms are greater than or equal to -2, while at least one term (the 2nd) is less than  $-2 + \epsilon$  for any  $\epsilon > 0$ .
  - (c)  $\limsup \ or \ \overline{\lim} = 1$ , since infinitely many terms of the sequence are greater than  $1 \epsilon$  for any  $\epsilon > 0$  (namely, all 1's in the sequence), while only a finite number of terms are greater than  $1 + \epsilon$  for any  $\epsilon > 0$  (namely, the 1st term).
  - (d)  $\lim \inf \text{or } \lim = -1$ , since infinitely many terms of the sequence are less than  $-1 + \epsilon$  for any  $\epsilon > 0$  (namely, all -1's in the sequence), while only a finite number of terms are less than  $-1 \epsilon$  for any  $\epsilon > 0$  (namely, the 2nd term).
- **2.21.** Find the (a) l.u.b., (b) g.l.b., (c) lim sup ( $\overline{\lim}$ ), and (d) lim inf ( $\underline{\lim}$ ) for the sequences in Problem 2.17.

The results are shown in the following table.

SEQUENCE	l.u.b.	g.l.b.	lim <u>sup</u> or lim	lim inf or <u>lim</u>
$2, 1.9, 1.8, 1.7, \ldots, 2 - (n-1)/10 \ldots$	2	none	_∞	_∞
$1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots$	1	-1	1	-1
$\frac{1}{2}$ , $-\frac{1}{3}$ , $\frac{1}{4}$ $-\frac{1}{5}$ ,, $(-1)^{n-1}/(n+1)$ ,	$\frac{1}{2}$	$-\frac{1}{3}$	0	0
.6, .66, .666,, $\frac{2}{3}$ $(1-1/10^n)$ ,	$\frac{2}{3}$	6	$\frac{2}{3}$	$\frac{2}{3}$
$-1, +2, -3, +4, -5, \ldots, (-1)^n n, \ldots$	none	none	+∞	_∞

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2.44. Evaluate each of the following, using theorems on limits:

(a) 
$$\lim_{n \to \infty} \frac{4 - 2n - 3n^2}{2n^2 + n}$$

(d) 
$$\lim_{n \to \infty} \frac{4 \cdot 10^n - 3 \cdot 10^{2n}}{3 \cdot 10^{n-1} + 2 \cdot 10^{2n-1}}$$

(b) 
$$\lim_{n \to \infty} \sqrt[3]{\frac{(3 - \sqrt{n})(\sqrt{n} + 2)}{8n - 4}}$$
 (e)  $\lim_{n \to \infty} (\sqrt{n^2 + n - n})$ 

(e) 
$$\lim_{n \to \infty} (\sqrt{n^2 + n - n})$$

$$(c) \quad \lim_{n \to \infty} \frac{\sqrt{3n^2 - 5n + 4}}{2n - 7}$$

(f) 
$$\lim_{n \to \infty} (2^n + 3^n)^{1/n}$$

Ans. (a) 
$$-3/2$$
 (b)  $-1/2$  (c)  $\sqrt{3}/2$  (d)  $-15$  (e)  $1/2$  (f) 3

2 (c) 
$$\sqrt{3}$$

$$(d) -15$$

**Bounded monotonic sequences** 

- × 2.45. Prove that the sequence with *n*th term  $u_n = u_n = \sqrt{n} / (n+1)$ . (a) is monotonic decreasing, (b) is bounded below, (c) is bounded above, and (d) has a limit.
  - If  $u_n = \frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \cdots + \frac{1}{n+n}$ , prove that  $\lim_{n \to \infty} u_n$  exists and lies between 0 and 1. 2.46.
- If  $u_n = \sqrt{u_n + 1}$ ,  $u_1 = 1$ , prove that  $\lim_{n \to \infty} u = \frac{1}{2}(1 + \sqrt{5})$ . × 2.47.
  - If  $u_{n+1} = \frac{1}{2} (u_n + p/u_n)$  where p > 0 and  $u_1 > 0$ , prove that  $\lim_{n \to \infty} u_n = \sqrt{p}$ . Show how this can be used to 2.48. determine  $\sqrt{2}$ .
  - 2.49. If  $u_n$  is monotonic increasing (or monotonic decreasing), prove that  $S_n/n$ , where  $S_n = u_1 + u_2 + \cdots + u_n$  is also monotonic increasing (or monotonic decreasing).

