Uniform Continuity

Let f be continuous in an interval. Then, by definition, at each point x_0 of the interval and for any $\epsilon > 0$, we can find $\delta > 0$ (which will in general depend on both ϵ and the particular point x_0) such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. If we can find δ for each ϵ which holds for all points of the interval (i.e., if δ depends *only* on ϵ and *not* on x_0), we say that f is *uniformly continuous* in the interval.

Alternatively, f is uniformly continuous in an interval if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$ where x_1 and x_2 are any two points in the interval.

Theorem If f is continuous in a *closed* interval, it is uniformly continuous in the interval.

SOLVED PROBLEMS

Functions

3.1. Let $f(x) = (x - 2)(8 - x)$ for $2 \leqq x \leqq 8$. (a) Find $f(6)$ and $f(-1)$. (b) What is the domain of definition of $f(x)$? (c) Find $f(1 - 2t)$ and give the domain of definition. (d) Find $f[f(3)], f[f(5)]$. (*e*) Graph $f(x)$.

(a) $f(6) = (6 - 2)(8 - 6) = 4 \cdot 2 = 8$
$f(-1)$ is not defined since $f(x)$ is defined only for $2 \leqq x \leqq 8$.

(b) The set of all x such that $2 \leqq x \leqq 8$.

(c) $f(1 - 2t) = \{(1 - 2t) - 2\}\{8 - (1 - 2t)\} = -(1 + 2t)(7 + 2t)$ where t is such that $2 \leqq 1 - 2t \leqq 8$; i.e., $-7/2 \leqq t \leqq -1/2$.

(d) $f(3) = (3 - 2)(8 - 3) = 5, f[f(3)] = f(5) = (5 - 2)(8 - 5) = 9.$
$f(5) = 9$ so that $f[f(5)] = f(9)$ is not defined.

(e) The following table shows $f(x)$ for various values of x.

x	2	3	4	5	6	7	8	2.5	7.5
$f(x)$	0	5	8	9	8	5	0	2.75	2.75

Plot points (2, 0), (3, 5), (4, 8), (5, 9), (6, 8), (7, 5), (8, 0), (2.5, 2.75), (7.5, 2.75). These points are only a few of the infinitely many points on the required graph shown in the adjoining Figure 3.5. This set of points defines a curve which is part of a *parabola*.

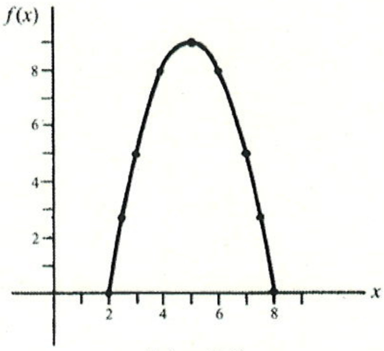

Figure 3.5

3.2. Let $g(x) = (x - 2)(8 - x)$ for $2 < x < 8$. (a) Discuss the difference between the graph of $g(x)$ and that of $f(x)$ in Problem 3.1. (b) What are the l.u.b. and g.l.b. of $g(x)$? (c) Does $g(x)$ attain its l.u.b. and g.l.b. for any value of x in the domain of definition? (d) Answer parts (b) and (c) for the function $f(x)$ of Problem 3.1.

(a) The graph of $g(x)$ is the same as that in Problem 3.1 except that the two points (2, 0) and 8, 0) are missing, since $g(x)$ is not defined at $x = 2$ and $x = 8$.

(b) The l.u.b. of $g(x)$ is 9. The g.l.b. of $g(x)$ is 0.

(c) The l.u.b. of $g(x)$ is attained for the value of $x = 5$. The g.l.b. of $g(x)$ is not attained, since there is no value of x in the domain of definition such that $g(x) = 0$.

(d) As in (*b*), the l.u.b. of $f(x)$ is 9 and the g.l.b. of $f(x)$ is 0. The l.u.b. of $f(x)$ is attained for the value $x = 5$ and the g.l.b. of $f(x)$ is attained at $x = 2$ and $x = 8$.

