Then by the absolute value property 2 on Page 4,

$$|l_1 - l_2| = |l_1 - f(x) + f(x) - l_2| \leqq |l_1 - f(x)| + |f(x) - l_2| < \epsilon/2 + \epsilon/2 = \epsilon$$

i.e., $|l_1 - l_2|$ is less than any positive number ϵ (however small) and so must be zero. Thus, $l_1 = l_2$.

3.18. If $\lim\limits_{x\to x_0} g(x) = B \neq 0$, prove that there exists $\delta > 0$ such that

$$|g(x)| > \frac{1}{2}|B| \qquad \text{for} \qquad 0 < |x - x_0| < \delta$$

Since $\lim\limits_{x\to x_0} g(x) = B$, we can find $\delta > 0$ such that $|g(x) - B| < \frac{1}{2}|B|$ for $0 < |x - x_0| < \delta$.

Writing $B = B - g(x) + g(x)$, we have

$$|B| \leqq |B - g(x)| + |g(x)| < \frac{1}{2}|B| + |g(x)|$$

i.e., $|B| < \frac{1}{2}|B| + |g(x)|$, from which $|g(x)| > \frac{1}{2}|B|$.

3.19. Given $\lim\limits_{x\to x_0} f(x) = A$ and $\lim\limits_{x\to x_0} g(x) = B$, prove (a) $\lim\limits_{x\to x_0} [f(x) + g(x)] = A + B$, (b) $\lim\limits_{x\to x_0} f(x)g(x) = AB$, (c) $\lim\limits_{x\to x_0} \frac{1}{g(x)} = \frac{1}{B}$ if $B \neq 0$, and (d) $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$ if $B \neq 0$.

(a) We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$|[f(x) + g(x)] - (A + B)| < \varepsilon \qquad \text{when} \qquad 0 < |x - x_0| < \delta$$

Using absolute value property 2, Page 4, we have

$$|[f(x) + g(x)] - (A + B)| = |[f(x) - A] + [g(x) - B]| \leqq |f(x) - A| + |g(x) - B| \tag{1}$$

By hypothesis, given $\epsilon > 0$ we can find $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - A| < \epsilon/2 \qquad \text{when} \qquad 0 < |x - x_0| < \delta_1 \tag{2}$$

$$|g(x) - B| < \epsilon/2 \qquad \text{when} \qquad 0 < |x - x_0| < \delta_2 \tag{3}$$

Then, from Equations (1), (2), and (3),

$$|[f(x) + g(x)] - (A + B)| < \epsilon/2 + \epsilon/2 = \epsilon \qquad \text{when} \qquad 0 < |x - x_0| < \delta$$

where δ is chosen as the smaller of δ_1 and δ_2.

(b) We have

$$\begin{aligned} |f(x)g(x) - AB| &= |f(x)[g(x) - B] + B[f(x) - A]| \\ &\leqq |f(x)|\,|g(x) - B| + |B|\,|f(x) - A| \\ &\leqq |f(x)|\,|g(x) - B| + (|B| + 1)\,|f(x) - A| \end{aligned} \tag{4}$$

Since $\lim\limits_{x\to x_0} f(x) = A$, we can find δ_1 such that $|f(x) - A| < 1$ for $0 < |x - x_0| < \delta_1$, i.e., $A - 1 < f(x) < A + 1$, so that $f(x)$ is bounded, i.e., $|f(x)| < P$ where P is a positive constant.

Since $\lim\limits_{x\to x_0} g(x) = B$, given $\epsilon > 0$, we can find $\delta_2 > 0$ such that $|g(x) - B| < \epsilon/2P$ for $0 < |x - x_0| < \delta_2$.

Since $\lim\limits_{x\to x_0} f(x) = A$, given $\epsilon > 0$, we can find $\delta_3 > 0$ such that $|f(x) - A| < \dfrac{\varepsilon}{2(|B| + 1)}$ for $0 < |x - x_0| < \delta_2$.

Using these in Equation (4), we have

$$|f(x)g(x) - AB| < P \cdot \frac{\varepsilon}{2P} + (|B|+1) \cdot \frac{\varepsilon}{2(|B|+1)} = \varepsilon$$

for $0 < |x - x_0| < \delta$, where δ is the smaller of δ_1, δ_1, δ_2, δ_3, and the proof is complete.

