Assume x_1 is such that $a < x_0 < x < x_1 < b$. By Cauchy's generalized mean value theorem,

$$\frac{f(x)-f(x_1)}{g(x)-g(x_1)} = \frac{f'(\xi)}{g'(\xi)} \qquad x<\xi<x_1$$

Hence,

$$\frac{f(x)-f(x_1)}{g(x)-g(x_1)} = \frac{f(x)}{g(x)}\cdot\frac{1-f(x_1)/f(x)}{1-g(x_1)/g(x)} = \frac{f'(\xi)}{g'(\xi)}$$

from which we see that

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}\cdot\frac{1-g(x_1)/g(x)}{1-f(x_1)/f(x)} \tag{1}$$

Let us now suppose that $\lim\limits_{x\to x_0+}\dfrac{f'(x)}{g'(x)} = L$ and write Equation (1) as

$$\frac{f(x)}{g(x)} = \left(\frac{f'(\xi)}{g'(x)} - L\right)\left(\frac{1-g(x_1)/g(x)}{1-f(x_1)/f(x)}\right) + L\left(\frac{1-g(x_1)/g(x)}{1-f(x_1)/f(x)}\right) \tag{2}$$

We can choose x_1 so close to x_0 that $|f'(\xi)/g'(\xi) - L| < \epsilon$. Keeping x_1 fixed, we see that

$$\lim_{x\to x_0+}\left(\frac{1-g(x_1)/g(x)}{1-f(x_1)/f(x)}\right) = 1 \text{ since } 1 \lim_{x\to x_0+} f(x)_1 = \infty \text{ and } \lim_{x\to x_0+} g(x) = \infty$$

Then taking the limit as $x \to x_0+$ on both sides of (2), we see that, as required,

$$\lim_{x\to x_0+}\frac{f(x)}{g(x)} = L = \lim_{x\to x_0+}\frac{f'(x)}{g'(x)}$$

Appropriate modifications of this procedure establish the result if $x \to x_0-$, $x \to x_0$, $x \to \infty$, or $x \to -\infty$.

4.27. Evaluate (a) $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x}$ and (b) $\lim\limits_{x\to 1}\dfrac{1+\cos\pi x}{x^2-2x+1}$.

All of these have the "indeterminate form" 0/0.

(a) $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x} = \lim\limits_{x\to 0}\dfrac{2e^{2x}}{1} = 2$

(b) $\lim\limits_{x\to 1}\dfrac{1+\cos\pi x}{x^2-2x+1} = \lim\limits_{x\to 1}\dfrac{-\pi\sin\pi x}{2x-2} = \lim\limits_{x\to 1}\dfrac{-\pi^2+\cos\pi x}{2} = \dfrac{\pi^2}{2}$

Note: Here L'Hospital's rule is applied twice, since the first application again yields the "indeterminate form" 0/0 and the conditions for L'Hospital's rule are satisfied once more.

4.28. Evaluate (a) $\lim\limits_{x\to\infty}\dfrac{3x^2-x+5}{5x^2-6x-3}$ and (b) $\lim\limits_{x\to\infty} x^2e^{-x}$.

All of these have or can be arranged to have the "indeterminate form" ∞/∞.

(a) $\lim\limits_{x\to\infty}\dfrac{3x^2-x+5}{5x^2-6x-3} = \lim\limits_{x\to\infty}\dfrac{6x-1}{10x+6} = \lim\limits_{x\to\infty}\dfrac{6}{10} = \dfrac{3}{5}$

(b) $\lim\limits_{x\to\infty} x^2e^{-x} = \lim\limits_{x\to\infty}\dfrac{x^2}{e^x} = \lim\limits_{x\to\infty}\dfrac{2x}{e^x} = \lim\limits_{x\to\infty}\dfrac{2}{e^x} = 0$

4.29. Evaluate $\lim\limits_{x\to 0+} x^2\ln x$.

$$\lim_{x\to 0+} x^2 \text{In } x = \lim_{x\to 0+}\frac{\text{In } x}{1/x^2} = \lim_{x\to 0+}\frac{1/x}{-2/x^3}\ \lim_{x\to 0+}\frac{-x^2}{2} = 0$$

The given limit has the "indeterminate form" $0\cdot\infty$. In the second step the form is altered so as to give the indeterminate form ∞/∞, and L'Hospital's rule is then applied.