**2.14.** Evaluate each of the following, using theorems on limits.

(a) $$\lim_{n\to\infty} \frac{3n^2 - 5n}{5n^2 + 2n - 6} = \lim_{n\to\infty} \frac{3 - 5/n}{5 + 2/n - 6/n^2} = \frac{3+0}{5+0+0} = \frac{3}{5}$$

(b) $$\lim_{n\to\infty} \left\{ \frac{n(n+2)}{n+1} - \frac{n^3}{n^2+1} \right\} = \lim_{n\to\infty} \left\{ \frac{n^3 + n^2 + 2n}{(n+1)(n^2+1)} \right\} = \lim_{n\to\infty} \left\{ \frac{1 + 1/n + 2/n^2}{(1+1/n)(1+1/n^2)} \right\}$$

$$= \frac{1+0+0}{(1+0)\cdot(1+0)} = 1$$

(c) $$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

(d) $$\lim_{n\to\infty} \frac{3n^2 + 4n}{2n - 1} = \lim_{n\to\infty} \frac{3 = 4/n}{2/n - 1/n^2}$$

Since the limits of the numerator and the denominator are 3 and 0, respectively, the limit does not exist.

Since $\dfrac{3n^2 + 4n}{2n-1} > \dfrac{3n^2}{2n} = \dfrac{3n}{2}$ can be made larger than any positive number $M$ by choosing $n > N$, we can write, if desired, $\displaystyle\lim_{n\to\infty} \frac{3n^2 + 4n}{2n-1} = \infty$.

(e) $$\lim_{n\to\infty} \left(\frac{2n-3}{2n+7}\right)^4 = \left(\lim_{n\to\infty} \frac{2 - 3/n}{3 + 7/n}\right)^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{18}$$

(f) $$\lim_{n\to\infty} \frac{2n^5 - 4n^2}{3n^7 + n^3 - 10} = \lim_{n\to\infty} \frac{2/n^2 - 4/n^5}{3 + 1/n^4 - 10/n^7} = \frac{0}{3} = 0$$

(g) $$\lim_{n\to\infty} \frac{1 + 2\cdot 10^n}{5 + 3\cdot 10^n} = \lim_{n\to\infty} \frac{10^{-n} + 2}{5\cdot 10^{-n} + 3} = \frac{2}{3}$$ (Compare with Problem 2.5.)

## Bounded monotonic sequences

**2.15.** Prove that the sequence with $n$th $u_n = \dfrac{2n-7}{3n+2}$ (a) is monotonic increasing, (b) is bounded above, (c) is bounded below, (d) is bounded, (e) has a limit.

(a) $\{u_n\}$ is monotonic increasing if $u_{n+1} \geqq u_n$, $n = 1, 2, 3, \ldots$ Now

$$\frac{2(n+1) - 7}{3(n+1) + 2} \geqq \frac{2n-7}{3n+2} \quad \text{if and only if} \quad \frac{2n - 5}{2n+5} \geqq \frac{2n-7}{3n+2}$$

or $(2n-5)(3n+2) \geqq (2n-7)(3n+5)$, $6n^2 - 11n - 10 \geqq 6n^2 - 11n - 35$, i.e., $-10 \geqq -35$, which is true. Thus, by reversal of steps in the inequalities, we see that $\{u_n\}$ is monotonic increasing. Actually, since $-10 > -35$, the sequence is strictly increasing.

(b) By writing some terms of the sequence, we may *guess* that an upper bound is 2 (for example). To *prove* this we must show that $u_n \leqq 2$. If $(2n-7)/(3n+2) \leqq 2$, then $2n - 7 \leqq 6n + 4$ or $-4n < 11$, which *is* true. Reversal of steps proves that 2 is an upper bound.

(c) Since this particular sequence is monotonic increasing, the first term $-1$ is a lower bound; i.e., $u_n \geqq -1$, $n = 1, 2, 3, \ldots$ Any number less than $-1$ is also a lower bound.

