1. $\mathbf{A} = \mathbf{B}$ if and only if $A_1 = B_1, A_2 = B_2, A_3 = B_3$
2. $\mathbf{A} + \mathbf{B} = (A_1 + B_1, A_2 + B_2, A_3 + B_3)$
3. $\mathbf{A} - \mathbf{B} = (A_1 - B_1, A_2 - B_2, A_3 - B_3)$
4. $\mathbf{0} = (0, 0, 0)$
5. $m\mathbf{A} = m(A_1, A_2, A_3) = (mA_1, mA_2, mA_3)$

In addition, two forms of multiplication are established.

6. $\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$
7. Length or magnitude of $\mathbf{A} = |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_1^2 + A_2^2 + A_3^2}$
8. $\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_3, A_1B_2 - A_2B_1)$

Unit vectors are defined to be (1, 0, 0), (0, 1, 0), (0, 0, 1) and then designated by **i**, **j**, **k**, respectively, thereby identifying the components axiomatically introduced with the geometric orthonormal basis elements.

If one wishes, this axiomatic formulation (which provides a component representation for vectors) can be used to reestablish the fundamental laws previously introduced geometrically; however, the primary reason for introducing this approach was to formalize a component representation of the vectors. It is that concept that will be used in the remainder of this chapter.

Note 1 One of the advantages of component representation of vectors is the easy extension of the ideas to all dimensions. In an n-dimensional space, the component representation is

$$\mathbf{A}(A_1, A_2, \ldots, A_n)$$

An exception is the cross product which is specifically restricted to three-dimensional space. There are generalizations of the cross product to higher dimensional spaces, but there is no direct extension.)

Note 2 The geometric interpretation of a vector endows it with an absolute meaning at any point of space. The component representation (as an ordered triple of numbers) in Euclidean three space is not unique; rather, it is attached to the coordinate system employed. This follows because the components are geometrically interpreted as the projections of the arrow representation on the coordinate directions. Therefore, the projections on the axes of a second coordinate system (rotated, for example) from the first one will be different. (See Figure 7.10.) Therefore, for theories where groups of coordinate systems play a role, a more adequate component definition of a vector is as a collection of ordered triples of numbers, each one identified with a coordinate system of the group, and any two related by a coordinate transformation. This viewpoint is indispensable in Newtonian mechanics, electromagnetic theory, special relativity, and so on.

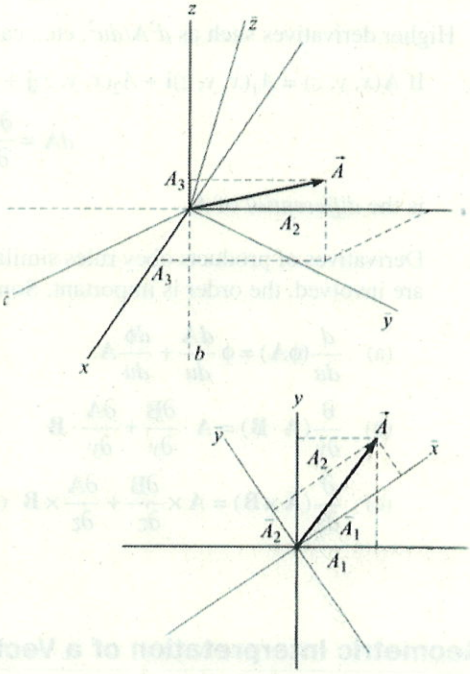

Figure 7.10

Vector Functions

If corresponding to each value of a scalar u we associate a vector $\mathbf{A}$, then $\mathbf{A}$ is called a *function* of u denoted by $\mathbf{A}(u)$. In three dimensions we can write $\mathbf{A}(u) = A_1(u)\mathbf{i} + A_2(u)\mathbf{j} + A_3(u)\mathbf{k}$.

