3.32. Prove Theorem 8, Page 52.

Suppose that $f(a) < 0$ and $f(b) > 0$. Since $f(x)$ is continuous, there must be an interval $(a, a + h)$, $h > 0$, for which $f(x) < 0$. The set of points $(a, a + h)$ has an upper bound and so has a least upper bound, which we call c. Then $f(c) \leqq 0$. Now we cannot have $f(c) < 0$, because if $f(c)$ were negative we would be able to find an interval about c (including values greater than c) for which $f(x) < 0$; but since c is the least upper bound, this is impossible, and so we must have $f(c) = 0$ as required.

If $f(a) > 0$ and $f(b) < 0$, a similar argument can be used.

3.33. (a) Given $f(x) = 2x^3 - 3x^2 + 7x - 10$, evaluate $f(1)$ and $f(2)$. (b) Prove that $f(x) = 0$ for some real number x such that $1 < x < 2$. (c) Show how to calculate the value of x in (b).

(a) $f(1) = 2(1)^3 - 3(1)^2 + 7(1) - 10 = -4$, $f(2) = 2(2)^3 - 3(2)^2 + 7(2) - 10 = 8$.

(b) If $f(x)$ is continuous in $a \leqq x \leqq b$ and if $f(a)$ and $f(b)$ have opposite signs, then there is a value of x between a and b such that $f(x) = 0$ (Problem 3.32).

To apply this theorem, we need only realize that the given polynomial is continuous in $1 \leqq x \leqq 2$, since we have already shown in (*a*) that $f(1) < 0$ and $f(2) > 0$. Thus, there *exists* a number c between 1 and 2 such that $f(c) = 0$.

(c) $f(1.5) = 2(1.5)^3 - 3(1.5)^2 + 7(1.5) - 10 = 0.5$. Then, applying the theorem of (*b*) again, we see that the required root lies between 1 and 1.5 and is "most likely" closer to 1.5 than to 1, since $f(1.5) = 0.5$ has a value closer to 0 than $f(1) = -4$ (this is not always a valid conclusion but is worth pursuing in practice).

Thus, we consider $x = 1.4$. Since $f(1.4) = 2(1.4)^3 - 3(1.4)^2 + 7(1.4) - 10 = -0.592$, we conclude that there is a root between 1.4 and 1.5 which is most likely closer to 1.5 than to 1.4.

Continuing in this manner, we find that the root is 1.46 to 2 decimal places.

3.34. Prove Theorem 10, Page 52.

Given any $\epsilon > 0$, we can find x such that $M - f(x) < \epsilon$ by definition of the l.u.b. M.

Then $\dfrac{1}{M - f(x)} > \dfrac{1}{\varepsilon}$, so that $\dfrac{1}{M - f(x)}$ is not bounded and, hence, cannot be continuous in view of Theorem 4, Page 52. However, if we suppose that $f(x) \neq M$, then, since $M - f(x)$ is continuous, by hypothesis we must have $\dfrac{1}{M - f(x)}$ also continuous. In view of this contradiction, we must have $f(x) = M$ for at least one value of x in the interval.

Similarly, we can show that there exists an x in the interval such that $f(x) = m$ (Problem 3.93).

SUPPLEMENTARY PROBLEMS

Functions

3.35. Give the largest domain of definition for which each of the following rules of correspondence supports the the construction of a function.

(a) $\sqrt{(3 - x)(2x + 4)}$ (b) $(x - 2)/(x^2 - 4)$ (c) $\sqrt{\sin 3x}$ (d) $\log_{10}(x^3 - 3x^2 - 4x + 12)$

Ans. (a) $-2 \leqq x \leqq 3$ (b) all $x \neq \pm 2$ (c) $2m\pi/3 \leqq x \leqq (2m + 1)\pi/3$, $m = 0, \pm 1, \pm 2, \ldots$ (d) $x > 3$, $-2 < x < 2$

3.36. If $f(x) = \dfrac{3x+1}{x-2}$, $x \neq 2$, find:

(a) $\dfrac{5f(-1)-2f(0)+3f(5)}{6}$ (b) $\left\{f\left(-\dfrac{1}{2}\right)\right\}^2$ (c) $f(2x-3)$ (d) $f(x)+f(4/x),\ x \neq 0$

(e) $\dfrac{f(h)-f(0)}{h}\ h \neq 0$ (f) $f(\{f(x)\}$

Ans. (a) $\dfrac{61}{81}$ (b) $\dfrac{1}{25}$ (c) $\dfrac{6x-8}{2x-5}, x \neq 0, \dfrac{5}{2}, 2$ (d) $\dfrac{5}{2}, x \neq 0, 2$ (e) $\dfrac{7}{2h-4}, h \neq 0, 2$ (f) $\dfrac{10x+1}{x+5}, x \neq -5, 2$

3.37. If $f(x) = 2x^2$, $0 < x \leqq 2$, find (*a*) the l.u.b. and (*b*) the g.l.b. of $f(x)$. Determine whether $f(x)$ attains its l.u.b. and g.l.b.

Ans. (a) 8 (b) 0

3.38. Construct a graph for each of the following functions.

(a) $f(x) = |x|, -3 \leqq x \leqq 3$

(b) $f(x) = 2 - \dfrac{|x|}{x}, -2 \leqq x \leqq 2$

(c) $f(x) = \begin{cases} 0, & x<0 \\ \dfrac{1}{2}, & x=0 \\ 1, & x>0 \end{cases}$

(d) $f(x) = \begin{cases} -x, & -2 \leqq x \leqq 0 \\ x, & 0 \leqq x \leqq 2 \end{cases}$

(e) $f(x) = x^2 \sin 1/x,\ x \neq 0$

(*f*) $\dfrac{x-[x]}{x}$ where $[x]$ = greatest integer $\leqq x$

(*g*) $f(x) = \cosh x$

(*h*) $f(x) = \dfrac{\sin x}{x}$

(*i*) $f(x) = \dfrac{x}{(x-1)(x-2)(x-3)}$

(*j*) $f(x) = \dfrac{\sin^2 x}{x^2}$

3.39. Construct graphs for (*a*) $x^2/a^2 + y^2/b^2 = 1$, (*b*) $x^2/a^2 - y^2/b^2 = 1$, (*c*) $y^2 = 2px$, and (*d*) $y = 2ax - x^2$, where a, b, and p are given constants. In which cases, when solved for y, is there exactly one value of y assigned to each value of x, thus making possible definitions of functions f and enabling us to write $y = f(x)$? In which cases must branches be defined?

3.40. (*a*) From the graph of $y = \cos x$, construct the graph obtained by interchanging the variables and from which $\cos^{-1} x$ will result by choosing an appropriate branch. Indicate possible choices of a principal value of $\cos^{-1} x$. Using this choice, find $\cos^{-1}(1/2) - \cos^{-1}(-1/2)$. Does the value of this depend on the choice? Explain.

3.41. Work parts (a) and (b) of Problem 3.40 for (a) $y = \sec^{-1} x$ and (b) $y = \cot^{-1} x$.

3.42. Given the graph for $y = f(x)$, show how to obtain the graph for $y = f(ax + b)$, where a and b are given constants. Illustrate the procedure by obtaining the graphs of (a) $y = \cos 3x$, (b) $y = \sin(5x + \pi/3)$, and (c) $y = \tan(\pi/6 - 2x)$.

3.43. Construct graphs for (a) $y = e^{-|x|}$, (b) $y = \ln |x|$, and (c) $y = e^{-|x|} \sin x$.