(d) Since the sequence has an upper and a lower bound, it is bounded. Thus, for example, we can write $|u_n| \leqq 2$ for all $n$.

(e) Since every bounded monotonic (increasing or decreasing) sequence has a limit, the given sequence has a limit. In fact, $\displaystyle\lim_{n\to\infty} \frac{2n-7}{3n+2} = \lim_{n\to\infty} \frac{2 - 7/n}{3 + 2/n} = \frac{2}{3}$.

**2.16.** A sequence $\{u_n\}$ is defined by the recursion formula $u_{n+1} \sqrt{3u_n}$, $u_1 = 1$. (a) Prove that $\lim_{n\to\infty} u_n$ exists. (b) Find the limit in (a).

(a) The terms of the sequence are $u_1 = 1$, $u_2 = \sqrt{3u_1} = 3^{1/2}$, $u_3 = \sqrt{3u_2} = 3^{1/2+1/4}, \ldots$.

The $n$th term is given by $u_n = 3^{1/2+1/4+\cdots+1/2n-1}$, as can be proved by mathematical induction (Chapter 1). Clearly, $u_{n+1} \geqq u_n$. Then the sequence is monotonic increasing.

By Problem 1.14, $u_n \leqq 3^1 = 3$, i.e., $u_n$ is bounded above. Hence, $u_n$ is bounded (since a lower bound is zero).

Thus, a limit exists, since the sequence is bounded and monotonic increasing.

(b) Let $x$ = required limit. Since $\lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} \sqrt{3u_n}$, we have $x = \sqrt{3x}$ and $x = 3$. (The other possibility, $x = 0$, is excluded, since $u_n \geqq 1$.)

**Another method:** $\lim_{n\to\infty} 3^{1/2+1/4+\cdots+1/2^{n-1}} = \lim_{n\to\infty} 3^{1-1/2^n} = 3 \lim_{n\to\infty}^{(1-1/2^n)} = 3^1 = 3$

**2.17.** Verify the validity of the entries in the following table.

| SEQUENCE | BOUNDED | MONOTONIC INCREASING | MONOTONIC DECREASING | LIMIT EXISTS |
|---|---|---|---|---|
| $2, 1.9, 1.8, 1.7, \ldots, 2-(n-1)/10 \ldots$ | No | No | Yes | No |
| $1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots$ | Yes | No | No | No |
| $\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots, (-1)^{n-1}/(n+1), \ldots$ | Yes | No | No | Yes (0) |
| $.6, .66, .666, \ldots, \frac{2}{3}(1-1/10^n), \ldots$ | Yes | Yes | No | Yes ($\frac{2}{3}$) |
| $-1, +2, -3, +4, -5, \ldots, (-1)^n n, \ldots$ | No | No | No | No |

**2.18.** Prove that the sequence with the $n$th term $u_n = \left(1+\frac{1}{n}\right)^n$ is monotonic, increasing, and bounded, and thus a limit exists. The limit is denoted by the symbol $e$.

*Note:* $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, where $e \cong 2.71828\cdots$ was introduced in the eighteenth century by Leonhart Euler as the base for a system of logarithms in order to simplify certain differentiation and integration formulas.

By the binomial theorem, if $n$ is a positive integer (see Problem 1.95),

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^2 + \cdots + \frac{n(n-1)\cdots(n-n+1)}{n!}x^n$$

Letting $x = 1/n$,

$$u^n = \left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots(n-n+1)}{n!}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$

$$+ \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

Since each term beyond the first two terms in the last expression is an increasing function of $n$, it follows that the sequence $u_n$ is a monotonic increasing sequence.

It is also clear that

$$\left(1+\frac{1}{n}\right)^n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} < 3$$

by Problem 1.14.

Thus, $u_n$ is bounded and monotonic increasing, and so has a limit which we denote by $e$. The value of $e$ = 2.71828 . . .