Least upper bound, greatest lower bound, limit superior, limit inferior

2.50. Find the l.u.b., g.l.b., lim sup ( lim ), and lim inf (lim) for each sequence:

(a) 
$$-1, \frac{1}{3}, -\frac{1}{5}, \frac{1}{7}, \dots, (-1)^n/(2n-1), \dots$$
 (c)  $1, -3, 5, -7, \dots, (-1)^{n-1}(2n-1), \dots$ 

(c) 
$$1, -3, 5, -7, \ldots, (-1)^{n-1} (2n-1), \ldots$$

(b) 
$$\frac{2}{3}$$
,  $-\frac{3}{4}$ ,  $\frac{4}{5}$ ,  $-\frac{5}{6}$ , ...,  $(-1)^{n+1}$   $(n+1)/(n+2)$ , ... (d) 1, 4, 1, 16, 1, 36, ...,  $n^{1+(-1)n}$ , ...

(d) 1, 4, 1, 16, 1, 36, ..., 
$$n^{1+(-1)n}$$
, ...

Ans. (a) 
$$\frac{1}{3}$$
, -1, 0,0

Ans. (a) 
$$\frac{1}{3}$$
, -1, 0,0 (b) 1, -1, 1, -1 (c) none, none,  $+\infty$ ,  $-\infty$  (d) none, 1,  $+\infty$ , 1

2.51. Prove that a bounded sequence  $\{u_n\}$  is convergent if and only if  $\overline{\lim} u_n = \underline{\lim} u_n$ .

Infinite series

Find the sum of the series  $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ . 2.52. Ans. 2

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- 3.63. If  $[x] = \text{largest integer} \le x$ , evaluate (a)  $\lim_{x \to 2+} \{x [x]\}$  and (b)  $\lim_{x \to 2-} \{x [x]\}$ .
- 3.64. If  $\lim_{x \to x_0} f(x) = A$ , prove that (a)  $\lim_{x \to x_0} \{f(x)\}^2 = A^2$  and (b)  $\lim_{x \to x_0} \sqrt[3]{f(x)} = \sqrt[3]{A}$ . What generalizations of these do you suspect are true? Can you prove them?
- 3.65. If  $\lim_{x \to x_0} f(x)$ , = A and  $\lim_{x \to x_0} g(x)$ , = B, prove that (a)  $\lim_{x \to x_0} \{f(x) g(x)\} = A B$  and (b)  $\lim_{x \to x_0} \{af(x) + bg(x)\} = aA + bB$ , where a, b = any constants.
- **3.66.** If the limits of f(x), g(x), and h(x) are A, B, and C respectively, prove that (a)  $\lim_{x \to x_0} \{f(x) + g(x) + h(x)\} = A + B + C$  and (b)  $\lim_{x \to x_0} f(x)g(x)h(x) = ABC$ . Generalize these results.
- × 3.67. Evaluate each of the following using the theorems on limits.

(a) 
$$\lim_{x \to 1/2} \left\{ \frac{2x^2 - 1}{(3x + 2)(5x - 3)} - \frac{2 - 3x}{x^2 - 5x + 3} \right\}$$

(b) 
$$\lim_{x \to \infty} \frac{(3x-1)(2x+3)}{(5x-3)(4x+5)}$$

(c) 
$$\lim_{x \to -\infty} \left( \frac{3x}{x-1} - \frac{2x}{x+1} \right)$$

(d) 
$$\lim_{x \to 1} \frac{1}{x - 1} \left( \frac{1}{x + 3} - \frac{2x}{3x + 5} \right)$$
  
Ans. (a)  $-8/21$  (b)  $3/10$  (c) 1 (d)  $1/32$ 

- X 3.68. Evaluate  $\lim_{h\to 0} \frac{\sqrt[3]{8+h}-2}{h}$ . (Hint: Let  $8+h=x^3$ .)

  Ans. 1/12
  - 3.69. If  $\lim_{x \to x_0} f(x) = A$  and  $\lim_{x \to x_0} g(x) = B \neq 0$ , prove directly that  $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$ .
- $\times$  3.70. Given  $\lim_{x\to 0} \frac{\sin 3x}{x} = 1$ , evaluate: Corpore The Limits:

(a) 
$$\lim_{x \to 0} \frac{\sin 3x}{x}$$

(e) 
$$\lim_{x \to 0} \frac{6x - \sin 2x}{2x + 3\sin 4x}$$

(b) 
$$\lim_{x \to 0} \frac{1 - \cos x}{x}$$

(f) 
$$\lim_{x \to 0} \frac{\cos ax - \cos bx}{r^2}$$

(c) 
$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

(g) 
$$\lim_{x\to 0} \frac{1-2\cos x + \cos 2x}{x^2}$$

(d) 
$$\lim_{x \to 3} (x-3) \csc \pi x$$

(h) 
$$\lim_{x \to 1} \frac{3 \sin \pi x - \sin 3\pi x}{x^3}$$

Ans. (a) 3 (b) 0 (c) 
$$\frac{1}{2}$$
 (d)  $-1/\pi$  (e)  $\frac{2}{7}$  (f)  $\frac{1}{2}$  (b<sup>2</sup> - a<sup>2</sup>) (g) -1 (h)  $4\pi^3$ 

