The multiplier λ is the *Lagrange multiplier*. If desired, we can consider equivalently $\phi = \lambda F + G$ where $\phi_x = 0, \phi_y = 0$.

8.26. Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225, z = 0$.

We must find the minimum value of $x^2 + y^2$ (the square of the distance from the origin to any point in the xy plane) subject to the constraint $x^2 + 8xy + 7y^2 = 225$.

According to the method of Lagrange multipliers, we consider $\phi = x^2 + 8xy + 7y^2 - 225 + \lambda(x^2 + y^2)$.

Then

$$\phi_x = 2x + 8y + 2\lambda x = 0 \quad \text{or} \quad (\lambda + 1)x + 4y = 0 \qquad (1)$$

$$\phi_y = 8x + 14y + 2\lambda y = 0 \quad \text{or} \quad 4x + (\lambda + 7)y = 0 \qquad (2)$$

From Equations (1) and (2), since $(x, y) \neq (0, 0)$, we must have

$$\begin{vmatrix} \lambda+1 & 4 \\ 4 & \lambda+7 \end{vmatrix} = 0, \text{ i.e., } \lambda^2 + 8\lambda - 9 = \text{ or } \lambda = 1, -9$$

Case 1: $\lambda = 1$. From Equation (*1*) or (2), $x = -2y$, and substitution in $x^2 + 8xy + 7y^2 = 225$ yields $-5y^2 = 225$, for which no real solution exists.

Case 2: $\lambda = -9$. From Equation (*1*) or (2), $y = 2x$, and substitution in $x^2 + 8xy + 7y^2 = 225$ yields $45x^2 = 225$. Then $x^2 = 5$, $y^2 = 4x^2 = 20$ and so $x^2 + y^2 = 25$. Thus, the required shortest distance is $\sqrt{25} = 5$.

8.27. (a) Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the constraint conditions $x^2/4 + y^2/5 + z^2/25 = 1$ and $z = x + y$. (b) Give a geometric interpretation of the result in (*a*).

(a) We must find the extrema of $F = x^2 + y^2 + z^2$ subject to the constraint conditions $\phi_1 = \frac{x^2}{4} + \frac{x^2}{5} + \frac{z^2}{25} - 1 = 0$ and $\phi_2 = x + y - z = 0$. In this case we use two Lagrange multipliers λ_1, λ_2 and consider the function

$$G = F + \lambda_1\phi_1 + \lambda_2\phi_2 = x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right) + \lambda_2(x + y - z)$$

Taking the partial derivatives of G with respect to x, y, z and setting them equal to zero, we find

$$G_x = 2x + \frac{\lambda_1 x}{2} + \lambda_2 = 0, \quad G_y = 2y + \frac{2\lambda_1 y}{5} + \lambda_2 = 0, \quad G_x = 2z + \frac{2\lambda_1 z}{25} - \lambda_2 = 0 \qquad (1)$$

Solving these equations for x, y, z, we find

$$x = \frac{-2\lambda_2}{\lambda_1 + 4}, \quad y = \frac{-5\lambda_2}{2\lambda_1 + 10}, \quad z = \frac{25\lambda_2}{2\lambda_1 + 50} \qquad (2)$$

From the second constraint condition, $x + y - z = 0$, we obtain, on division by λ_2, assumed different from zero (this is justified, since otherwise we would have $x = 0$, $y = 0$, $z = 0$, which would not satisfy the first constraint condition), the result

$$\frac{2}{\lambda_1 + 4} + \frac{5}{2\lambda_1 + 10} + \frac{25}{2\lambda_1 + 50} = 0$$

Multiplying both sides by $2(\lambda_1 + 4)(\lambda_1 + 5)(\lambda_1 + 5)(\lambda_1 + 25)$ and simplifying yields

$$17\lambda_1^2 + 245\lambda_1 + 750 = 0 \text{ or } (\lambda_1 + 10)(17\lambda_1 + 75) = 0$$

from which $\lambda_1 = -10$ or $-75/17$.

Case 1: $\lambda_1 = -10$.