**2.19.** Prove that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$, where $x \to \infty$ in any manner whatsoever (i.e., not necessarily along the positive integers, as in Problem 2.18).

If $n$ = largest integer $\leqq x$, then $n \leqq x \leqq n+1$ and $\left(1+\frac{1}{n+1}\right)^n \leqq \left(1+\frac{1}{x}\right)^x \leqq \left(1+\frac{1}{n}\right)^{n+1}$. Since

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}\bigg/\left(1+\frac{1}{n+1}\right) = e \text{ and } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) = e,$$

it follows that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

## Least upper bound, greatest lower bound, limit superior, limit inferior

**2.20.** Find the (a) l.u.b., (b) g.l.b., (c) lim sup ( $\overline{\lim}$ ), and (d) lim inf ($\underline{\lim}$) for the sequence 2, –2, 1, –1, 1, –1, 1, –1, . . . .

(a) l.u.b. = 2, since all terms are less than equal to 2, while at least one term (the 1st) is greater than $2 - \epsilon$ for any $\epsilon > 0$.

(b) g.l.b. = –2, since all terms are greater than or equal to –2, while at least one term (the 2nd) is less than $-2 + \epsilon$ for any $\epsilon > 0$.

(c) lim sup or $\overline{\lim}$ = 1, since infinitely many terms of the sequence are greater than $1 - \epsilon$ for any $\epsilon > 0$ (namely, all 1's in the sequence), while only a finite number of terms are greater than $1 + \epsilon$ for any $\epsilon > 0$ (namely, the 1st term).

(d) lim inf or $\underline{\lim}$ = –1, since infinitely many terms of the sequence are less than $-1 + \epsilon$ for any $\epsilon > 0$ (namely, all –1's in the sequence), while only a finite number of terms are less than $-1 - \epsilon$ for any $\epsilon > 0$ (namely, the 2nd term).

**2.21.** Find the (a) l.u.b., (b) g.l.b., (c) lim sup ( $\overline{\lim}$ ), and (d) lim inf ($\underline{\lim}$) for the sequences in Problem 2.17.

The results are shown in the following table.

| SEQUENCE | l.u.b. | g.l.b. | lim sup or $\overline{\lim}$ | lim inf or $\underline{\lim}$ |
|---|---|---|---|---|
| 2, 1.9, 1.8, 1.7, . . ., $2-(n-1)/10$ . . . | 2 | none | $-\infty$ | $-\infty$ |
| 1, –1, 1, –1, . . . , $(-1)^{n-1}$, . . . | 1 | –1 | 1 | –1 |
| $\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots, (-1)^{n-1}/(n+1), \ldots$ | $\frac{1}{2}$ | $-\frac{1}{3}$ | 0 | 0 |
| .6, .66, .666, . . ., $\frac{2}{3}(1-1/10^n)$, . . . | $\frac{2}{3}$ | 6 | $\frac{2}{3}$ | $\frac{2}{3}$ |
| –1, + 2, –3, +4, –5, . . ., $(-1)^n n$, . . . | none | none | $+\infty$ | $-\infty$ |

**2.44.** Evaluate each of the following, using theorems on limits:

(a) $\lim_{n\to\infty} \dfrac{4-2n-3n^2}{2n^2+n}$

(b) $\lim_{n\to\infty} \sqrt[3]{\dfrac{(3-\sqrt{n})(\sqrt{n}+2)}{8n-4}}$

(c) $\lim_{n\to\infty} \dfrac{\sqrt{3n^2-5n+4}}{2n-7}$

(d) $\lim_{n\to\infty} \dfrac{4\cdot 10^n - 3\cdot 10^{2n}}{3\cdot 10^{n-1}+2\cdot 10^{2n-1}}$

(e) $\lim_{n\to\infty} (\sqrt{n^2+n}-n)$

(f) $\lim_{n\to\infty} (2^n+3^n)^{1/n}$

*Ans.* (a) $-3/2$ (b) $-1/2$ (c) $\sqrt{3}/2$ (d) $-15$ (e) $1/2$ (f) $3$

## Bounded monotonic sequences

**2.45.** Prove that the sequence with $n$th term $u_n = u_n = \sqrt{n}/(n+1)$, (a) is monotonic decreasing, (b) is bounded below, (c) is bounded above, and (d) ~~has a~~ limit.