3.71. If 
$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$
, prove that (a)  $\lim_{x\to 0} \frac{e^{-ax} - e^{-bx}}{x} = b - a$ , (b)  $\lim_{x\to 0} \frac{a^x - b^x}{x} = \ln \frac{a}{b}$ ,  $a, b > 0$ , and (c)  $\lim_{x\to 0} \frac{\tanh ax}{x} = a$ .

**3.72.** Prove that 
$$\lim_{x \to x_0} f(x) = l$$
 if and only if  $\lim_{x \to x_0^+} f(x) = l$ .

### Continuity

In the following problems, assume the largest possible domain unless otherwise stated.

3.73. Prove that 
$$f(x) = x^2 - 3x + 2$$
 is continuous at  $x = 4$ .

**3.74.** Prove that 
$$f(x) = 1/x$$
 is continuous (a) at  $x = 2$  and (b) in  $1 \le x \le 3$ .

(a) 
$$f(x) = \frac{\sin x}{x}$$
;  $x \neq 0$ ,  $f(0) = 0$ ;  $x = 0$ 

(a) 
$$f(x) = \frac{\sin x}{x}$$
;  $x \neq 0$ ,  $f(0) = 0$ ;  $x = 0$  (c)  $f(x) = \frac{x^3 - 8}{x^2 - 4}$ ;  $x \neq 2$ ,  $f(2) = 3$ ;  $x = 2$ 

(b) 
$$f(x) = x - |x|$$
;  $x = 0$ 

(d) 
$$f(x) = \begin{cases} \sin \pi x, & 0 < x < 1 \\ \ln & 1 < x < 2 \end{cases}$$
;  $x = 1$ 

Ans. (a) discontinuous, (b) continuous, (c) continuous, (d) discontinuous

3.76. If 
$$[x]$$
 = greatest integer  $\leq x$ , investigate the continuity of  $f(x) = x - [x]$  in the interval (a)  $1 < x < 2$  and (b)  $1 \leq x \leq 2$ .

**3.77.** Prove that 
$$f(x) = x^3$$
 is continuous in every finite interval.

3.78. If 
$$f(x)/g(x)$$
 and  $g(x)$  are continuous at  $x = x_0$ , prove that  $f(x)$  must be continuous at  $x = x_0$ .

**3.79.** Prove that 
$$f(x) = (\tan^{-1} x)/x$$
,  $f(0) = 1$  is continuous at  $x = 0$ .

**3.81.** If 
$$f(x)$$
 and  $g(x)$  are polynomials, prove that  $f(x)/g(x)$  is continuous at each point  $x = x_0$  for which  $g(x_0) \neq 0$ .

(a) 
$$f(x) = \frac{x}{(x-2)(x-4)}$$

(c) 
$$f(x) = \sqrt{(x-3)(6-x)}$$
,  $3 \le x \le 6$ 

(b) 
$$f(x) = x^2 \sin 1/x, x \neq 0, f(0) = 0$$

$$(d) f(x) = \frac{1}{1 + 2\sin x}$$

Ans. (a) 
$$x = 2$$
, 4 (b) none (c) none (d)  $x = 7\pi/6 \pm 2m\pi$ ,  $11\pi/6 \pm 2m\pi$ ,  $m = 0, 1, 2, ...$ 

## SUPPLEMENTARY PROBLEMS

#### **Derivatives**

- 4.37. Use the definition to compute the derivatives of each of the following functions at the indicated point: (a) (3x 4)/(2x + 3), x = 1, (b)  $x^3 3x^2 + 2x 5$ , x = 2, (c)  $\sqrt{x}$ , x = 4, and (d)  $\sqrt[3]{6x 4}$ , x = 2.