Note that a function, such as $f(x)$, which is *continuous* in a closed interval attains its l.u.b. and g.l.b. at some point of the interval. However, a function, such as $g(x)$, which is not continuous in a closed interval need not attain its l.u.b. and g.l.b. See Problem 3.34.

3.3. Let

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number} \end{cases}$$

(a) Find $f\left(\frac{2}{3}\right)$, $f(-5)$, $f(1.41423)$, $f(\sqrt{2})$. (b) Construct a graph of $f(x)$ and explain why it is misleading by itself.

(a) $f\left(\frac{2}{3}\right) = 1$ since $\frac{2}{3}$ is a rational number

$f(-5) = 1$ since -5 is a rational number

$f(1.41423) = 1$ since 1.41423 is a rational number

$f(\sqrt{2}) = 0$ since $\sqrt{2}$ is an irrational number

(b) The graph is shown in Figure 3.6. Because the sets of both rational numbers and irrational numbers are dense, the visual impression is that there are two images corresponding to each domain value. In actuality, each domain value has only one corresponding range value.

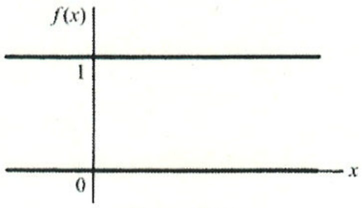

Figure 3.6

3.4. Referring to Problem 3.1: (a) Draw the graph with axes interchanged, thus illustrating the two possible choices available for definition of f^{-1}. (b) Solve for x in terms of y to determine the equations describing the two branches, and then interchange the variables.

(a) The graph of $y = f(x)$ is shown in Figure 3.5 of Problem 3.1(a). By interchanging the axes (and the variables), we obtain the graphical form of Figure 3.7. This figure illustrates that there are two values of y corresponding to each value of x, and, hence, two branches. Either may be employed to define f^{-1}.

(b) We have $y = (x - 2)(8 - x)$ or $x^2 - 10x + 16 + y = 0$. The solution of this quadratic equation is

$$x = 5 \pm \sqrt{9 - y}$$

After interchanging variables

$$y = 5 \pm \sqrt{9 - x}$$

In Figure 3.7, AP represents $y = 5 + \sqrt{9 - x}$, and BP designates $y = 5 - \sqrt{9 - x}$. Either branch may represent f^{-1}.

Note: The point at which the two branches meet is called a *branch point*.

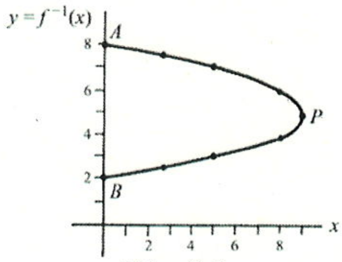

Figure 3.7

3.5. (a) Prove that $g(x) = 5 + \sqrt{9 - x}$ is strictly decreasing in $0 \leqq x \leqq 9$. (b) Is it monotonic decreasing in this interval? (c) Does $g(x)$ have a single-valued inverse?

(a) $g(x)$ is strictly decreasing if $g(x_1) > g(x_2)$ whenever $x_1 < x_2$. If $x_1 < x_2$, the $9 - x_1 > 9 - x_2$, $\sqrt{9 - x_1} > \sqrt{9 - x_2}$, and $5 + \sqrt{9 - x_1} > 5 + \sqrt{9 - x_2}$, showing that $g(x)$ is strictly decreasing.

(b) Yes, any strictly decreasing function is also monotonic decreasing, since if $g(x_1) > g(x_2)$ it is also true that $g(x_1) \geqq g(x_2)$. However, if $g(x)$ is monotonic decreasing, it is not necessarily strictly decreasing.

(c) If $y = 5 + \sqrt{9 - x}$, then $y - 5 = \sqrt{9 - x}$ or, squaring, $x = -16 + 10y - y^2 = (y - 2)(8 - y)$ and x is a single-valued function of y; i.e., the inverse function is single-valued.

In general, any strictly decreasing (or increasing) function has a single-valued inverse (see Theorem 6, Page 52).

The results of this problem can be interpreted graphically using Figure 3.7.