**2.46.** If $u_n = \dfrac{1}{1+n} + \dfrac{1}{2+n} + \dfrac{1}{3+n} + \cdots + \dfrac{1}{n+n}$, prove that $\lim_{n\to\infty} u_n$ exists and lies between 0 and 1.

**2.47.** If $u_n = \sqrt{u_n + 1}$, $u_1 = 1$, prove that $\lim_{n\to\infty} u = \frac{1}{2}(1+\sqrt{5})$.

**2.48.** If $u_{n+1} = \frac{1}{2}(u_n + p/u_n)$ where $p > 0$ and $u_1 > 0$, prove that $\lim_{n\to\infty} u_n = \sqrt{p}$. Show how this can be used to determine $\sqrt{2}$.

**2.49.** If $u_n$ is monotonic increasing (or monotonic decreasing), prove that $S_n/n$, where $S_n = u_1 + u_2 + \cdots + u_n$ is also monotonic increasing (or monotonic decreasing).

## Least upper bound, greatest lower bound, limit superior, limit inferior

**2.50.** Find the l.u.b., g.l.b., lim sup ($\overline{\lim}$), and lim inf ($\underline{\lim}$) for each sequence:

(a) $-1, \frac{1}{3}, -\frac{1}{5}, \frac{1}{7}, \ldots, (-1)^n/(2n-1), \ldots$

(b) $\frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots, (-1)^{n+1}(n+1)/(n+2), \ldots$

(c) $1, -3, 5, -7, \ldots, (-1)^{n-1}(2n-1), \ldots$

(d) $1, 4, 1, 16, 1, 36, \ldots, n^{1+(-1)^n}, \ldots$

*Ans.* (a) $\frac{1}{3}, -1, 0, 0$ (b) $1, -1, 1, -1$ (c) none, none, $+\infty, -\infty$ (d) none, $1, +\infty, 1$

**2.51.** Prove that a bounded sequence $\{u_n\}$ is convergent if and only if $\overline{\lim}\, u_n = \underline{\lim}\, u_n$.

## Infinite series

**2.52.** Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$.

*Ans.* 2

**3.63.** If $[x]$ = largest integer $\leqq x$, evaluate (a) $\lim_{x\to 2+} \{x - [x]\}$ and (b) $\lim_{x\to 2-} \{x - [x]\}$.

*Ans.* (a) 0 and (b) 1

**3.64.** If $\lim_{x\to x_0} f(x) = A$, prove that (a) $\lim_{x\to x_0} \{f(x)\}^2 = A^2$ and (b) $\lim_{x\to x_0} \sqrt[3]{f(x)} = \sqrt[3]{A}$. What generalizations of these do you suspect are true? Can you prove them?

**3.65.** If $\lim_{x\to x_0} f(x)$, = A and $\lim g(x)$, = $B$, prove that (a) $\lim_{x\to x_0} \{f(x) - g(x)\} = A - B$ and (b) $\lim_{x\to x_0} \{af(x) + bg(x)\} = aA + bB$, where $a, b$ = any constants.

**3.66.** If the limits of $f(x)$, $g(x)$, and $h(x)$ are $A$, $B$, and $C$ respectively, prove that (a) $\lim_{x\to x_0} \{f(x) + g(x) + h(x)\} = A + B + C$ and (b) $\lim_{x\to x_0} f(x)g(x)h(x) = ABC$. Generalize these results.

X **3.67.** Evaluate each of the following using the theorems on limits.