The function concept is easily extended. Thus, if to each point (x, y, z) there corresponds a vector **A**, then **A** is a function of (x, y, z), indicated by $\mathbf{A}(x, y, z) = A_1(x, y, z)\mathbf{i} + A_2(x, y, z)\mathbf{j}, + A_3(x, y, z)\mathbf{k}$.

We sometimes say that a vector function **A** defines a *vector field* since it associates a vector with each point of a region. Similarly, $\phi(x, y, z)$ defines a *scalar field* since it associates a scalar with each point of a region.

Limits, Continuity, and Derivatives of Vector Functions

Limits, continuity, and derivatives of vector functions follow rules similar to those for scalar functions already considered. The following statements show the analogy which exists.

1. The vector function represented by $\mathbf{A}(u)$ is said to be *continuous* at u_0 if, given any positive number δ, we can find some positive number δ such that $|\mathbf{A}(u) - \mathbf{A}(u_0)| < \delta$ whenever $|u - u_0| < \delta$. This is equivalent to the statement $\lim_{u \to u_0} \mathbf{A}(u) = \mathbf{A}(u_0)$.
2. The derivative of $\mathbf{A}(u)$ is defined as

$$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{A}(u + \Delta u) - \mathbf{A}(u)}{\Delta u}$$

provided this limit exists. In case $\mathbf{A}(u) = A_1(u)\mathbf{i} + A_2\, A_2(u)\mathbf{j} + A_3(u)\mathbf{k}$; then

$$\frac{d\mathbf{A}}{du} = \frac{dA_1}{du}\mathbf{i} + \frac{dA_2}{du}\mathbf{j} + \frac{dA_3}{du}\mathbf{k}$$

Higher derivatives such as $d^2\mathbf{A}/du^2$, etc., can be similarly defined.

3. If $\mathbf{A}(x, y, z) = A_1(x, y, z)\mathbf{i} + A_2(x, y, z)\mathbf{j} + A_3(x, y, z)\mathbf{k}$; then

$$d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial x}dx + \frac{\partial \mathbf{A}}{\partial y}dy + \frac{\partial \mathbf{A}}{\partial z}dz$$

is the *differential* of **A**.

4. Derivatives of products obey rules similar to those for scalar functions. However, when cross products are involved, the order is important. Some examples are

(a) $\dfrac{d}{du}(\phi\mathbf{A}) = \phi\dfrac{d\mathbf{A}}{du} + \dfrac{d\phi}{du}\mathbf{A}.$

(b) $\dfrac{\partial}{\partial y}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\dfrac{\partial \mathbf{B}}{\partial y} + \dfrac{\partial \mathbf{A}}{\partial y}\cdot\mathbf{B}$

(c) $\dfrac{\partial}{\partial z}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\dfrac{\partial \mathbf{B}}{\partial z\cdot} + \dfrac{\partial \mathbf{A}}{\partial z}\times\mathbf{B}$ (maintain the order of **A** and **B**)

Geometric Interpretation of a Vector Derivative

If **r** is the vector joining the origin O of a coordinate system and the point (x, y, z), then specification of the vector function $\mathbf{r}(u)$ defines x, y, and z as functions of u (**r** is called a *position vector*). As u changes, the terminal point of **r** describes a *space curve* (see Figure 7.11) having parametric equations $x = x(u)$, $y = y(u)$, $z = z(u)$. If the parameter u is the are length s measured from some fixed point on the curve, then recall from the discussion of arc length that $ds^2 = d\mathbf{r}\cdot d\mathbf{r}$. Thus,

$$\frac{d\mathbf{r}}{ds} = \mathbf{T} \tag{7}$$

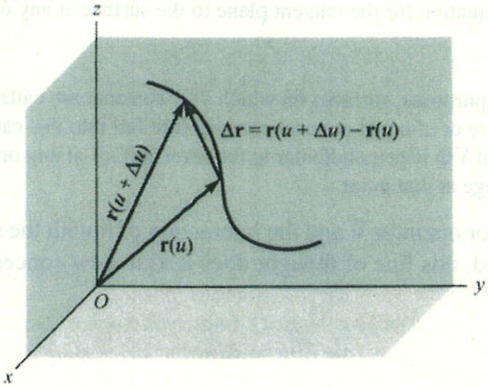

Figure 7.11

is a unit vector in the direction of the tangent to the curve and is called the *unit tangent vector*. If u is the time t, then