From (2), $x = \frac{1}{3}\lambda_2$, $y = \frac{1}{2}\lambda_2$, $z = 5/6\lambda_2$. Substituting in the first constraint condition, $x^2/4 + y^2/5 + z^2/25 = 1$, yields $\lambda_2^2 = 180/19$ or $\lambda_2 = \pm 6\sqrt{5/19}$. This gives the two critical points

$$\left(2\sqrt{5/19}, 3\sqrt{5/19}, 5\sqrt{5/19}\right) \quad \left(-2\sqrt{5/19}, -3\sqrt{5/19}, -5\sqrt{5/19}\right)$$

The value of $x^2 + y^2 + z^2$ corresponding to these critical points is $(20 + 45 + 125)/19 = 10$.

Case 2: $\lambda_1 = -75/17$.

From (2), $x = 34/7\,\lambda_2$, $y = -17/4\lambda_2$, $z = 17/28\lambda_2$. Substituting in the first constraint condition, $x^2/4 + y^2/5 + z^2/25 = 1$, yields $\lambda_2 = \pm 140/(17\sqrt{646})$ which give the critical points

$$\left(40/\sqrt{646}, -35\sqrt{646}, 5/\sqrt{646}\right) \quad \left(-40/\sqrt{646}, -35\sqrt{646}, -5/\sqrt{646}\right)$$

The value of $x^2 + y^2 + z^2$ corresponding to these is $(1600 + 1225 + 25)/646 = 75/17$.

Thus, the required maximum value is 10 and the minimum value is 75/17.

(b) Since $x^2 + y^2 + z^2$ represents the square of the distance of (x, y, z) from the origin $(0, 0, 0)$, the problem is equivalent to determining the largest and smallest distances from the origin to the curve of intersection of the ellipsoid $x^2/4 + y^2/5 + z^2/25 = 1$ and the plane $z = x + y$. Since this curve is an ellipse, we have the interpretation that $\sqrt{10}$ and $\sqrt{75/17}$ are the lengths of the semimajor and semiminor axes of this ellipse.

The fact that the maximum and minimum values happen to be given by $-\lambda_1$ in both Case 1 and Case 2 is more than a coincidence. It follows, in fact, on multiplying Equations (*1*) by x, y, and z in succession and adding, for we then obtain

$$2x^2 + \frac{\lambda_1 x^2}{2} + \lambda_2 x + 2y^2 + \frac{2\lambda_1 y^2}{5} + \lambda_2 y + 2z^2 + \frac{2\lambda_1 z^2}{25} - \lambda_2 z = 0$$

i.e.,

$$x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25}\right) + \lambda_2\,(x + y - z) = 0$$

Then, using the constraint conditions, we find $x^2 + y^2 + z^2 = -\lambda_1$.

For a generalization of this problem, see Problem 8.76.

Applications to errors

8.28. The period T of a simple pendulum of length l is given by $T = 2\sqrt{l/g}$. Find (a) the error and (b) the percent error made in computing T by using $l = 2$ m and $g = 9.75$ m/ sec^2, if the true values are $l = 19.5$m and $g = 9.81$ m/sec^2.

(a) $T = 2\pi l^{1/2} g^{-1/2}$. Then

$$dT = (2\pi\, g^{-1/2}(\frac{1}{2}\, l^{-1/2}\, dl) + (2\pi\, l^{1/2})(-\frac{1}{2} g^{-3/2}\, dg) = \frac{\pi}{\sqrt{lg}}\, dl - \pi\sqrt{\frac{1}{g^3}}\, dg \qquad (1)$$

Error in $g = \Delta g = dg = +0.06$; error in $l = \Delta l = dl = -0.5$

The error in T is actually ΔT, which is in this case approximately equal to dT. Thus, we have from Equation (1),

$$\text{Error in } T = dT = \frac{\pi}{\sqrt{(2)(9.75)}}(-0.05) - \pi\sqrt{\frac{2}{(9.75)^3}}\,(+0.06) = -0.0444 \text{ sec (approx.)}$$

The value of T for $l = 2$, $g = 9.75$ is $T = 2\pi\sqrt{\frac{2}{9.75}} = 2.846$ sec (approx.)