(a) $\lim_{x\to 1/2} \left\{\dfrac{2x^2 - 1}{(3x+2)(5x-3)} - \dfrac{2-3x}{x^2 - 5x + 3}\right\}$

(b) $\lim_{x\to\infty} \dfrac{(3x-1)(2x+3)}{(5x-3)(4x+5)}$

(c) $\lim_{x\to-\infty} \left(\dfrac{3x}{x-1} - \dfrac{2x}{x+1}\right)$

(d) $\lim_{x\to 1} \dfrac{1}{x-1}\left(\dfrac{1}{x+3} - \dfrac{2x}{3x+5}\right)$

*Ans.* (a) $-8/21$ (b) 3/10 (c) 1 (d) 1/32

X **3.68.** Evaluate $\lim_{h\to 0} \dfrac{\sqrt[3]{8+h} - 2}{h}$. (Hint: Let $8 + h = x^3$.)

*Ans.* 1/12

**3.69.** If $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B \neq 0$, prove directly that $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{A}{B}$.

X **3.70.** Given $\lim_{x\to 0} \dfrac{\sin 3x}{x} = 1$, evaluate: COMPUTE THE LIMITS:

(a) $\lim_{x\to 0} \dfrac{\sin 3x}{x}$

(b) $\lim_{x\to 0} \dfrac{1 - \cos x}{x}$

(c) $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$

(d) $\lim_{x\to 3} (x-3) \csc \pi x$

(e) $\lim_{x\to 0} \dfrac{6x - \sin 2x}{2x + 3 \sin 4x}$

(f) $\lim_{x\to 0} \dfrac{\cos ax - \cos bx}{x^2}$

(g) $\lim_{x\to 0} \dfrac{1 - 2\cos x + \cos 2x}{x^2}$

(h) $\lim_{x\to 1} \dfrac{3 \sin \pi x - \sin 3\pi x}{x^3}$

*Ans.* (a) 3 (b) 0 (c) $\dfrac{1}{2}$ (d) $-1/\pi$ (e) $\dfrac{2}{7}$ (f) $\dfrac{1}{2}(b^2 - a^2)$ (g) $-1$ (h) $4\pi^3$

COMPUTE THE FOLLOWING LIMITS:

X **3.71.** If $\lim_{x\to 0}\dfrac{e^x - 1}{x} = 1$, prove that (a) $\lim_{x\to 0}\dfrac{e^{-ax} - e^{-bx}}{x} = b - a$, (b) $\lim_{x\to 0}\dfrac{a^x - b^x}{x} = \ln\dfrac{a}{b}$, $a, b > 0$, and (c) $\lim_{x\to 0}\dfrac{\tanh ax}{x} = a$.

**3.72.** Prove that $\lim_{x\to x_0} f(x) = l$ if and only if $\lim_{x\to x_0^+} f(x) = l$.

## Continuity

In the following problems, assume the largest possible domain unless otherwise stated.

**3.73.** Prove that $f(x) = x^2 - 3x + 2$ is continuous at $x = 4$.

**3.74.** Prove that $f(x) = 1/x$ is continuous (a) at $x = 2$ and (b) in $1 \leqq x \leqq 3$.

**3.75.** Investigate the continuity of each of the following functions at the indicated points:

(a) $f(x) = \dfrac{\sin x}{x}$; $x \neq 0$, $f(0) = 0$; $x = 0$

(c) $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$; $x \neq 2$, $f(2) = 3$; $x = 2$

(b) $f(x) = x - |x|$; $x = 0$

(d) $f(x) = \begin{cases} \sin \pi x, & 0 < x < 1 \\ \ln & 1 < x < 2 \end{cases}$; $x = 1$

*Ans.* (a) discontinuous, (b) continuous, (c) continuous, (d) discontinuous

**3.76.** If $[x]$ = greatest integer $\leqq x$, investigate the continuity of $f(x) = x - [x]$ in the interval (a) $1 < x < 2$ and (b) $1 \leqq x \leqq 2$.