  Ans. (a) 17/25, (b) 2, (c)  $\frac{1}{4}$ , (d)  $\frac{1}{2}$
- **4.38.** Show from definition that (a)  $\frac{d}{dx}x^4 = 4x^3$  and (b)  $\frac{d}{dx}\frac{3+x}{3-x} = \frac{6}{(3-x)^2}, x \neq 3$ .
- **4.39.** Let  $f(x) = \begin{cases} x^3 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$  Prove that (a) f(x) is continuous at x = 0. (b) f(x) has a derivative at x = 0, and (c) f'(x) is continuous at x = 0.
- 4.40. Let  $f(x) = \begin{cases} xe^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ . Determine whether f(x) (a) is is continuous at x = 0, and (b) has a derivative at Ans. (a) Yes (b) Yes, 0
  - **4.41.** Give an alternative proof of the theorem in Problem 4.3, using " $\epsilon$ ,  $\delta$ " definitions.
  - **4.42.** If  $f(x) = e^x$ , show that  $f'(x_0) = e^{x0}$  depends on the result  $\lim_{h \to 0} (e^h 1)/h = 1$ .
  - **4.43.** Use the results  $\lim_{h\to 0} (\sin h)/h = 1$ .  $\lim_{h\to 0} (1-\cos h)/h = 0$  to prove that if  $f(x) = \sin x$ ,  $f'(x_0) = \cos x_0$ .

### Right-and left-hand derivatives

4.44. Let f(x) = x |x|. (a) Calculate the right-hand derivative of f(x) at x = 0. (b) Calculate the left-hand derivative of f(x) at x = 0. (c) Does f(x) have a derivative at x = 0? (d) Illustrate the conclusions in (a), (b), and (c) from a graph.

- **4.45.** Discuss the (a) continuity and (b) differentiability of  $f(x) = x^p \sin 1/x$ , f(0) = 0, where p is any positive number. What happens in case p is any real number?
- - **4.47.** Prove that the derivative of f(x) at  $x = x_0$  exists if and only if  $f'_+(x_0) = f'_-(x_0)$ .
  - 4.48. (a) Prove that  $f(x) = x^3 x^2 + 5x 6$  is differentiable in  $a \le x \le b$ , where a and b are any constants. (b) Find equations for the tangent lines to the curve  $y = x^3 x^2 + 5x 6$  at x = 0 and x = 1. Illustrate by means of a graph. (c) Determine the point of intersection of the tangent lines in (b). (d) Find f'(x), f''(x), f'''(x), f'''(x), . . .

Ans. (b) 
$$y = 5x - 6$$
,  $y = 6x - 7$  (c)  $(1, -1)$  (d)  $3x^2 - 2x + 5$ ,  $6x - 2$ ,  $6$ ,  $0$ ,  $0$ ,  $0$ , ...

**4.49.** If  $f(x) = x^2 |x|$ , discuss the existence of successive derivatives of f(x) at x = 0.

#### **Differentials**

- **4.50.** If y = f(x) = x + 1/x, find (a)  $\Delta y$ , (b) dy, (c)  $\Delta y dy$ , (d)  $(\Delta y dy)/\Delta x$ , and (e) dy/dx.

  Ans. (a)  $\Delta x \frac{\Delta x}{x^2(x + \Delta x)}$  (b)  $\left(1 \frac{1}{x^2}\right)\Delta x$  (c)  $\frac{(\Delta x)^2}{x^2(x + \Delta x)}$  (d)  $\frac{\Delta x}{x^2(x + \Delta x)}$  (e)  $1 \frac{1}{x^2}$
- **4.51.** If  $f(x) = x^2 + 3x$ , find (a)  $\Delta y$ , (b) dy, (c)  $\Delta y/\Delta x$ , (d) dy/dx, and (e)  $(\Delta y dy)/\Delta x$ , if x = 1 and  $\Delta x = .01$ . Ans. (a) .0501, (b) .05, (c) 5.01, (d) 5, (e) .01
- **4.52.** Using differentials, compute approximate values for each of the following: (a)  $\sin 31^\circ$ , (b)  $\ln(1.12)$ , (c)  $\sqrt[5]{36}$ . Ans. (a) 0.515, (b) 0.12, (c) 2.0125
- **4.53.** If  $y = \sin x$ , evaluate (a)  $\Delta y$  and (b) dy. (c) Prove that  $(\Delta y dy)/\Delta x \to 0$  as  $\Delta x \to 0$ .

### **Differentiation rules and elementary functions**

- **4.54.** Prove the following:
  - (a)  $\frac{d}{dx} \{f(x) + g(x)\} = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$
  - (b)  $\frac{d}{dx} \left\{ f(x) g(x) \right\} = \frac{d}{dx} f(x) \frac{d}{dx} g(x)$
  - (c)  $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0.$
- **4.55.** Evaluate (a)  $\frac{d}{dx} \left\{ x^3 \ln \left( x^2 2x + 5 \right) \right\}$  at x = 1 and (b)  $\frac{d}{dx} \left\{ \sin^2 \left( 3x + \pi / 6 \right) \right\}$  at x = 0.