(c) We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \frac{|g(x) - B|}{|B||g(x)|} < \varepsilon \quad \text{when} \quad 0 < |x - x_0| < \delta \tag{5}$$

By hypothesis, given $\epsilon > 0$, we can find $\delta_1 > 0$ such that

$$|g(x) - B| < \frac{1}{2}B^2\epsilon \qquad \text{when} \qquad 0 < |x - x_0| < \delta_1$$

By Problem 3.18, since $\lim\limits_{x\to 0} g(x) = B \neq 0$, we can find $\delta_2 > 0$ such that

$$|g(x)| > \frac{1}{2}|B| \qquad \text{when} \qquad 0 < |x - x_0| < \delta_2$$

Then, if δ is the smaller of δ_1 and δ_2, we can write

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \frac{|g(x) - B|}{|B||g(x)|} < \frac{\frac{1}{2}B^2\varepsilon}{|B| \cdot \frac{1}{2}|B|} = \varepsilon \quad \text{whenever} \quad 0 < |x - x_0| < \delta$$

and the required result is proved.

(d) From parts (b) and (c),

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} f(x) \cdot \frac{1}{g(x)} = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} \frac{1}{g(x)} = A \cdot \frac{1}{B} = \frac{A}{B}$$

This can also be proved directly (see Problem 3.69).

These results can also be proved in the cases $x \to x_0+$, $x \to x_0-$, $x \to \infty$, $x \to -\infty$.

Note: In the proof of (a) we have used the results $|f(x) - A| < \epsilon/2$ and $|g(x) - B| < \epsilon/2$, so that the final result would come out to be $|f(x) + g(x) - (A + B)| < \epsilon$. Of course, the proof would be *just as valid* if we had used 2ϵ (or any other positive multiple of ϵ) in place of ϵ. A similar remark holds for the proofs of (b), (c), and (d).

3.20. Evaluate each of the following, using theorems on limits.

(a)
$$\begin{aligned}\lim_{x\to 2}(x^2 - 6x + 4) &= \lim_{x\to 2} x^2 + \lim_{x\to 2}(-6x) + \lim_{x\to 2} 4\\ &= (\lim_{x\to 2} x)(\lim_{x\to 2} x) + (\lim_{x\to 2} -6)(\lim_{x\to 2} x) + \lim_{x\to 2} 4\\ &= (2)(2) + (-6)(2) + 4 = -4\end{aligned}$$

In practice, the intermediate steps are omitted.

(b)
$$\lim_{x\to -1} \frac{(x+3)(2x-1)}{x^2+3x-2} = \frac{\lim\limits_{x\to -1}(x+3)\lim\limits_{x\to -1}(2x-1)}{\lim\limits_{x\to -1}(x^2+3x-2)} = \frac{2\cdot(-3)}{-4} = \frac{3}{2}$$

(c)
$$\lim_{x\to\infty} \frac{2x^4 - 3x^2 + 1}{6x^4 + x^3 - 3x} = \lim_{x\to\infty} \frac{2 - \dfrac{3}{x^2} + \dfrac{1}{x^4}}{6 + \dfrac{1}{x} - \dfrac{3}{x^3}}$$

$$= \frac{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \dfrac{-3}{2} + \lim\limits_{x\to\infty} \dfrac{1}{x^4}}{\lim\limits_{x\to\infty} 6 + \lim\limits_{x\to\infty} \dfrac{1}{x} + \lim\limits_{x\to\infty} \dfrac{-3}{x^3}} = \frac{2}{6} = \frac{1}{3}$$

by Problem 3.19.

(d) $$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$
$$= \lim_{h\to 0}\frac{4+h-4}{h(\sqrt{4+h}+2} = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{2+2} = \frac{1}{4}$$

(e) $$\lim_{x\to 0+}\frac{\sin x}{\sqrt{x}} = \lim_{x\to 0+}\frac{\sin x}{x}\cdot\sqrt{x} = \lim_{x\to 0-}\frac{\sin x}{x}\cdot\lim_{x\to 0+}\sqrt{x} = 1\cdot 0 = 0.$$

Note that in (c), (d), and (e) if we use the theorems on limits indiscriminately we obtain the so-called *indeterminate forms* ∞/∞ and 0/0. To avoid such predicaments, note that in each case the form of the limit is suitably modified. For other methods of evaluating limits, see Chapter 4.

Continuity

(Assume that values at which continuity is to be demonstrated are interior domain values unless otherwise stated.)

3.21. Prove that $f(x) = x^2$ is continuous at $x = 2$.

Method 1: By Problem 3.10, $\lim_{x\to 2} f(x) = f(2) = 4$ and so $f(x)$ is continuous at $x = 2$.

Method 2: We must show that, given any $\epsilon > 0$, we can find $\delta > 0$ (depending on ϵ) such that $|f(x) - f(2)| = |x^2 - 4| < \epsilon$ when $|x - 2| < \delta$. The proof patterns are given in Problem 3.10.

3.22. (a) Prove that $f(x) = \begin{cases} x\sin 1/x, & x \neq 0 \\ 5, & x = 0 \end{cases}$ is not continuous at $x = 0$. (b) Can we redefine $f(0)$ so that $f(x)$ is continuous at $x = 0$?