3.6. Construct graphs for the following functions:

$$\text{(a)}\ f(x) = \begin{cases} x \sin 1/x, & x > 0 \\ 0, & x = 0 \end{cases}$$

$$\text{(b)}\ f(x) = [x] = \text{greatest integer} \leqq x$$

(a) The required graph is shown in Figure 3.8. Since $|x \sin 1/x| \leqq |x|$, the graph is included between $y = x$ and $y = -x$. Note that $f(x) = 0$ when $\sin 1/x = 0$ or $1/x =, m\pi$, $m = 1, 2, 3, 4, \ldots$, i.e., where $x = 1/\pi, 1/2\pi, 1/3\pi, \ldots$ The curve oscillates infinitely often between $x = 1/\pi$ and $x = 0$.

(b) The required graph is shown in Figure 3.9. If $1 \leqq x < 2$, then $[x] = 1$. Thus, $[1.8] = 1$, $[\sqrt{2}] = 1$, $[1.99999] = 1$. However, $[2] = 2$. Similarly, for $2 \leqq x < 3$, $[x] = 2$, etc. Thus, there are *jumps* at the integers. The function is sometimes called the *staircase function* or *step function*.

3.7. (a) Construct the graph of $f(x) = \tan x$. (b) Construct the graph of some of the infinite number of branches available for a definition of $\tan^{-1} x$. (c) Show graphically why the relationship of x to y is multivalued. (d) Indicate possible principal values for $\tan^{-1} x$. (e) Using your choice, evaluate $\tan^{-1}(-1)$.

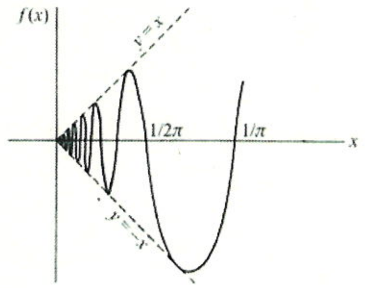

Figure 3.8

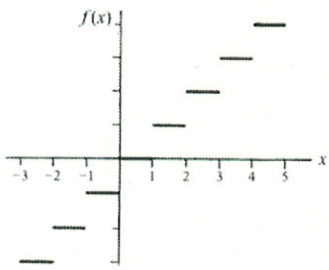

Figure 3.9

(a) The graph of $f(x) = \tan x$ appears in Figure 3.10.

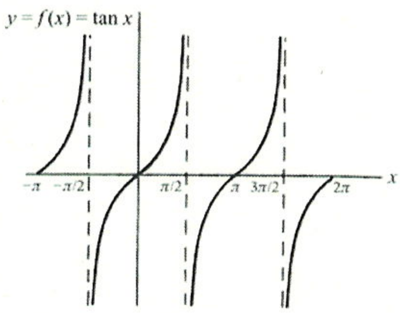

Figure 3.10

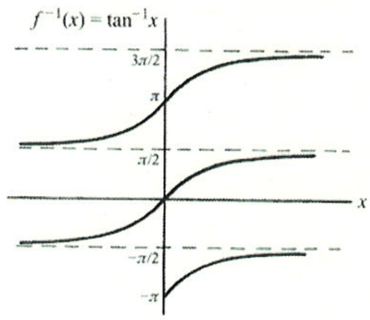

Figure 3.11

(b) The required graph is obtained by interchanging the x and y axes in the graph of (*a*). The result, with axes oriented as usual, appears in Figure 3.11.

(c) In Figure 3.11, any vertical line meets the graph in infinitely many points. Thus, the relation of y to x is multivalued and infinitely many branches are available for the purpose of defining $\tan^{-1} x$.

(d) To define $\tan^{-1} x$ as a single-valued function, it is clear from the graph that we can do so only by restricting its value to any of the following: $-\pi/2 < \tan^{-1} x < \pi/2$, $\pi/2 < \tan^{-1} x < 3\pi/2$, etc. We agree to take the first as defining the principal value.

Note that no matter which branch is used to define $\tan^{-1} x$, the resulting function is strictly increasing.

(e) $\tan^{-1}(-1) = -\pi/4$ is the only value lying between $-\pi/2$ and $\pi/2$; i.e., it is the principal value according to our choice in (*d*).