  Ans. (a)  $3 \ln 4$  (b)  $\frac{3}{2} \sqrt{3}$ 
  - **4.56.** Derive these formulas: (a)  $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$ , a > 0,  $a \ne 1$ ;  $\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$ ; and (c)  $\frac{d}{dx}\tanh u = \sec h^2 u \frac{du}{dx}$  where u is a differentiable function of x.
- $\times$  4.57. Compute (a)  $\frac{d}{dx} \tan^{-1} x$ , (b)  $\frac{d}{dx} \csc^{-1} x$ , (c)  $\frac{d}{dx} \sinh^{-1} x$ , and (d)  $\frac{d}{dx} \coth^{-1} x$ , paying attention to the use of principal values.
  - **4.58.** If  $y = x^x$ , compute dy/dx. (Hint: Take logarithms before differentiating.) Ans.  $x^x(1 + \ln x)$

4.59. If 
$$y = \{\ln(3x+2)\}^{\sin-1(2x+5)}$$
, find  $dy/dx$  at  $x = 0$ .  
Ans.  $\left(\frac{\pi}{4\ln 2} + \frac{2\ln \ln 2}{\sqrt{3}}\right) (\ln 2)^{\pi/6}$ 

- **4.60.** If y = f(u), where u = g(v) and v = h(x), prove that  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$  assuming f, g, and h are differentiable.
- **4.61.** Calculate (a) dy/dx and (b)  $d^2 y/dx^2$  if  $xy \ln y = 1$ . Ans. (a)  $y^2/(1 - xy)$  (b)  $(3y^3 - 2xy^4)/(1 - xy)^3$ , provided  $xy \ne 1$
- **4.62.** If  $y = \tan x$ , prove that  $y''' = 2(1 + y^2)(1 + 3y^2)$ .
  - **4.63.** If  $x = \sec t$  and  $y = \tan t$ , evaluate (a) dy/dx, (b)  $d^2y/dx^2$ , and (c)  $d^3y/dx^3$ , at  $t = \pi/4$ . Ans. (a)  $\sqrt{2}$  (b) -1 (c)  $3\sqrt{2}$
  - **4.64.** Prove that  $\frac{d^2y}{dx^2} = -\frac{d^2x}{dy^2} / \left(\frac{dx}{dy}\right)^3$ , stating precise conditions under which it holds.
  - **4.65.** Establish formulas (a) 7 and (b) 18 on Pages 73 and 78.

### Mean value theorems

- **4.66.** Let  $f(x) = 1 (x 1)^{2/3}$ ,  $0 \le x \le 2$ . (a) Construct the graph of f(x). (b) Explain why Rolle's theorem is not applicable to this functions; i.e., there is no value  $\xi$  for which  $f'(\xi) = 0$ ,  $0 < \xi < 2$ .
- **4.67.** Verify Rolle's theorem for  $f(x) = x^2(1-x)^2$ ,  $0 \le x \le 1$ .
- **4.68.** Prove that between any two real roots of  $e^x \sin x = 1$  there is at least one real root of  $e^x \cos x = -1$ . (Hint: Apply Rolle's theorem to the function  $e^{-x} \sin x$ .)
- **4.69.** (a) If 0 < a < b, prove that  $(1 a/b) < \ln b/a < (b/a 1)$ . (b) Use the result of (a) to show that  $\frac{1}{6} < \ln 1.2 \frac{1}{5}$ .
- **4.70.** Prove that  $(\pi/6 + \sqrt{3}/15) < \sin^{-1} 6 < (\pi/6 + 1/8)$  by using the mean value theorem.
- **4.71.** Show that the function F(x) in Problem 4.20 represents the difference in ordinates of curve ACB and line AB at any point x in (a, b).
- **4.72.** (a) If  $f'(x) \le 0$  at all points of (a, b), prove that f(x) is monotonic decreasing in (a, b). (b) Under what conditions is f(x) strictly decreasing in (a, b)?
- **4.73.** (a) Prove that  $(\sin x)/x$  is strictly decreasing in  $(0, \pi/2)$ . (b) Prove that  $0 \le \sin x \le 2x/\pi$  for  $0 \le x \le \pi/2$ .
- 4.74. (a) Prove that  $\frac{\sin b \sin a}{\cos a \cos b} = \cot \xi$ , where  $\xi$  is between a and b. (b) By placing a = 0 and b = x in (a), show that  $\xi = x/2$ . Does the result hold if x < 0?

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