(a) From Problem 3.13, $\lim_{x\to 0} f(x) = 0$. But this limit is not equal to $f(0) = 5$, so $f(x)$ is discontinuous at $x = 0$.

(b) By redefining $f(x)$ so that $f(0) = 0$, the function becomes continuous. Because the function can be made continuous at a point simply by redefining the function at the point, we call the point a *removable discontinuity.*

3.23. Is the function $f(x) = \dfrac{2x^4 - 6x^3 + x^2 + 3}{x-1}$ continuous at $x = 1$?

$f(1)$ does not exist, so $f(x)$ is not continuous at $x = 1$. By redefining $f(x)$ so that $f(1) = \lim_{x\to 1} f(x) = -8$ (see Problem 3.11), it becomes continuous at $x = 1$; i.e., $x = 1$ is a removable discontinuity.

3.24. Prove that if $f(x)$ and $g(x)$ are continuous at $x = x_0$, so also are (a) $f(x)$, + $g(x)$, (b) $f(x)g(x)$, and (c) $\dfrac{f(x)}{g(x)}$ if $f(x_0) \neq 0$.

These results follow at once from the proofs given in Problem 3.19 by taking $A = f(x_0)$ and $B = g(x_0)$ and rewriting $0 < |x - x_0| < \delta$ as $|x - x_0| < \delta$, i.e., *including* $x = x_0$.

3.25. Prove that $f(x) = x$ is continuous at any point $x = x_0$.

We must show that, given any $\epsilon > 0$, we can find $\delta > 0$ such that $|f(x) - f(x_0)| = |x - x_0| < \epsilon$ when $|x - x_0| < \delta$. By choosing $\delta = \epsilon$, the result follows at once.

3.26. Prove that $f(x) = 2x^3 + x$ is continuous at any point $x = x_0$.

Since x is continuous at any point $x = x_0$ (Problem 3.25), so also is $x \cdot x = x^2$, $x^2 \cdot x = x^3$, $2x^3$, and, finally, $2x^3 + x$, using the theorem (Problem 3.24) that sums and products of continuous functions are continuous.

3.27. Prove that if $f(x) = \sqrt{x-5}$ for $5 \leqq x \leqq 9$, then $f(x)$ is continuous in this interval.

If x_0 is any point such that $5 < x_0 < 9$, then $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} \sqrt{x-5} = \sqrt{x_0-5} = f(x_0)$. Also, $\lim\limits_{x\to 5+} \sqrt{x-5} = 0 = f(5)$ and $\lim\limits_{x\to 9-} \sqrt{x-5} = 2\ f(9)$. Thus the result follows.

Here we have used the result that $\lim\limits_{x\to x_0} \sqrt{f(x)} = \sqrt{\lim\limits_{x\to x_0} f(x)} = \sqrt{f(x_0)}$ if $f(x)$ is continuous at x_0. An ϵ, δ proof, directly from the definition, can also be employed.

3.28. For what values of x in the domain of definition is each of the following functions continuous?

(a) $f(x) = \dfrac{1}{x^2-1}$

(b) $f(x) = \dfrac{1+\cos x}{3+\sin x}$

(c) $f(x) = \dfrac{1}{\sqrt[4]{10+4}}$

(d) $f(x) = 10^{-1/(x-3)2}$

(e) $f(x) = \begin{cases} 10^{-1(x-3)^2}, & x \neq 3 \\ 0, & x = 3 \end{cases}$

(f) $f(x) = \dfrac{x-|x|}{x}$

(g) $f(x) = \begin{cases} \dfrac{x-|x|}{x}, & x < 0 \\ 2, & x = 0 \end{cases}$

(h) $f(x) = x \csc x = \dfrac{x}{\sin x}$.

(i) $f(x) = x \csc x, f(0) = 1$.

(a) All x except $\cdot\ x = \pm 1$ (where the denominator is zero)

(b) All x

(c) All $x > -10$

(d) All $x \neq 3$ (see Problem 3.55)

(e) All x, since $\lim\limits_{x\to 3} f(x) = f(3)$

(f) If $x > 0$, $f(x) = \dfrac{x-x}{x} = 0$. If $x < 0$, $f(x) = \dfrac{x+x}{x} = 2$. At $x = 0$, $f(x)$ is undefined. Then $f(x)$ is continuous for all x except $x = 0$.

(g) As in (f), $f(x)$ is continuous for $x < 0$. Then, since

$$\lim_{x\to 0-} \frac{x-|x|}{x} = \lim_{x\to 0-} \frac{x+x}{x} = \lim_{x\to 0-} 2 = 2 = f(0)$$

it follows that $f(x)$ is continuous (from the left) at $x = 0$.

Thus, $f(x)$ is continuous for all $x \leqq 0$, i.e., everywhere in its domain of definition.