SOLVED PROBLEMS

Derivatives

4.1. (a) Let $f(x) = \frac{3+x}{3-x}$, $x \neq 3$. Evaluate $f'(2)$ from the definition.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{5+h}{1-h} - 5\right) = \lim_{h\to 0} \frac{1}{h} \cdot \frac{6h}{1-h} = \lim_{h\to 0} \frac{6}{1-h} = 6$$

Note: By using rules of differentiation we find

$$f'(x) = \frac{(3-x)\frac{d}{dx}(3+x) - (3+x)\frac{d}{dx}(3-x)}{(3-x)^2} = \frac{(3-x)(1) - (3+x)(-1)}{(3-x)^2} = \frac{6}{(3-x)^2}$$

at all points x where the derivative exists. Putting $x = 2$, we find $f'(2) = 6$. Although such rules are often useful, one must be careful not to apply them indiscriminately (see Problem 4.5).

(b) Let $f(x) = \sqrt{2x-1}$. Evaluate $f'(5)$ from the definition.

$$f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{\sqrt{9+2h} - 3}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{9+2h} - 3}{h} \cdot \frac{\sqrt{9+2h} + 3}{\sqrt{9+2h} + 3} = \lim_{h\to 0} \frac{9+2h-9}{h(\sqrt{9+2h} + 3} = \lim_{h\to 0} \frac{2}{\sqrt{9+2h} + 3} = \frac{1}{3}$$

By using rules of differentiation we find $f'(x) = \frac{d}{dx}(2x-1)^{1/2} = \frac{1}{2}(2x-1)^{-1/2}\frac{d}{dx}(2x-1) = (2x-1)^{-1/2}$. Then $f'(5) = 9^{-1/2} = \frac{1}{3}$.

4.2. (a) Show directly from definition that the derivative of $f(x) = x^3$ is $3x^2$.

(b) Show from definition that $\frac{d}{dx}\sqrt{x}) = \frac{1}{2\sqrt{x}}$.

(a) $$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}[(x+h)^3 - x^3]$$

$$= \frac{1}{h}[x^3 + 3x^2h + 3xh^2 + h^2] - x^3] = 3x^2 + 3xh + h^2$$

Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = 3x^2$$

(b) $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

The result follows by multiplying numerator and denominator by $\sqrt{x+h} - \sqrt{x}$ and then letting $h \to 0$.

4.3. If $f(x)$ has a derivative at $x = x_0$, prove that $f(x)$ must be continuous at $x = x_0$.

$$f(x_0 + h) - f(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} \cdot h, \quad h \neq 0$$

Then

$$\lim_{h\to 0} f(x_0 + h) - f(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot \lim_{h\to 0} h = f'(x_0) \cdot 0 = 0$$

Right- and left-hand derivatives

4.7. Let $f(x) = |x|$. (a) Calculate the right-hand derivatives of $f(x)$ at $x = 0$. (b) Calculate the left-hand derivative of $f(x)$ at $x = 0$. (c) Does $f(x)$ have a derivative at $x = 0$? (d) Illustrate the conclusions in (a), (b), and (c) from a graph.

(a) $f'_+(0) = \lim\limits_{h\to 0+} \dfrac{f(h) - f(0)}{h} = \lim\limits_{h\to 0+} \dfrac{|h| - 0}{h} = \lim\limits_{h\to 0+} \dfrac{h}{h} = 1$

since $|h| = -h$ for $h > 0$.

(b) $f'_-(0) = \lim\limits_{h\to 0-} \dfrac{f(h) - f(0)}{h} = \lim\limits_{h\to 0-} \dfrac{|h| - 0}{h} = \lim\limits_{h\to 0-} \dfrac{-h}{h} = -1$

since $|h| = -h$ for $h < 0$.

(c) No. The derivative at 0 does not exist if the right- and left-hand derivatives are unequal.

(d) The required graph is shown in Figure 4.8. Note that the slopes of the lines $y = x$ and $y = -x$ are 1 and -1, respectively, representing the right- and left-hand derivatives at $x = 0$. However, the derivative at $x = 0$ does not exist.

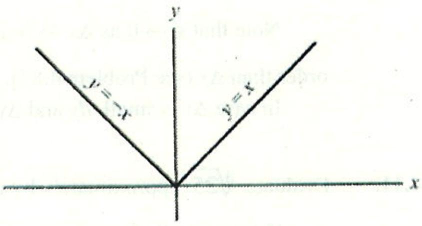

Figure 4.8

4.8. Prove that $f(x) = x^2$ is differentiable in $0 \leqq x \leqq 1$.

Let x_0 be any value such that $0 < x_0 < 1$. Then

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0 + h)^2 - x_0^2}{h} = \lim_{h\to 0} (2x_0 + h) = 2x_0$$

At the endpoint $x = 0$,

$$f'_+(0) = \lim_{h\to 0+} \frac{f(0 + h) - f(0)}{h} = \lim_{h\to 0+} \frac{h^2 - 0}{h} = \lim_{h\to 0+} h = 0$$

At the end point $x = 1$,

$$f'_-(1) = \lim_{h\to 0-} \frac{f(1 + h) - f(1)}{h} = \lim_{h\to 0-} \frac{(1 + h)^2 - 1}{h} = \lim_{h\to 0-} (2 + h) = 2$$

Then $f(x)$ is differentiable in $0 \leqq x \leqq 1$. We may write $f'(x) = 2x$ for any x in this interval. It is customary to write $f'_+(0) = f'(0)$ and $f'_-(1) = f'(1)$ in this case.