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \tag{8}$$

is the *velocity* with which the terminal point of **r** describes the curve. We have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\mathbf{T} = \upsilon\mathbf{T} \tag{9}$$

from which we see that the magnitude of **v** is $\upsilon = ds/dt$. Similarly,

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a} \tag{10}$$

is the *acceleration* with which the terminal point of **r** describes the curve. These concepts have important applications in *mechanics* and *differential geometry*.

A primary objective of vector calculus is to express concepts in an intuitive and compact form. Success is nowhere more apparent than in applications involving the partial differentiation of scalar and vector fields. [Illustrations of such fields include implicit surface representation $\Phi\{x, y, z(x, y) = 0$, the electromagnetic potential function $\Phi(x, y, z)$, and the electromagnetic vector field $\mathbf{F}(x, y, z)$.] To give mathematics the capability of addressing theories involving such functions, William Rowan Hamilton and others of the nineteenth century introduced derivative concepts called *gradient, divergence*, and *curl*, and then developed an analytic structure around them.

An intuitive understanding of these entities begins with examination of the differential of a scalar field, i.e.,

$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{dy}dy + \frac{\partial\Phi}{\partial z}dz$$

Now suppose the function Φ is constant on a surface S and that C; $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$ is a curve on S. At any point of this curve, $\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dx}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$ lies in the tangent plane to the surface. Since this statement is true for every surface curve through a given point, the differential $d\mathbf{r}$ spans the tangent plane. Thus, the triple $\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}$ represents a vector perpendicular to S. With this special geometric characteristic in mind we define

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \frac{\partial\Phi}{\partial z}\mathbf{k}$$

to be the *gradient of the scalar field* Φ.

Furthermore, we give the symbol ∇ a special significance by naming it *del*.

EXAMPLE 1. f $\Phi(x, y, z) = 0$ is an implicitly defined surface, then, because the function always has the value zero for points on it, the condition of constancy is satisfied and $\nabla\phi$ is normal to the surface at any of its points.

This allows us to form an equation for the tangent plane to the surface at any one of its points. See Problem 7.36.

EXAMPLE 2. For certain purposes, surfaces on which Φ is constant are called *level surfaces*. In meteorology, surfaces of equal temperature or of equal atmospheric pressure fall into this category. From the previous development, opment, we see that $\nabla\Phi$ is perpendicular to the level surface at any one of its points and, hence, has the direction of maximum change at that point.

The introduction of the vector operator ∇ and the interaction of it with the multiplicative properties of dot and cross come to mind. Indeed, this line of thought does lead to new concepts called *divergence* and *curl*. A summary follows.

Gradient, Divergence, and Curl

Consider the vector operator ∇ (*del*) defined by

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{11}$$

Then if $\phi(x, y, z)$ and $\mathbf{A}(x, y, z)$ have continuous first partial derivatives in a region (a condition which is in many cases stronger than necessary), we can define the following.