**3.77.** Prove that $f(x) = x^3$ is continuous in every finite interval.

**3.78.** If $f(x)/g(x)$ and $g(x)$ are continuous at $x = x_0$, prove that $f(x)$ must be continuous at $x = x_0$.

**3.79.** Prove that $f(x) = (\tan^{-1} x)/x$, $f(0) = 1$ is continuous at $x = 0$.

**3.80.** Prove that a polynomial is continuous in every finite interval.

**3.81.** If $f(x)$ and $g(x)$ are polynomials, prove that $f(x)/g(x)$ is continuous at each point $x = x_0$ for which $g(x_0) \neq 0$.

**3.82.** Give the points of discontinuity of each of the following functions.

(a) $f(x) = \dfrac{x}{(x-2)(x-4)}$

(c) $f(x) = \sqrt{(x-3)(6-x)}$, $3 \leqq x \leqq 6$

(b) $f(x) = x^2 \sin 1/x$, $x \neq 0$, $f(0) = 0$

(d) $f(x) = \dfrac{1}{1 + 2\sin x}$

*Ans.* (a) $x = 2, 4$ (b) none (c) none (d) $x = 7\pi/6 \pm 2m\pi$, $11\pi/6 \pm 2m\pi$, $m = 0, 1, 2, \ldots$

## SUPPLEMENTARY PROBLEMS

### Derivatives

**4.37.** Use the definition to compute the derivatives of each of the following functions at the indicated point: (a) $(3x - 4)/(2x + 3)$, $x = 1$, (b) $x^3 - 3x^2 + 2x - 5$, $x = 2$, (c) $\sqrt{x}$, $x = 4$, and (d) $\sqrt[3]{6x - 4}$, $x = 2$.

*Ans.* (a) 17/25, (b) 2, (c) $\frac{1}{4}$, (d) $\frac{1}{2}$

**4.38.** Show from definition that (a) $\frac{d}{dx}x^4 = 4x^3$ and (b) $\frac{d}{dx}\frac{3+x}{3-x} = \frac{6}{(3-x)^2}$, $x \neq 3$.

**4.39.** Let $f(x) = \begin{cases} x^3 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ Prove that (a) $f(x)$ is continuous at $x = 0$. (b) $f(x)$ has a derivative at $x = 0$, and (c) $f'(x)$ is continuous at $x = 0$.

**4.40.** Let $f(x) = \begin{cases} xe^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Determine whether $f(x)$ (a) is is continuous at $x = 0$, and (b) has a derivative at $x = 0$.

*Ans.* (a) Yes (b) Yes, 0

**4.41.** Give an alternative proof of the theorem in Problem 4.3, using "$\epsilon$, $\delta$" definitions.

**4.42.** If $f(x) = e^x$, show that $f'(x_0) = e^{x_0}$ depends on the result $\lim_{h \to 0}(e^h - 1)/h = 1$.

**4.43.** Use the results $\lim_{h \to 0}(\sin h)/h = 1$. $\lim_{h \to 0}(1 - \cos h)/h = 0$ to prove that if $f(x) = \sin x$, $f'(x_0) = \cos x_0$.

### Right-and left-hand derivatives

**4.44.** Let $f(x) = x|x|$. (a) Calculate the right-hand derivative of $f(x)$ at $x = 0$. (b) Calculate the left-hand derivative of $f(x)$ at $x = 0$. (c) Does $f(x)$ have a derivative at $x = 0$? (d) Illustrate the conclusions in (a), (b), and (c) from a graph.

*Ans.* (a) 0 (b) 0 (c) Yes, 0

**4.45.** Discuss the (*a*) continuity and (*b*) differentiability of $f(x) = x^p \sin 1/x$, $f(0) = 0$, where $p$ is any positive number. What happens in case $p$ is any real number?

**4.46.** Let $f(x) = \begin{cases} 2x - 3, & 0 \leqq x \leqq 2 \\ x^2 - 3, & 2 < x \leqq 4 \end{cases}$. Discuss the (*a*) continuity and (*b*) differentiability of $f(x)$ in $0 \leqq x \leqq 4$.