4.9. Find an equation for the tangent line to $y = x^2$ at the point where (a) $x = 1/3$ and (b) $x = 1$.

(a) From Problem 4.8. $f'(x_0) = 2x_0$ so that $f'(1/3) = 2/3$. Then the equation of the tangent line is

$$y - f(x_0) = f(x_0)(x - x_0) \quad \text{or} \quad y - \frac{1}{9} = \frac{2}{3}(x - \frac{1}{3}). \qquad \text{i.e., } y = \frac{2}{3}x - \frac{1}{9}$$

(b) As in part (*a*), $y - f(1) = f'(1)(x - 1)$ or $y - 1 = 2(x - 1)$, i.e., $y = 2x - 1$.

4.13. If $y = f(u)$ where $u = g(x)$, prove that $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$, assuming that f and g are differentiable.

Let x be given an increment $\Delta x \neq 0$. Then, as a consequence, u and y take on increments Δu and Δy, respectively, where

$$\Delta y = f(u + \Delta u) - f(u), \qquad \Delta u = g(x + \Delta x) - g(x) \tag{1}$$

Note that as $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta u \to 0$.

If $\Delta u \neq 0$, let us write $\epsilon = \dfrac{\Delta y}{\Delta u} - \dfrac{dy}{du}$ so that $\epsilon \to 0$ as $\Delta u \to 0$ and

$$\Delta y = \frac{dy}{du}\Delta u + \epsilon\, \Delta u \tag{2}$$

If $\Delta u = 0$ for values of Δx, then Equation (1) shows that $\Delta y = 0$ for these values of Δx. For such cases, we define $\epsilon = 0$.

It follows that in both cases. $\Delta u \neq 0$ or $\Delta u = 0$, Equation (2) holds. Dividing Equation (2) by $\Delta x \neq 0$ and taking the limit as $\Delta x \to 0$, we have

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{dy}{du}\frac{\Delta u}{\Delta x} + \epsilon\frac{\Delta u}{\Delta x}\right) = \frac{dy}{du}.\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} + \lim_{\Delta x\to 0}\epsilon\,.\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} \tag{3}$$

$$= \frac{dy}{du}\frac{du}{dx} + 0\,.\,\frac{du}{dx}\frac{dy}{du}\cdot\frac{du}{dx}$$

4.14. Given $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$, derive the following formulas:

(a) $\dfrac{d}{dx}(\tan x) = \sec^2 x$ (b) $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

× (a) $$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\dfrac{d}{dx}(\sin x) - \sin x\dfrac{d}{dx}}{\cos^2 x}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = {}^2\, x$$

× (b) If $y = \sin^{-1} x$, then $x = \sin y$. Taking the derivative with respect to x,

$$1 = \cos y\frac{dy}{dx} \quad\text{or}\quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

We have supposed here that the principal value $-\pi/2 \leqq \sin^{-1} x \leqq \pi/2$ is chosen so that $\cos y$ is positive, thus accounting for our writing $\cos y = \sqrt{1-\sin^2 y}$ rather than $\cos y = \pm\sqrt{1-\sin^2 y}$.

$y = \sin x$

$D \sin^{-1} y = \dfrac{1}{D \sin x} = \dfrac{1}{\cos x} = \dfrac{1}{\sqrt{1-\sin^2 x}} = \dfrac{1}{\sqrt{1-y^2}}$

4.15. Derive the formula $\dfrac{d}{dx}(\log_a u) = \dfrac{\log_a e}{u}\dfrac{du}{dx}\,(a > 0, a \neq 1)$, where u is a differentiable function of x.

Consider $y = f(u) = \log_a u$. By definition,

$$\frac{dy}{du} = \lim_{\Delta u\to 0}\frac{f(u+\Delta u) - f(u)}{\Delta u} = \lim_{\Delta u\to 0}\frac{\log_a(u+\Delta u) - \log_a u}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\frac{1}{\Delta u}\log_a\left(\frac{u+\Delta u}{u}\right) = \lim_{\Delta u\to 0}\frac{1}{u}\log_a\left(1+\frac{\Delta u}{u}\right)^{u/\Delta u}$$

Since the logarithm is a continuous function, this can be written