1. **Gradient.** The *gradient* of ϕ is defined by

$$\begin{aligned}\text{grad } \phi = \nabla\phi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \\ &= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\end{aligned} \tag{12}$$

2. **Divergence.** The *divergence* of $\mathbf{A}$ is defined by

$$\begin{aligned}\text{div } \mathbf{A} = \nabla\cdot\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \\ &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\end{aligned} \tag{13}$$

3. **Curl.** The *curl* of $\mathbf{A}$ is defined by

$$\begin{aligned}\text{curl } \mathbf{A} = \nabla\times\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \\ &= \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_2 & A_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ A_1 & A_2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ A_1 & A_2 \end{vmatrix} \\ &= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\end{aligned}$$

Note that in the expansion of the determinant, the operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ must precede A_1, A_2, A_3. In other words, ∇ is a vector operator, not a vector. When employing it, the laws of vector algebra either do not apply or at the very least must be validated. In particular, $\nabla \times \mathbf{A}$ is a new vector obtained by the specified partial differentiation on $\mathbf{A}$, while $\mathbf{A} \times \nabla$ is an operator waiting to act upon a vector or a scalar.

Formulas Involving ∇

If the partial derivatives of $\mathbf{A}$, $\mathbf{B}$, U, and V are assumed to exist, then

1. $\nabla(U + V) = \nabla U + \nabla V$ or grad $(U + V) =$ grad u + grad V
2. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$ or div $(\mathbf{A} + \mathbf{B})$ + div $\mathbf{A}$ + div $\mathbf{B}$
3. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$ or curl $(\mathbf{A} + \mathbf{B})$ = curl $\mathbf{A}$ + curl $\mathbf{B}$
4. $\nabla \cdot (U\mathbf{A}) = (\nabla U) \cdot \mathbf{A} + U(\nabla \cdot \mathbf{A})$
5. $\nabla \times (U\mathbf{A}) = (\nabla U) \times \mathbf{A} + U(\nabla \times \mathbf{A})$
6. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
7. $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})$
8. $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$
9. $\nabla.(\nabla U) \equiv \nabla^2 U \equiv \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$ is called the *Laplacian* of U.

   and $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the *Lapacian operator*.
10. $\nabla \times (\nabla U) = 0$. The curl of the gradient of U is zero.
11. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. The divergence of the curl of $\mathbf{A}$ is zero.
12. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

Vector Interpretation of Jacobians and Orthogonal Curvilinear Coordinates

The transformation equations

$$x = f(u_1, u_2, u_3), \qquad y = g(u_1, u_2, u_3), \qquad z = h(u_1, u_2, u_3) \tag{15}$$

(where we assume that f, g, h are continuous, have continuous partial derivatives, and have a single-valued inverse) establish a one-to-one correspondence between points in an xyz and $u_1\, u_2 u_3$ rectangular coordinate system. In vector notation, the transformation (15) can be written

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = f(u_1, u_2, u_3)\mathbf{i} + g(u_1, u_2, u_3)\mathbf{j} + h(u_1, u_2, u_3)\mathbf{k} \tag{16}$$

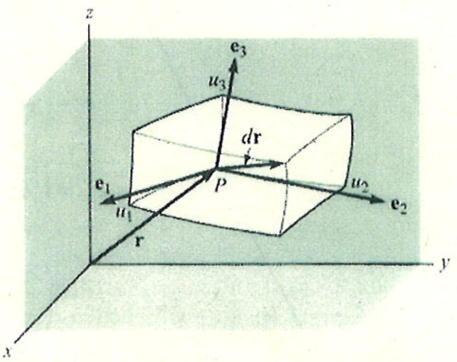

Figure 7.12

A point P in Figure 7.12 can then be defined not only by *rectangular coordinates* (x, y, z) but by coordinates (u_1, u_2, u_3) as well. We call (u_1, u_2, u_3) the *curvilinear coordinates* of the point.

If u_2 and u_3 are constant, then as u_1 varies, $\mathbf{r}$ describes a curve which we call the u_1 *coordinate curve*. Similarly, we define the u_2 and u_3 coordinate curves through P.

From Equation (16), we have

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 \tag{17}$$

The collection of vectors $\frac{\partial \mathbf{r}}{\partial u_1}, \frac{\partial \mathbf{r}}{\partial u_2}, \frac{\partial \mathbf{r}}{\partial u_3}$ is a basis for the vector structure associated with the curvilinear system. If the cur-