**4.47.** Prove that the derivative of $f(x)$ at $x = x_0$ exists if and only if $f'_+(x_0) = f'_-(x_0)$.

**4.48.** (a) Prove that $f(x) = x^3 - x^2 + + 5x - 6$ is differentiable in $a \leqq x \leqq b$, where $a$ and $b$ are any constants. (b) Find equations for the tangent lines to the curve $y = x^3 - x^2 + 5x - 6$ at $x = 0$ and $x = 1$. Illustrate by means of a graph. (c) Determine the point of intersection of the tangent lines in (b). (d) Find $f'(x), f''(x), f'''(x), f^{(IV)}(x), \ldots$

*Ans.* (b) $y = 5x - 6$, $y = 6x - 7$ (c) $(1, -1)$ (d) $3x^2 - 2x + 5$, $6x - 2$, $6, 0, 0, 0, \ldots$

**4.49.** If $f(x) = x^2|x|$, discuss the existence of successive derivatives of $f(x)$ at $x = 0$.

### Differentials

**4.50.** If $y = f(x) = x + 1/x$, find (a) $\Delta y$, (b) $dy$, (c) $\Delta y - dy$, (d) $(\Delta y - dy)/\Delta x$, and (e) $dy/dx$.

*Ans.* (a) $\Delta x - \dfrac{\Delta x}{x(x+\Delta x)}$ (b) $\left(1 - \dfrac{1}{x^2}\right)\Delta x$ (c) $\dfrac{(\Delta x)^2}{x^2(x+\Delta x)}$ (d) $\dfrac{\Delta x}{x^2(x+\Delta x)}$ (e) $1 - \dfrac{1}{x^2}$

*Note:* $\Delta x = dx$.

**4.51.** If $f(x) = x^2 + 3x$, find (a) $\Delta y$, (b) $dy$, (c) $\Delta y/\Delta x$, (d) $dy/dx$, and (e) $(\Delta y - dy)/\Delta x$, if $x = 1$ and $\Delta x = .01$.

*Ans.* (a) .0501, (b) .05, (c) 5.01, (d) 5, (e) .01

**4.52.** Using differentials, compute approximate values for each of the following: (a) sin 31°, (b) ln(1.12), (c) $\sqrt[5]{36}$.

*Ans.* (a) 0.515, (b) 0.12, (c) 2.0125

**4.53.** If $y = \sin x$, evaluate (a) $\Delta y$ and (b) $dy$. (c) Prove that $(\Delta y - dy)/\Delta x \to 0$ as $\Delta x \to 0$.

### Differentiation rules and elementary functions

**4.54.** Prove the following:

(a) $\dfrac{d}{dx}\{f(x) + g(x)\} = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$

(b) $\dfrac{d}{dx}\{f(x) - g(x)\} = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$

(c) $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0.$

**4.55.** Evaluate (a) $\dfrac{d}{dx}\{x^3 \ln(x^2 - 2x + 5)\}$ at $x = 1$ and (b) $\dfrac{d}{dx}\{\sin^2(3x + \pi/6)\}$ at $x = 0$.

*Ans.* (a) $3\ln 4$ (b) $\dfrac{3}{2}\sqrt{3}$

**4.56.** Derive these formulas: (a) $\dfrac{d}{dx}a^u = a^u \ln a \dfrac{du}{dx}$, $a > 0$, $a \neq 1$; $\dfrac{d}{dx}\csc u = -\csc u \cot u \dfrac{du}{dx}$; and (c) $\dfrac{d}{dx}\tanh u = \operatorname{sech}^2 u \dfrac{du}{dx}$ where $u$ is a differentiable function of $x$.

**4.57.** Compute (a) $\dfrac{d}{dx}\tan^{-1} x$, (b) $\dfrac{d}{dx}\csc^{-1} x$, (c) $\dfrac{d}{dx}\sinh^{-1} x$, and (d) $\dfrac{d}{dx}\coth^{-1} x$, paying attention to the use of principal values.

**4.58.** If $y = x^x$, compute $dy/dx$. (Hint: Take logarithms before differentiating.)

*Ans.* $x^x(1 + \ln x)$

**4.59.** If $y = \{\ln(3x+2)\}^{\sin^{-1}(2x+.5)}$, find $dy/dx$ at $x = 0$.

*Ans.* $\left(\dfrac{\pi}{4\ln 2} + \dfrac{2\ln\ln 2}{\sqrt{3}}\right)(\ln 2)^{\pi/6}$

**4.60.** If $y = f(u)$, where $u = g(\upsilon)$ and $\upsilon = h(x)$, prove that $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}$ assuming $f$, $g$, and $h$ are differentiable.

**4.61.** Calculate (*a*) $dy/dx$ and (*b*) $d^2y/dx^2$ if $xy - \ln y = 1$.

*Ans.* (a) $y^2/(1-xy)$ (b) $(3y^3 - 2xy^4)/(1-xy)^3$, provided $xy \neq 1$

**4.62.** If $y = \tan x$, prove that $y''' = 2(1+y^2)(1+3y^2)$.

**4.63.** If $x = \sec t$ and $y = \tan t$, evaluate (a) $dy/dx$, (b) $d^2y/dx^2$, and (c) $d^3y/dx^3$, at $t = \pi/4$.

*Ans.* (a) $\sqrt{2}$ (b) $-1$ (c) $3\sqrt{2}$

**4.64.** Prove that $\dfrac{d^2y}{dx^2} = -\dfrac{d^2x}{dy^2}\Big/\left(\dfrac{dx}{dy}\right)^3$, stating precise conditions under which it holds.

**4.65.** Establish formulas (a) 7 and (b) 18 on Pages 73 and 78.

## Mean value theorems

**4.66.** Let $f(x) = 1 - (x-1)^{2/3}$, $0 \leqq x \leqq 2$. (*a*) Construct the graph of $f(x)$. (*b*) Explain why Rolle's theorem is not applicable to this functions; i.e., there is no value $\xi$ for which $f'(\xi) = 0$, $0 < \xi < 2$.

**4.67.** Verify Rolle's theorem for $f(x) = x^2(1-x)^2$, $0 \leqq x \leqq 1$.

**4.68.** Prove that between any two real roots of $e^x \sin x = 1$ there is at least one real root of $e^x \cos x = -1$. (Hint: Apply Rolle's theorem to the function $e^{-x} - \sin x$.)

**4.69.** (a) If $0 < a < b$, prove that $(1 - a/b) < \ln b/a < (b/a - 1)$. (b) Use the result of (*a*) to show that $\dfrac{1}{6} < \ln 1.2 \dfrac{1}{5}$.

**4.70.** Prove that $(\pi/6 + \sqrt{3}/15) < \sin^{-1}.6 < (\pi/6 + 1/8)$ by using the mean value theorem.

**4.71.** Show that the function $F(x)$ in Problem 4.20 represents the difference in ordinates of curve *ACB* and line *AB* at any point $x$ in $(a, b)$.

**4.72.** (a) If $f'(x) \leqq 0$ at all points of $(a, b)$, prove that $f(x)$ is monotonic decreasing in $(a, b)$. (b) Under what conditions is $f(x)$ strictly decreasing in $(a, b)$?

**4.73.** (a) Prove that $(\sin x)/x$ is strictly decreasing in $(0, \pi/2)$. (b) Prove that $0 \leqq \sin x \leqq 2x/\pi$ for $0 \leqq x \leqq \pi/2$.

**4.74.** (a) Prove that $\dfrac{\sin b - \sin a}{\cos a - \cos b} = \cot\xi$, where $\xi$ is between $a$ and $b$. (b) By placing $a = 0$ and $b = x$ in (*a*), show that $\xi = x/2$. Does the result hold if $x < 0$?